ON L-FUNCTIONS FOR THE SPACE OF BINARY QUADRATIC FORMS

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This note is a summary of my talk in “Workshop on L-FUNCTIONS” at Fukuoka in 22 April 2011. I thank Professor Weng for his invitation to the conference. In this note, we give explicit forms of $L$-functions with nontrivial quadratic characters for the space of binary quadratic forms.

1. Main results

Let $F$ be an algebraic number field. We set $X = \{x \in M(2) \mid x = t^tx\}$ and $G = \text{GL}(1) \times \text{PGL}(2)$ over $F$. The group $G$ acts on $X$ by
\[
g \cdot x = \frac{a}{\det h}hx^t, \quad g = (a, h) \in G, \ a \in \text{GL}(1), \ h \in \text{PGL}(2), \ x \in X.
\]
This action is faithful. Let $A_F$ be the adele ring of $F$, $| |$ the idele norm of $A_F \times F$, $A^1_F = \{a \in A_F \mid |a|_F = 1\}$, $\omega$ a character on $A^1_F/F$, and $\mathcal{S}(X(A_F))$ the Schwartz space on $X(A_F)$. We assume that $\omega$ is a quadratic character, that is, $\omega^2 = 1$. We put $\omega(g) = \omega\left(\frac{a}{\det h}\right)$ and $\chi(g) = a^2$ for $g = (a, h) \in G$.

We set $X^*(F) = \{x \in X(F) \mid \det x \neq 0 \text{ and } -\det x \not\in (F^\times)^2\}$.

Let $dg$ be the Tamagawa measure on $G(A)$. We define the zeta integral $Z(\Phi, s, \omega)$ by
\[
Z(\Phi, s, \omega) = \int_{G(A_F)/G(F)} |\chi(g)|^s \omega(g) \sum_{x \in X^*(F)} \Phi(g \cdot x) \, dg
\]
for $s \in \mathbb{C}$ and $\Phi \in \mathcal{S}(X(A_F))$. $Z(\Phi, s, \omega)$ is absolutely convergent for Re($s$) $> 3/2$. We will give an explicit form of $Z(\Phi, s, \omega)$ for any non-trivial quadratic character $\omega$. Let $\Sigma$ denote the set of places of $F$. For any $v \in \Sigma$, we denote by $F_v$ the completion of $F$ at $v$ and $| |_v$ the normal valuation of $F_v$. Let $\mathcal{S}(X(F_v))$ be the Schwartz space on $X(F_v)$. For each $v < \infty$, we denote by $\mathcal{O}_v$ the ring of integers of $F_v$, $p_v$ the maximal ideal of $\mathcal{O}_v$, and $\pi_v$ a prime element. We set $q_v = |\mathcal{O}_v/p_v|$. Let $d$ be an element of $F^\times$ such that the quadratic extension $F(\sqrt{d})$ corresponds to $\omega$ via the class field theory. We set $\omega = \prod_{v \in \Sigma} \omega_v$ where $\omega_v$ is a character on $F_v^\times$. Let $\zeta_F(s)$ denote the Dedekind zeta function defined by $\zeta_F(s) = \prod_{v < \infty} (1 - q_v^{-s})^{-1}$ and $L(s, \omega)$ the Hecke $L$-function defined by $L(s, \omega) = \prod_{v \in \Sigma} L_v(s, \omega_v)$ where
\[
L_v(s, \omega_v) = \begin{cases} 
(1 - \omega_v(\pi_v)q_v^{-s})^{-1} & \text{if } v < \infty \text{ and } \omega_v \text{ is unramified} \\
1 & \text{otherwise}
\end{cases}
\]
Let $\Delta_F$ be the discriminant of $F$, $e_v$ the ramification index of $F_v$ for $v|2$, and $\Phi_{0,v}$ the characteristic function of

$$\left\{ \left( \begin{array}{ccc} a & b \\ c & d \end{array} \right) \in X(F_v) \mid a, c \in \mathcal{O}_v \text{ and } b \in \frac{1}{2} \mathcal{O}_v \right\}$$

