ANALYTIC TORSIONS OF SPHERES

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The notion of analytic torsion was first introduced by Ray and Singer [3]. Since then, there are two major breakthroughs in this field. The first one, given independently by Cheeger and Müller, relates the Reidemeister torsion and the analytic torsion. The second one, given by Quillen, defines the so-called Quillen metric, which leads to the study of its curvature, in particular, the arithmetic Riemann–Roch formula of Gillet and Soulé.

While a lot of progresses have been made in the qualitative study of analytic torsion, comparably few results in the quantative study are known. In fact, the latter is also very important, for example, in order to find the right arithmetic Todd genus [1], we need to use the analytic torsions of the structure sheaves of projective spaces with the Fubini–Study metric.

The value of the analytic torsion, by definition, depends on the eigenvalues and their multiplicities of the associated Laplacians. In practice, there are two ways to compute it. One is done by a certain general formalism. For example, the (logarithmic) analytic torsion for an even dimensional compact Riemannian manifold is zero by duality, and in [5], the analytic torsions of all line bundles over projective spaces with respect to the Fubini–Study metric are given by the arithmetic Riemann–Roch formula. The other is done by using the precise eigenvalues and their multiplicities, which are, in general, hard to find. In this paper, we will adopt the second method.

The main result of this paper is to give the analytic torsions of unit spheres in Euclidean spaces with the standard Riemannian metric.

Theorem. *For $M = S^{2m-1}$ with the standard metric, the analytic torsion for M is*

$$T = \frac{2\pi^m}{(m-1)!}\,.$$

Such a result is somehow quite surprising as the expression is extremely simple and the only term involving the zeta function is $\zeta'(0)$: T can be naturally written as

$$\frac{1}{2^{m-1}(m-1)!}e^{-2m\zeta'(0)}\,,$$

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and a similar result for projective spaces $\mathbf{P}^n_{\mathbf{C}}$ in complex geometry does involve $\zeta(i)$ and $\zeta'(i)$ with i odd and $i = 1, \dots, n$.

As it is known, analytic torsions for (complex) projective spaces are used in an essential way to find out the exact form of the arithmetic Riemann–Roch theorem in complex geometry [1]. Therefore we hope that our result could play a similar role in Riemannian geometry.

The paper is organized as follows. In the first section, we introduce the analytic torsion of Ray and Singer for a Riemannian manifold. For the sphere S^n we state a theorem of Ikeda and Taniguchi, which gives the eigenvalues and their multiplicities of the associated Laplacians in terms of representation theory. In Sec. 2, we compute explicitly the multiplicities using the Weyl dimension formula. In Sec. 3, we state two technical results. One is for the derivative of the generalized *zeta* function, and the other is purely combinatorial. Both will play crucial roles in proving the main result. Finally, Sec. 4 is devoted to giving the proof of the above theorem.

1. Preliminary

For a compact Riemannian manifold (M, g) of dimension n, let $\mathcal{D} = \sum \mathcal{D}^q(M)$ be the space of C^∞-differential forms. We have the usual exterior differential $d : \mathcal{D}^q(M) \to \mathcal{D}^{q+1}(M)$. The metric g defines a dual operator d^* of d. Let $\Delta = dd^* + d^*d$ be the associated Laplacian. For $0 \le q \le n$, let $\lambda_{q,j}$ be the eigenvalues of Δ on $\mathcal{D}^q(M)$. Then associated with them we have the zeta function $\zeta_q(s)$ of Δ defined by

$$\zeta_q(s) = \sum (\lambda_{q,j})^{-s} .$$

This is a well-defined function for $\mathrm{Re}(s) > \frac{n}{2}$. And we extend it to the whole complex plane meromorphically. The resulting function is holomorphic at $s = 0$. Following Ray and Singer, the *analytic torsion* of M is defined by

$$T(M, g) = \exp\Big[\frac{1}{2} \sum_{q=0}^{n} (-1)^q q \zeta_q'(0)\Big] . \tag{1.1}$$

It is known that if M is an oriented even dimensional compact manifold without boundary, the (logarithmic) analytic torsion is identically zero [3]. We are interested in computing the analytic torsions of unit spheres S^n with the standard Riemannian metric. From the above remark, we need only consider the case that n is odd.

For doing so, we need the following work of [2]. For the sphere S^n, $S^n = G/K$ where $G = SO(n+1)$ and

$$K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in M_{n+1}(R);\ A \in SO(n) \right\} .$$

It is well known that the Laplacian in this case is exactly (up to a possible scalar) the Casimir operator of the Lie algebra of G. In general the space $\mathcal{D}^q$ contains a dense subspace which is decomposed into a sum of irreducible representations of G, all of them are of highest weight. In [2], the explicit irreducible representations of the group G occurring in $\mathcal{D}^p$ are given. The result there is (the notations in the statement will be explained in the next section)

Theorem 1.1. [2] (a) *Suppose* $p \leq \frac{n}{2}$. *The highest weight* Λ *of the irreducible representation* ρ *intervening in* $\mathcal{D}$, *with multiplicity at least one, are as follows:*

(i) *In case* $n = 2m$,

$$\Lambda = \begin{cases} k\Lambda_1 + \Lambda_p, k\Lambda_1 + \Lambda_{p+1}, & (0 \leq p \leq m-2) \\ k\Lambda_1 + \Lambda_{m-1}, k\Lambda_1 + \Lambda_m, & (p = m-1) \\ k\Lambda_1 + \Lambda_m. & (p = m) \end{cases}$$

where k *runs over all non-negative integers.*

(ii) *In case* $n = 2m - 1$

$$\Lambda = \begin{cases} k\Lambda_1 + \Lambda_p \, , k\Lambda_1 + \Lambda_{p+1}, & (0 \leq p \leq m-3) \\ k\Lambda_1 + \Lambda_{m-2} \, , k\Lambda_1 + \Lambda_{m-1}, & (p = m-2) \\ k\Lambda_1 + \Lambda_m^+ \, , k\Lambda_1 + \Lambda_{m-1} \, , k\Lambda_1 + \Lambda_m^-, & (p = m-1) \end{cases}$$

where k *runs over all non-negative integers.*

Further, the multiplicity of the above ρ *is exactly one except for the case* $n = 2m$ *and* $p = m$, *in which case the multiplicity is two.*

(b) *The Laplacian has eigenvalue* C_Λ *on a* $SO(n+1)$*- irreducible submodule of differential forms on* S^n *with the highest weight* Λ.

We will give the precise values of C_Λ, as well as their multiplicities in the following section.

2. Representation Aspect

For a compact semi-simple Lie group G, let g_C be the complexified Lie algebra of G. Fix a maximal torus, say, the dimension is l. Denote by Π the corresponding root system, and we choose a positive root system Π_+. For an invariant bilinear form $(\cdot|\cdot)$ on g_C (which is a scalar of the Killing form), we have the Casimir operator:

$$C = \sum_i u_i u_i^*$$

where $\{u_i\}$ is any basis of g_C over $\mathbf{C}$ and $\{u_i^*\}$ is the dual basis with respect to $(\cdot|\cdot)$. Let $\lambda_1, \ldots, \lambda_l$ be the fundamental dominant weights. Given an irreducible representation $V(\Lambda)$ of G with highest weight Λ, the Casimir operator C acts as a scalar $C = C_\Lambda = (\Lambda + 2\rho|\Lambda)$, where ρ is as usual the half of the sum of the positive roots. The dimension of $V(\Lambda)$ is given by the Weyl dimension formula:

$$\dim V(\Lambda) = \prod_{\alpha \in \Pi_+} \frac{(\Lambda + \rho|\alpha)}{(\rho|\alpha)}.$$

For $G = SO(n)$, we choose the invariant form $(\cdot|\cdot)$ so that the scalars C_Λ for the Λ's occurring in Theorem 1.1 is the same as the eigenvalues of the Laplacian on the sphere listed in [2], which we will give below as well. (For $n = 2$, although the group G is not semi-simple, the final result in this section is still valid.)

Due to the nature of the two different types of orthogonal groups, we consider the case for n even and odd respectively.

For $G = SO(2m)$, let h be an Euclidean space of dimension m with orthogonal basis e_i, then Π can be chosen so that $\Pi = \{\pm e_i \pm e_j \,|\, i \neq j\}$. Choose the positive roots so that $\Pi_+ = \{e_i - e_j, e_i + e_j \,|\, i < j\}$ and let $\{e_1 - e_2, e_2 - e_3, \ldots, e_{m-1} - e_m, e_{m-1} + e_m\}$ be the simple roots. Then the fundamental dominant weights are given by $\lambda_i = e_1 + e_2 + \ldots + e_i$ for $1 \leq i \leq m-2$ and $\lambda_{m-1} = \frac{1}{2}(e_1 + e_2 + \ldots + e_{m-1} + e_m)$, $\lambda_m = \frac{1}{2}(e_1 + e_2 + \ldots + e_{m-1} - e_m)$. It is known that the infinitesimal representations with highest weights $\Lambda_1, \ldots, \Lambda_{m-2}, \Lambda_{m-1}, \Lambda_m^{\pm}$ generate all the irreducible finite dimensional representation of G, where $\Lambda_i = \lambda_i$ for $i \leq m-2$, $\Lambda_{m-1} = \lambda_{m-1} + \lambda_m$, $\Lambda_m^+ = 2\lambda_{m-1}$ and $\Lambda_m^- = 2\lambda_m$. We have

$$\rho = (m-1)e_1 + (m-2)e_2 + \ldots + e_{m-1}\,.$$

For the highest weight Λ occurring in the representation of G on the sphere S^n, $n = 2m-1$, we give the scalars C_Λ and $\dim V(\Lambda)$ below.