for $v < \infty$. For each $v \in \Sigma$, we define the local zeta function $Z(\Phi_v, s, \omega_v, d)$ by

$$Z(\Phi_v, s, \omega_v, d) = \frac{2c_v}{L_v(1, \omega_v)} \times (\det x_v)^{-\frac{3}{2}} \omega_v(x_v) \Phi(x_v) \, dx_v$$

where $x_v = \left( \begin{array}{ccc} 1 & 0 \\ 0 & -d \end{array} \right)$, $c_v = \left\{ \begin{array}{ll} (1 - q_v^{-1})^{-1} & \text{if } v < \infty \\ 1 & \text{if } v|\infty \end{array} \right.$, $dx_v$ is the Haar measure on $X(F_v)$ normalized by $\int_{X(\mathcal{O}_v)} dx_v = 1$, and $\omega_v(x_v) = \omega_v\left(\frac{a_v - h_v}{\det a_v} \right)$ for $x_v = g_v \cdot x_d$, $g_v = (a_v, h_v) \in G(F_v)$. The following formula is deduced from Saito’s works [Saito1, Saito2].

**Theorem 1.** We assume that $\Phi = \prod_{v \in \Sigma} \Phi_v$ where $\Phi_v \in S(X(F_v))$. Let $S$ be any finite subset of $\Sigma$, which contains $\{ v \in \Sigma \mid v|\infty \text{ or } \omega_v \text{ is ramified or } \Phi_v \neq \Phi_{0,v} \}$. Then, we have

$$Z(\Phi, s, \omega) = \frac{L(1, \omega) |\Delta_F|^{-3/2}}{\text{Residue}_{s=1} \zeta_F(s)} \times \prod_{v|2, v \notin S} \frac{q_v^{(2s-1)e_v} \zeta_F^{S}((\mathcal{O}_v^\times)^2)}{\zeta_F^{S}((\mathcal{O}_v^\times)^2)} \times \prod_{v \in S} Z(\Phi_v, s, \omega_v, d)$$

where we set $\zeta_F^{S}(s) = \prod_{v \notin S}(1 - q_v^{-s})^{-1}$.

From Theorem 1 and some results for local zeta functions (cf. [Igusa1, Igusa2, SS]) we find that $Z(\Phi, s, \omega)$ is meromorphically continued to the whole complex $s$-plane. It is proved by [Yukie] in general. We define the function $\Phi_{1,v}$ as

$$\Phi_{1,v}(x_v) = \begin{cases} \omega_v(a) & \text{if } \exists h \text{ s.t. } x_v = h \left( \begin{array}{ccc} a & b \\ c & d \end{array} \right), a \in \mathcal{O}_v^\times, b, c \in \mathfrak{p}_v \\ 0 & \text{otherwise} \end{cases}$$

for each $v < \infty$. By Theorem 1 and local computations we have the following.

**Theorem 2.** We set

$$S_{\text{fin}} = \{ v \in \Sigma \mid v < \infty \text{ and } \omega_v \text{ is ramified} \},$$

$$S_{\text{fin},2} = \{ v \in S_{\text{fin}} \mid v \not\in 2 \},$$

$$S_{\text{fin},2,1} = \{ v \in S_{\text{fin}} \mid v|2 \text{ and } d \in \mathcal{O}_v^\times(F_v^\times)^2 \},$$

$$S_{\text{fin},2,\pi_v} = \{ v \in S_{\text{fin}} \mid v|2 \text{ and } d \in \pi_v \mathcal{O}_v^\times(F_v^\times)^2 \},$$

$$\Sigma_{\infty} = \{ v \in \Sigma \mid v|\infty \}.$$

We have the disjoint union $S_{\text{fin}} = S_{\text{fin},2} \cup S_{\text{fin},2,1} \cup S_{\text{fin},2,\pi_v}$. We assume $\Phi = \prod_{v \in \Sigma} \Phi_v$. If we set $\Phi_v = \Phi_{0,v}$ for $v \notin S_{\text{fin}} \cup \Sigma_{\infty}$ and $\Phi_v = \Phi_{1,v}$ for $v \in S_{\text{fin}}$, then we obtain