Case 1. $\Lambda = k\Lambda_1$, $C_\Lambda = k(k+n-1)$, $n = 2m-1$ and

$$\dim V(\Lambda) = \frac{k+m-1}{m-1}\binom{2m+k-3}{k}\,.$$

Case 2. $\Lambda = k\Lambda_1 + \Lambda_p$, $2 \leq p \leq m-2$, $C_\Lambda = (k+p)(k+n+1-p)$ and

$$\dim V(\Lambda) = 2\frac{(k+m)}{(k+p)}\frac{(2m+k-1)}{(2m+k-p)}\begin{pmatrix} k & \begin{matrix} 2m+k-2 \\ p-1 \end{matrix} & 2m-p-1 \end{pmatrix}\,.$$

Case 3. $\Lambda = k\Lambda_1 + \Lambda_m^{\pm}$, $C_\Lambda = (k+m)^2$

$$\dim V(\Lambda) = \binom{m+k-1}{k}\binom{2m+k-1}{m-1}\,.$$

Case 4. $\Lambda = k\Lambda_1 + \Lambda_{m-1}$, $C_\Lambda = (k+m)^2 - 1$

$$\dim V(\Lambda) = 2\frac{(k+m)}{(k+m+1)}\binom{m+k-2}{k}\binom{2m+k-1}{m}\,.$$

Similarly, for $G = SO(2m+1)$, let h be an Euclidean space of dimension m with the orthogonal basis e_i, then Π can be chosen so that $\Pi = \{\pm e_i \pm e_j \,|\, i \neq j\} \cup \{\pm e_i\}$. Choose the positive roots so that $\Pi_+ = \{e_i - e_j, e_i + e_j \,|\, i < j\} \cup \{e_i\}$ and let $\{e_1 - e_2, e_2 - e_3, \ldots, e_{m-1} - e_m, e_m\}$ be the simple roots. Then the fundamental dominant weights are given by $\lambda_i = e_1 + e_2 + \ldots + e_i$ for $1 \leq i \leq m-1$ and $\lambda_m = \frac{1}{2}(e_1 + e_2 + \ldots + e_{m-1} + e_m)$. It is known that the infinitesimal representations with highest weights $\Lambda_1, \ldots, \Lambda_{m-2}, \Lambda_{m-1}, \Lambda_m$ generate all the irreducible dimensional representation of G, where $\Lambda_i = \lambda_i$ for $i \leq m-1$, $\Lambda_m = 2\lambda_m$. We have

$$\rho = \frac{(2m-1)}{2}e_1 + \frac{(2m-3)}{2}e_2 + \ldots + \frac{1}{2}e_m\,.$$

For the highest weight Λ occurring in the representation of G on the sphere S^n, $n = 2m$, we give the scalars C_Λ and $\dim V(\Lambda)$ below.

Case 1. $\Lambda = k\Lambda_1$, $C_\Lambda = k(k+n-1)$ and

$$\dim V(\Lambda) = \frac{2k+2m-1}{2m-1}\binom{2m+k-2}{k}.$$

Case 2. $\Lambda = k\Lambda_1 + \Lambda_p$, $2 \le p \le m-1$, $C_\Lambda = (k+p)(k+n+1-p)$ and

$$\dim V(\Lambda) = \frac{(2k+2m+1)}{(2m+k-p+1)}\binom{2m+k}{k+p}\binom{m+k-1}{k}.$$

Case 3. $\Lambda = k\Lambda_1 + \Lambda_m$, $C_\Lambda = (k+m)(k+m+1)$

$$\dim V(\Lambda) = \binom{m+k-1}{k}\binom{2m+k+1}{m}\frac{2m+2k+1}{2m+k+1}.$$

3. Two Technical Results

We define the generalized ζ function by

$$\zeta_{\alpha,c}(s) = \sum_{k=1}^{\infty} \frac{1}{k^{s-c}(k+\alpha)^{s-c}} \tag{3.1}$$

where α and c are complex numbers. The series (3.1) is convergent for $\mathrm{Re}(s) >> 0$, and we extend it to a meromorphic function which is holomorphic at $s = 0$. We are interested in computing the derivative of $\zeta_{\alpha,c}$ at zero.

Lemma 3.1. *Assume α and c are positive integers, then we have*

$$\zeta'_{\alpha,c}(0) = 2\sum_{l=0}^{[\frac{c}{2}]}\binom{c}{2l}\alpha^{2l}\zeta'(2l-2c) + \sum_{k=1}^{\alpha}(k^c(k-\alpha)^c)\log k$$

where ζ' denotes the derivative of the usual Riemann zeta function.

Proof. We have

$$\begin{aligned}
\zeta'_{\alpha,c}(s) &= -\sum_{k=1}^{\infty}\frac{1}{k^{s-c}(k+\alpha)^{s-c}}\log k - \sum_{k=1}^{\infty}\frac{1}{k^{s-c}(k+\alpha)^{s-c}}\log(k+\alpha) \\
&= -\sum_{k=1}^{\infty}\frac{\log k}{k^{2s-2c}}(1+\frac{\alpha}{k})^{-s+c} - \sum_{k=1}^{\infty}\frac{1}{(1-\frac{\alpha}{k+\alpha})^{s-c}(k+\alpha)^{2s-2c}}\log(k+\alpha) \\
&= -\sum_{k=1}^{\alpha}\frac{\log k}{k^{s-c}(k+\alpha)^{s-c}} - \sum_{k=\alpha+1}^{\infty}\frac{\log k}{k^{2s-2c}}(1+\frac{\alpha}{k})^{-s+c}
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{k=1}^{\infty} \frac{\log(k+\alpha)}{(k+\alpha)^{2s-2c}}(1-\frac{\alpha}{k+\alpha})^{-s+c} \\
= & -\sum_{k=1}^{\alpha} \frac{\log k}{k^{s-c}(k+\alpha)^{s-c}} - \sum_{k=\alpha+1}^{\infty} \frac{\log k}{k^{2s-2c}}(1+\frac{\alpha}{k})^{-s+c} \\
& \quad - \sum_{k=\alpha+1}^{\infty} \frac{\log k}{k^{2s-2c}}(1-\frac{\alpha}{k})^{-s+c} \\
= & -\sum_{k=1}^{\alpha} \frac{\log k}{k^{s-c}(k+\alpha)^{s-c}} - \sum_{k=\alpha+1}^{\infty} \frac{\log k}{k^{2s-2c}} \sum_{l=0}^{\infty} \binom{-s+c}{l} [1+(-1)^l](\frac{\alpha}{k})^l \\
= & -\sum_{k=1}^{\alpha} \frac{\log k}{k^{s-c}(k+\alpha)^{s-c}} - 2\sum_{k=\alpha+1}^{\infty} \frac{\log k}{k^{2s-2c}} \sum_{l \text{ even}} \binom{-s+c}{l} (\frac{\alpha}{k})^l \\
= & -\sum_{k=1}^{\alpha} \frac{\log k}{k^{s-c}(k+\alpha)^{s-c}} + 2\sum_{k=1, l \text{ even}}^{\alpha} \frac{\log k}{k^{2s-2c}} \binom{-s+c}{l} (\frac{\alpha}{k})^l \\
& -2\sum_{k=1, l \text{ even}}^{\infty} \frac{\log k}{k^{2s-2c}} \binom{-s+c}{l} (\frac{\alpha}{k})^l \\
= & -\sum_{k=1}^{\alpha} \frac{\log k}{k^{s-c}(k+\alpha)^{s-c}} + 2\sum_{k=1}^{\alpha} \frac{\log k}{k^{2s-2c}} \sum_{l \text{ even}} \binom{-s+c}{l})(\frac{\alpha}{k})^l \\
& + \sum_{l \text{ even}} \binom{-s+c}{l} \frac{d}{ds}[\alpha^l \zeta(2s-2c+l)] \\
= & \ I_1 + I_2 + I_3
\end{aligned}
$$

Let $s \to 0$, then

$$
I_3 \to 2[\sum_{l \text{ even}} \binom{c}{l} \alpha^l \zeta'(-2c+l)] = 2\sum_{l=0}^{[\frac{c}{2}]} \binom{c}{2l} \alpha^{2l} \zeta'(2l-2c),
$$

and the limit of $I_1 + I_2$ is

$$
\begin{aligned}
& -\sum_{k=1}^{\alpha}(k^c(k+\alpha)^c)\log k + 2\sum_{k=1}^{\alpha} k^{2c}\log k \sum_{l \text{ even}} \binom{c}{l}(\frac{\alpha}{k})^l \\
& \quad = -\sum_{k=1}^{\alpha}(k^c(k+\alpha)^c) + \sum_{k=1}^{\alpha} k^{2c}\log k[(1+\frac{\alpha}{k})^c + (1-\frac{\alpha}{k})^c] \\
& \quad = \sum_{k=1}^{\alpha}(k^c(k-\alpha)^c)\log k\,.
\end{aligned}
$$

Next, we give a combinatorial identity which will be used later.

Theorem 3.2. *As polynomials of x, we have the following:*

$$\sum_{p=1}^{m-1}(-1)^p\binom{2m-2}{p-1}\Big[\prod_{i=1,i\neq p}^{m}(x+2m-p-i)(x+i-p)$$
$$+\prod_{i=1,i\neq p}^{m}(-x+2m-p-i)(-x+i-p)\Big]$$
$$+(-1)^m\binom{2m-2}{m-1}\prod_{i=1}^{m-1}(x+i)(x-i)+m[(2m-2)!]=0\,.$$

Proof. The original one is equivalent to the following:

$$\sum_{p=1}^{m-1}(-1)^p\binom{2m-2}{p-1}\Big[\prod_{i=1,i\neq p}^{m}((k+m-p)^2-(m-i)^2)$$
$$+\prod_{i=1,i\neq p}^{m}((-k+m-p)^2-(m-i)^2)\Big]$$
$$+(-1)^m\binom{2m-2}{m-1}\prod_{i=1}^{m-1}(k^2-(m-i)^2)+m[(2m-2)!]=0\,.$$

After changing $m-p$ to p and $m-i$ to i, we need to show that

$$\sum_{p=1}^{m-1}(-1)^{m-p}\binom{2m-2}{m-p-1}\Big[\prod_{i=0,i\neq p}^{m-1}((k+p)^2-i^2)+\prod_{i=0,i\neq p}^{m-1}((-k+p)^2-i^2)\Big]$$
$$+(-1)^m\binom{2m-2}{m-1}\prod_{i=1}^{m-1}(k^2-i^2)+m[(2m-2)!]=0\,.$$