$$Z(\Phi, s, \omega) = \frac{L(1, \omega) |\Delta_F|^{-3/2} 2^{-|F:Q|}}{\text{Residue}_{s=1} \zeta_F(s)} \times \prod_{v \in \Sigma} \frac{\zeta_F^{S}(2s - 1)}{\zeta_F(2)} \times \prod_{v \in S_{\text{fin}}} Z(\Phi_v, s, \omega_v, d)$$

$$\times \prod_{v|2, v \notin S_{\text{fin}}} q_v^{(2s-1)e_v} \times \prod_{v \in S_{\text{fin},2}} q_v^{-s^2 + \frac{1}{2}} \times \prod_{v \in S_{\text{fin},2,1}} q_v^{-2s+1} \times \prod_{v \in S_{\text{fin},2,\pi_v}} q_v^{-s^2 + \frac{1}{2}}.$$
We will show that the formula for \( L(s, L^*_2, \psi) \) in [IS, Theorem 1] is derived from Theorem 2. Hence, Theorem 2 is a generalization of the formula. We can similarly deduce the other formulas of [IS, Theorem 1] from Theorem 1.

We assume \( F = \mathbb{Q} \). Let \( m \) be a square-free integer. We assume that \( m \) is not 0 and 1. Let \( \omega_m \) be the character on \( \mathbb{A}_{\mathbb{Q}}^1/\mathbb{Q}^\times \) which corresponds to the quadratic field \( \mathbb{Q}(\sqrt{m}) \), \( D \) the discriminant of \( \mathbb{Q}(\sqrt{m}) \), and \( \psi_m \) the Dirichlet character on \( \mathbb{Z}/D\mathbb{Z} \) which corresponds to the quadratic field \( \mathbb{Q}(\sqrt{m}) \). We set

\[
L = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in X(\mathbb{Q}) \mid a, c \in \mathbb{Z} \text{ and } b \in \frac{1}{2} \mathbb{Z} \right\},
\]

\[
L_1 = \{ x \in L \mid x \text{ is positive definite} \},
\]

\[
L_2 = \{ x \in L \mid \det x < 0, -\det(x) \not\in (\mathbb{Q}^\times)^2 \}.
\]

For \( x \in L \), we set

\[
\psi_m(x) = \begin{cases} 
\psi_m(a) & \text{if there exists an element } g \in \text{SL}(2, \mathbb{Z}) \text{ such that } gx^t g = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \\
& b \in \frac{1}{2} m \mathbb{Z} \text{ and } c \in m \mathbb{Z} \text{ if } m \equiv 1 \mod 4, \\
& b, c \in m \mathbb{Z} \text{ if } m \equiv 2 \mod 4, \\
& \text{and } b, c \in 2m \mathbb{Z} \text{ if } m \equiv 3 \mod 4, \\
0 & \text{otherwise.}
\end{cases}
\]

For each \( x \in L \), we denote by \( \Omega_x \) the maximal order of \( \mathbb{Q}(\sqrt{-\det x}) \) and \( \varepsilon_x > 1 \) the fundamental unit of \( \mathbb{Q}(\sqrt{-\det x}) \). We set

\[
\mu(x) = \begin{cases} 
\pi |\Omega_x|^{-1} & \text{if } x \in L_1, \\
\log \varepsilon_x & \text{if } x \in L_2.
\end{cases}
\]

Let \( G_x \) denote the stabilizer of \( x \in X(F) \) in \( G \) and let \( G_x^0 \) denote the connected component of 1 in \( G_x \). We define the \( L \)-function \( L(s, m, i) \) by

\[
L(s, m, i) = \sum_{x \in G(\mathbb{Z}) \setminus G} \mu(x) \psi_m(x) [G_x(\mathbb{Z}) : G_x^0(\mathbb{Z})] |\det x|^s
\]

except for the case \( m < 0 \) and \( i = 2 \). We put \( \zeta(s) = \zeta_Q(s) \). From Theorem 2 we deduce the following.