Note that

$$\binom{2m}{m-p}=\binom{2m}{m+p}$$

and changing p to $-q$ for the last term, we need to show that for all m,

$$-(m+1)[(2m)!]+(-1)^m\binom{2m}{m}\prod_{i=1}^{m}(k^2-i^2)$$
$$+\sum_{p=1}^{m}(-1)^{m+p}\binom{2m}{m+p}\prod_{i=0,i\neq p}^{m}((k+p)^2-i^2)$$
$$+\sum_{q=-1}^{-m}(-1)^{m+q}\binom{2m}{m+q}\prod_{i=0,i\neq -q}^{m}((k+q)^2-i^2)=0\,.$$

That is,

$$(m+1)[(2m)!] = \sum_{p=-m}^{m} (-1)^{m+p} \binom{2m}{m+p} \prod_{i=0, i\neq |p|}^{m} \left((k+p)^2 - i^2\right) = 0\,.$$

After changing $k+m$ to k and $m+p$ to p, we only need to show that

$$(m+1)[(2m)!] = \sum_{p=0}^{2m} (-1)^p \binom{2m}{p} \prod_{i=0, i\neq |m-p|}^{m} \left((k+p)^2 - i^2\right).$$

That is,

$$(m+1)[(2m)!] = \frac{1}{2}\sum_{p=0}^{2m} (-1)^p \binom{2m}{p} \frac{(k+p+m)!}{(k+p-m-1)!} \left(\frac{1}{k+m} + \frac{1}{k+2p-m}\right),$$

or equivalently,

$$2(m+1) = \sum_{p=0}^{2m} (-1)^p \frac{(k+p+m)!}{p!(2m-p)!(k+p-m-1)!} \left(\frac{1}{k+m} + \frac{1}{k+2p-m}\right).$$

After changing $k+m$ to k, the left-hand side is just

$$\begin{aligned}
&\sum_{p=0}^{2m} (-1)^p \frac{(k+p)!}{p!(2m-p)!(k+p-2m-1)!} \left(\frac{1}{k} + \frac{1}{k+2p-2m}\right) \\
&= \frac{1}{k}\sum_{p=0}^{2m} (-1)^p \frac{(k+p)!}{p!(2m-p)!(k+p-2m-1)!} \\
&\quad + \sum_{p=0}^{2m} (-1)^p \frac{(k+p)!}{p!(2m-p)!(k+p-2m-1)!(k+2p-2m)} \\
&= \frac{1}{k}\sum_{p=0}^{2m} (-1)^p \frac{(k+p)!(2m)!}{p!(2m-p)!(k+p-2m-1)!(2m)!} \\
&\quad + \sum_{p=0}^{2m} (-1)^p \frac{(k+p)!(k+2p-2m-1)!}{p!(2m-p)!(k+p-2m-1)!(k+2p-2m)!} \\
&= \frac{2m+1}{k}\sum_{p=0}^{2m} (-1)^p \binom{2m}{p}\binom{k+p}{2m+1} + \sum_{p=0}^{2m} (-1)^p \binom{k+p}{2m-p}\binom{k+2p-2m-1}{p} \\
&= \frac{2m+1}{k}\sum_{p=0}^{2m} (-1)^p \binom{2m}{p}\binom{k+p}{2m+1} + \sum_{p=0}^{2m} (-1)^p \binom{2m}{p}\binom{k+p}{2m+1}\frac{2m+1}{k+2p-2m}
\end{aligned}$$

Hence, it suffices to show the following:

Lemma 3.3. *With the notation as above, we have*

$$\sum_{p=0}^{m} (-1)^p \binom{m}{p}\binom{k+p}{m+1} = (-1)^m k \tag{3.2}$$

$$\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{k+p}{m+1}\frac{m+1}{k+2p-m}=\frac{1+(-1)^m}{2}. \tag{3.3}$$

The Proof of the Lemma. We prove (3.2) and (3.3) by induction. Obviously, the identities are true for $m=0$ and 1. On the other hand,

$$\begin{aligned}
&\sum_{p=0}^{m+1}(-1)^p\binom{m+1}{p}\binom{k+p}{m+2}\\
&\quad=\sum_{p=1}^{m+1}(-1)^p\binom{m}{p-1}\binom{k+p}{m+2}+\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{k+p}{m+2}\\
&\quad=-\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{k+p+1}{m+2}+\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{k+p}{m+2}\\
&\quad=-\sum_{p=0}^{m}(-1)^p\binom{m}{p}\left\{\binom{k+p+1}{m+2}-\binom{k+p}{m+2}\right\}\\
&\quad=-\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{k+p}{m+1}.
\end{aligned}$$

So by induction, we have the first identity.

Also, we have,

$$\begin{aligned}
&\sum_{p=0}^{m+1}(-1)^p\binom{m+1}{p}\binom{k+p}{m+2}\frac{m+2}{k-m-1+2p}\\
&\quad=\sum_{p=1}^{m+1}(-1)^p\binom{m}{p-1}\binom{k+p}{m+2}\frac{m+2}{k-m-1+2p}\\
&\qquad+\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{k+p}{m+2}\frac{m+2}{k-m-1+2p}\\
&\quad=-\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{k+p+1}{m+2}\frac{m+2}{k-m+1+2p}\\
&\qquad+\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{k+p}{m+2}\frac{m+2}{k-m-1+2p}\\
&\quad=\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{k+p}{m+1}\left[\frac{k+p-m-1}{k-m-1+2p}+\frac{k+p+1}{k-m+1+2p}\right]
\end{aligned}$$

$$= -\sum_{p=0}^{m}(-1)^p\binom{m}{p}\binom{k+p}{m+1}\left[\frac{p}{k-m-1+2p}+\frac{m-p}{k-m+1+2p}\right]$$

$$= -\sum_{p=0}^{m}(-1)^p\binom{k+p}{m+1}\left[\frac{m!p}{p!(m-p)!(k-m-1+2p)}\right.$$

$$\left.+\frac{m!(m-p)}{p!(m-p)!(k-m+1+2p)}\right]$$

$$= -m\sum_{p=1}^{m}(-1)^p\binom{k+p}{m+1}\left[\binom{m-1}{p-1}\frac{1}{k-m-1+2p}\right.$$

$$\left.+\binom{m-1}{p}\frac{1}{k-m+1+2p}\right]$$

$$= m\sum_{p=0}^{m-1}(-1)^p\binom{m-1}{p}\left[\binom{k+p+1}{m+1}\frac{1}{k-m+1+2p}\right.$$

$$\left.-\binom{k+p}{m+1}\frac{1}{k-m+1+2p}\right]$$

$$= m\sum_{p=0}^{m-1}(-1)^p\binom{m-1}{p}\frac{1}{k-m+1+2p}\left[\binom{k+p+1}{m+1}-\binom{k+p}{m+1}\right]$$

$$= \sum_{p=0}^{m-1}(-1)^p\binom{m-1}{p}\binom{k+p}{m}\frac{m}{k-m+1+2p}.$$

Hence by the induction assumption, we have our result.

Second proof. We may write Eq. (3.3) as

$$\sum_{p=0}^{m}(-1)^p\binom{k+p}{m+1}\binom{m}{p}\frac{1}{k-m+2p}=\frac{1+(-1)^m}{2(m+1)}\,. \tag{3.4}$$

For each fixed k, consider

$$a_m(x)=\sum_{p=0}^{m}(-1)^p\binom{k+p}{m+1}\binom{m}{p}\frac{x^{k+2p-m}}{k-m+2p}\,.$$

This turns out to be a polynomial in x. Then the left-hand sides of Eq. (3.2) and (3.4) are just $a'_m(1)$ and $a_m(1)$. Notice that $k+2p-m=k+p-(m-p)$, we have that $a'_m(x)$ is the coefficient of u^m in the product $G_m(x,u)H_m(x,u)$ where

$$G_m(x,u)=\sum_{p=0}^{\infty}\binom{k+p}{m+1}x^{k+p-1}u^p$$

$$H_m(x,u)=\sum_{p=0}^{\infty}\binom{m}{p}x^{-p}u^p\,.$$

Using $\sum_{p=0}^{\infty}\binom{p}{m}u^p = \frac{u^m}{(1-u)^{m+1}}$, we get

$$G_m(x,u) = x^{-1}u^{-k}\frac{(ux)^{m+1}}{(1-ux)^{m+2}} - x^{-1}u^{-k}\sum_{p=0}^{k-1}\binom{p}{m+1}(ux)^p$$

$$H_m(x,u) = (\frac{u-x}{x})^m ,$$

so that

$$G_m(x,u)H_m(x,u) = \frac{(u-x)^m u^{-k+m+1}}{(1-ux)^{m+2}} - x^{-1}u^{-k}\sum_{p=0}^{k-1}\binom{p}{m+1}(ux)^p(\frac{u-x}{x})^m .$$

The second term does not contribute anything to u^m, hence

$$a'_m(x) = \text{coeff. of } u^{k-1} \text{ in} \frac{(u-x)^m}{(1-ux)^{m+2}} . \tag{3.5}$$

Setting $x = 1$ in (3.5), we have

$$a'_m(1) = \text{ coeff. of } u^{k-1} \text{ in } \frac{(-1)^m}{(1-u)^2} = (-1)^m k$$

which is (3.2). Also, integrating both sides over $[-1,1]$ in (3.5), we obtain

$$a_m(1) - a_m(-1) = \text{coeff. of } u^{k-1} \text{ in } \int_{-1}^{1}\frac{(u-x)^m}{(1-ux)^{m+2}}dx .$$

Equivalently,

$$\begin{aligned}\sum_{m=0}^{\infty}(a_m(1) - a_m(-1))t^m &= \text{coeff. of } u^{k-1} \text{ in } \int_{-1}^{1}\frac{1}{(1-ux)[1-ux-(u-x)t]}dx \\ &= \text{coeff. of } u^{k-1} \text{ in } \frac{1}{t(1-u^2)}\log\frac{1+t}{1-t} .\end{aligned}$$

Observe that $a_m(1) = a_m(-1)$ unless $k+m$ is odd, and

$$\frac{1}{t(1-u^2)}\log\frac{1+t}{1-t} = \sum_{n=0}^{\infty}u^{2n}\sum_{m=0}^{\infty}\frac{1+(-1)^m}{m+1}t^m .$$

It follows that for k even

$$\sum_{m\,\text{odd}} 2a_m(1)t^m = 0 ,$$

and for k odd

$$\sum_{m\,\text{even}} 2a_m(1)t^m = \frac{2}{m+1} .$$

Therefore (3.2) and (3.4) hold for infinitely many choices of k. Since it is polynomial in k, they must be identically true.