**Theorem 3.** We have

\[
L(s, m, \frac{3 + \text{sgn}(m)}{2}) = L(1, \omega_m) \times \zeta(2s - 1) \times |m|^{-s + \frac{1}{2}} \times \begin{cases} 
2^{-2s} & \text{if } m \equiv 3 \mod 4, \\
2^{-1} & \text{if } m \equiv 2 \mod 4, \\
2^{2s-2} & \text{if } m \equiv 1 \mod 4.
\end{cases}
\]

We also have \( L(s, m, 1) = 0 \) for \( m > 0 \).

If we substitute \( m = -p \) (\( p \equiv 3 \mod 4 \)) into Theorem 3, then the above formula is the same as the formula for \( L(s, L^*_2, \psi) \) in [IS, Theorem 1].
2. ENDOSCOPY

In this section, we explain why we are interested in the zeta integrals with quadratic characters.

First, we review a formula given by Labesse-Langlands [LL, (5.11)]. We consider the parabolic subgroup

\[
\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \mid a \in F^\times, v \in F \right\}
\]

of \( \text{SL}(2, F) \). By the adjoint action of the Levi subgroup \( a \in \mathbb{G}_a \) on the unipotent radical \( v \in \mathbb{G}_a \), we can define the prehomogeneous zeta function \( \xi(\phi, s) \) as

\[
\xi(\phi, s) = \int_{\mathbb{A}_F^\times} |a|^s \sum_{v \in F} \phi(a^2v) \, d^\times a
\]

where \( \phi \in \mathcal{S}(\mathbb{A}_F) \) and \( \text{Re}(s) > 1 \). They proved the following formula in [LL].

**Theorem 4** (Labesse-Langlands). For \( \text{Re}(s) > 1 \), we have

\[
\xi(\phi, s) = \frac{1}{2} \sum_{\omega} \zeta(\phi, s, \omega)
\]

where \( \omega \) runs over all quadratic characters on \( \mathbb{A}_F^1/F^\times \) and \( \zeta(\phi, s, \omega) \) is the Tate integral, that is,

\[
\zeta(\phi, s, \omega) = \int_{\mathbb{A}_F^\times} |a|^s \omega(a) \phi(a) \, d^\times a.
\]

This theorem is proved by the Poisson summation formula for \( \mathbb{A}_F^1/(F^\times)^2 \) and \( F^\times/(F^\times)^2 \). They used this formula to stabilize the trace formula for \( \text{SL}(2) \). We can understand the meaning of the formula from point of view of trace formula.

Let \( f \in C_c^\infty(\text{SL}(2, \mathbb{A}_F)) \) and \( \phi(v) = \int_K f(k^{-1} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} k) \, dk \) where \( K \) is the standard maximal compact subgroup of \( \text{SL}(2, \mathbb{A}_F) \). Then, the unipotent term in the geometric side of the trace formula for \( f \) is

\[
\lim_{s \to 1} \frac{d}{ds} (s - 1) \xi(\phi, s) = \frac{1}{2} \lim_{s \to 1} (s - 1) \zeta(\phi, s, 1) + \frac{1}{2} \sum_{\omega \neq 1} \zeta(\phi, 1, \omega).
\]

The first term corresponds to an unipotent term of trace formula for \( \text{GL}(2) \). Hence, it is stable. If \( L \) is the quadratic extension of \( F \) which corresponds to \( \omega \neq 1 \), then \( H = R_{L/F}^1 \mathbb{G}_m \) is an elliptic endoscopic group of \( \text{SL}(2) \). Furthermore, if \( f^H \) is the transfer of \( f \) to \( H \), then we have

\[
\zeta(\phi, 1, \omega) = L(1, \omega) f^H(1).
\]

From this we have obtained a stabilization of the unipotent term of \( \text{SL}(2) \). The stabilization directly followed from Theorem 4. Hence, Theorem 4 looks like a stabilization of the zeta function \( \xi(\phi, s) \). We are interested in stabilizations of prehomogeneous zeta functions as a generalization of Theorem 4. In addition, we also want to know relations between such stabilizations and explicit forms of prehomogeneous zeta functions, which were studied by Ibukiyama and Saito.