4. Analytic Torsion of Spheres

Recall the analytic torsion of a Riemannian manifold (M,g) is defined by

$$T(M,g) = \exp[\frac{1}{2}\sum_{q=0}^{n}(-1)^q q\zeta_q'(0)]\,. \tag{4.1}$$

From Sec. 1, to compute the analytic torsion of the unit sphere S^n, we need only consider the case $n = 2m-1$ (for n even it is zero, or, we can see the cancellation directly from the lists in Sec. 1 and Sec. 2).

By definition, using Theorem 1.1, we can evaluate the analytic torsion T by the relation $2\log T = \mathcal{Z}'(0)$ where for $\mathrm{Re}(s) \gg 0$,

$$\mathcal{Z}(s) = \sum_{k\geq 0}\sum_{i=1}^{m-2} 2(-1)^i \frac{\dim V(k\Lambda_1+\Lambda_i)}{C^s_{k\Lambda_1+\Lambda_i}} + 2(-1)^{m+1}\frac{\dim V(k\Lambda_1+\Lambda_{m-1})}{C^s_{k\Lambda_1+\Lambda_{m-1}}} + (-1)^m \frac{\dim V(k\Lambda_1+\Lambda_m^+)}{C^s_{k\Lambda_1+\Lambda_m^+}} + (-1)^m \frac{\dim V(k\Lambda_1+\Lambda_m^-)}{C^s_{k\Lambda_1+\Lambda_m^-}}\,,$$

which can be extended to a meromorphic function on the whole complex plane, holomorphic at $s=0$. Using the duality and the formulae we gave in Sec. 2 (notice that the dimensions corresponding to the spin representations are the same), we find that

$$\mathcal{Z}(s) = 4\sum_{p=1}^{m-1}\frac{(-1)^i}{(p-1)!(2m-p-1)!}\sum_{k=0}^{\infty}\frac{\prod_{i=1,i\neq p}^{m}(k+i)(k+2m-i)}{(k+p)^s(k+2m-p)^s} + \frac{2(-1)^m}{(m-1)!^2}\sum_{k=0}^{\infty}\frac{\prod_{i=1,i\neq p}^{m}(k+i)(k+2m-i)}{(k+m)^{2s}}\,.$$

Set

$$\mathcal{Z}_p(s) = \sum_{k=0}^{\infty}\frac{\prod_{i=1,i\neq p}^{m}(k+i)(k+2m-i)}{(k+p)^s(k+2m-p)^s} = \sum_{k=p}^{\infty}\frac{\prod_{i=1,i\neq p}^{m}(k-p+i)(k+2m-p-i)}{k^s(k+2m-2p)^s} \tag{4.2}$$

so that

$$\mathcal{Z}(s) = 4\sum_{p=1}^{m-1}\frac{(-1)^p}{(p-1)!(2m-p-1)!}\mathcal{Z}_p(s) + 2\frac{(-1)^m}{(m-1)!^2}\mathcal{Z}_m(s) \tag{4.3}$$

Consider the function $\mathcal{Z}_p(s)$ first. Using

$$\prod_{j=1}^{m}(x+a_j)=\sum_{i=0}^{m}e_{m-i}(a_1,\ldots,a_m)x^i$$

where $e_1,\ldots,e_m$ are the elementary symmetric polynomials in $a_1,\ldots,a_m$, we see that

$$\mathcal{Z}_p(s)=\sum_{k=p}^{\infty}\sum_{j=0}^{m-1}\frac{e_{m-1-j}(d_1^p,\ldots,d_m^p)}{k^{s-j}(k+2m-2p)^{s-j}}$$

where

$$d_i^p=:(i-p)(2m-p-i)$$

for $i=1,2,\ldots,m, i\neq p$. Use the notation in Sec. 3, we have

$$\mathcal{Z}_p(s)=\sum_{j=0}^{m-1}e_{m-j-1}(d_1^p,\ldots,d_m^p)\left(\zeta_{2m-2p,j}(s)-\sum_{k=1}^{p-1}\frac{1}{k^{s-j}(k+2m-2p)^{s-j}}\right). \tag{4.4}$$

Lemma 4.1. *Let $S=\{-d_i^p|i=1,2,\ldots,m,i\neq p\}$. Then $S=S_1\cup S_2$ where*

$$S_1=\{k(k-2m+2p)|1\leq k\leq 2m-2p-1\}$$

and

$$S_2=\{k(k+2m-2p)|1\leq k\leq p-1\}.$$

Proof. The index i and the number k are related by $k=p-i$ if $i<p$ and $k=2m-p-i$ if $p>i$, which enumerate the elements in the two sets S_1 and S_2 respectively.