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If we see the Siegel parabolic subgroup
\[
\left\{ \begin{pmatrix} h & O_2 \\ O_2 & I_2 \end{pmatrix} \begin{pmatrix} I_2 \\ O_2 & I_2 \end{pmatrix} \right\} \in \text{Sp}(2, F) \quad | h \in \text{GL}(2, F) \quad \text{and} \quad S \in X(F) \}
\]
of \text{Sp}(2, F), we can define the prehomogeneous zeta function \( \Xi(\Phi, s) \) by
\[
\Xi(\Phi, s) = \int_{\text{GL}(2, \mathbb{A}_F)/\text{GL}(2, F)} | \det h |^{2s} \sum_{x \in X^*(F)} \Phi(h x^t h) \, dh
\]
where \( \Phi \in S(X(\mathbb{A}_F), \text{Re}(s) > 3/2) \), and \( dh \) is the Tamagawa measure on \( \text{GL}(2) \). If we apply the above-mentioned argument of [LL] to \( \Xi(\Phi, s) \) and the center of \( \text{GL}(2) \), then we have
\[
\Xi(\Phi, s) = \sum_\omega Z(\Phi, s, \omega)
\]
where \( \omega \) runs over all quadratic characters on \( \mathbb{A}_F/F^\times \). Let \( f \in C_\infty(\text{Sp}(2, \mathbb{A}_F)) \) and \( \Phi(x) = \int_K f(k^{-1} \begin{pmatrix} I_2 & x \\ O_2 & I_2 \end{pmatrix}) \, dk \), where \( K \) is a suitable maximal compact subgroup of \( \text{Sp}(2, \mathbb{A}_F) \). We denote by \( \{ \gamma \}_{\text{Sp}(2, F)} \) the \( \text{Sp}(2, F) \)-conjugacy class of \( \gamma \in \text{Sp}(2, F) \).

Hoffmann and I proved that the unipotent term for \( \cup_{x \in X^*(F)} \{ \begin{pmatrix} I_2 & x \\ O_2 & I_2 \end{pmatrix} \}_{\text{Sp}(2, F)} \) in the geometric side of the trace formula for \( f \) is equal to
\[
\lim_{s \to 3/2} \frac{d}{ds} (s - \frac{3}{2}) \Xi(\Phi, s).
\]
Furthermore, it follows from the above-mentioned equality that
\[
\lim_{s \to 3/2} \frac{d}{ds} (s - \frac{3}{2}) \Xi(\Phi, s) = \lim_{s \to 3/2} \frac{d}{ds} (s - \frac{3}{2}) Z(\Phi, s, 1) + \sum_{\omega \neq 1} Z(\Phi, \frac{3}{2}, \omega).
\]
The first term should be unstable. However, I do not know how to stabilize it. If we substitute \( s = 3/2 \) into Theorem 1, then we have
\[
Z(\Phi, \frac{3}{2}, \omega) = L(1, \omega) |\Delta_F|^{-3/2} \times \prod_{v \mid 2} 2q_v^{2e_v} |\Omega_v^\times : (\Omega_v^\times)^2| \times \prod_{v \in S} Z(\Phi_v, \frac{3}{2}, \omega_v, d).
\]
If we see the result of [Assem], then it seems that \( Z(\Phi, \frac{3}{2}, \omega) \) is related to the elliptic endoscopic group \( H = (\mathbb{R}_{L/F}G_m) \times \text{SL}(2) \text{ of Sp}(2) \), where \( L \) is the quadratic extension of \( F \) corresponding to \( \omega \neq 1 \). Spallone and I are studying this stabilization now.

REFERENCES