Using the S_2 part of Lemma 4.1, we see that the second term in (4.4) actually vanishes. Hence

$$\mathcal{Z}_p(s)=\sum_{j=0}^{m-1}e_{m-j-1}(d_1^p,\ldots,d_m^p)\zeta_{2m-2p,j}(s). \tag{4.5}$$

We have by using the technical result Lemma 3.1

$$\mathcal{Z}_p'(0)=\sum_{j=0}^{m-1}e_{m-j-1}(d_1^p,\ldots,d_m^p)\left\{2\sum_{l\leq[\frac{j}{2}]}\binom{j}{2l}(2m-2p)^{2l}\zeta'(2l-2j)\right.$$
$$\left.+\sum_{k=1}^{2m-2p}[k(k-2m+2p)]^j\log k\right\}.$$

Consider the logarithmic term first. When $p=m$, there is no logarithmic term at all. For $p<m$, by the S_1 part of Lemma 4.1.

$$\sum_{j=0}^{m-1}e_{m-j-1}(d^p)[k(k-2m+2p)]^j=0$$

for all k except $k = 2m - 2p$, where $d^p = (d_1^p, \dots, d_m^p)$. So the only nonzero logarithmic term is

$$\sum_{j=0}^{m} e_{m-j-1}(d^p)[(2m-2p)(2m-2p-2m+2p)]^j \log(2m-2p) = e_{m-1}(d^p)\log(2m-2p)\,.$$

Since

$$e_{m-1}(d^p) = \prod_{i \neq p} d_i^p = \prod_{i \neq p}(i-p)(2m-p-i) = (-1)^{p-1}\frac{(p-1)!(2m-p-1)!}{2}\,,$$

the logarithmic term in $\mathcal{Z}'(0)$ is

$$-2\sum_{p=1}^{m-1}\log(2m-2p) = -2\sum_{p=1}^{m-1}\log(2p)\,. \tag{4.6}$$

Next we need to compute the zeta part of $\mathcal{Z}'(0)$. We need to evaluate

$$\sigma := 4\sum_{p=1}^{m-1}\frac{(-1)^p}{(p-1)!(2m-p-1)!}\sum_{j=0}^{m-1} e_{m-j-1}(d^p)2\sum_{l}\binom{j}{2l}(2m-2p)^{2l}\zeta'(2l-2j)$$
$$+2\frac{(-1)^m}{(m-1)!^2}\sum_{j=0}^{m-1} e_{m-j-1}(d^m)2\zeta'(-2j)\,.$$

Multiplying $\frac{(2m-2)!}{4}$ to both sides, we have

$$\begin{aligned}\frac{(2m-2)!}{4}\sigma &= 2\sum_{p=1}^{m-1}(-1)^p\binom{2m-2}{p-1}\sum_{s=0}^{m-1} e_{m-s-1}(d^p)\\ &\quad\times\sum_{l}\binom{s}{2l}(2m-2p)^{2l}\zeta'(2l-2s)\\ &\quad+(-1)^m\binom{2m-2}{m-1}\sum_{j=0}^{m-1} e_{m-j-1}(d^m)\zeta'(-2j)\\ &=: \sum_{j=0}^{m-1}\sigma_j\zeta'(-2j)\end{aligned}$$

where

$$\sigma_j =: \sum_{p=1,s=0}^{m-1} 2(-1)^p\binom{2m-2}{p-1}e_{m-s-1}(d^p)\binom{s}{2s-2j}(2m-2p)^{2s-2j}$$
$$+(-1)^m\binom{2m-2}{m-1}e_{m-j-1}(d^m)\,.$$

To compute each σ_j, we consider the generating function

$$\sigma(x) =: \sum_{k=0}^{m-1}\sigma_k x^{2k}\,.$$

We have

$$\sigma(x) = \sum_{k=0}^{m-1}\sum_{p=1}^{m-1}\sum_{j=0}^{m-1} 2(-1)^p \binom{2m-2}{p-1} e_{m-j-1}(d^p) \binom{j}{2j-2k} (2m-2p)^{2j-2k} x^{2k}$$

$$+ \sum_{k}^{m-1} (-1)^m \binom{2m-2}{m-1} e_{m-k-1}(d^m) x^{2k}.$$

Since

$$2\sum_k \binom{j}{2j-2k}(2m-2p)^{2j-2k}x^{2k}$$

$$= 2(2m-2p)^{2j}\sum_k \binom{j}{2j-2k}\left(\frac{x}{2m-2p}\right)^{2k}$$

$$= 2(2m-2p)^{2j}\sum_l \binom{j}{2l}\left(\frac{x}{2m-2p}\right)^{2j-2l}$$

$$= 2x^{2j}\sum_l \binom{j}{2l}\left(\frac{2m-2p}{x}\right)^{2l}$$

$$= x^{2j}\left[(1+\frac{2m-2p}{x})^j + (1-\frac{2m-2p}{x})^j\right]$$

$$= [x^2+(2m-2p)x]^j + [x^2-(2m-2p)x]^j,$$

we deduce that

$$\sigma(x) = \sum_{p=1}^{m-1}\sum_{j=0}^{m-1}(-1)^p\binom{2m-2}{p-1}e_{m-j-1}(d^p)$$

$$\times [(x^2+(2m-2p)x)^j + (x^2-(2m-2p)x)]^j$$

$$+ (-1)^m \sum_k^{m-1}\binom{2m-2}{m-1}e_{m-k-1}(d^m)x^{2k}.$$

As

$$\sum_{j=0}^{m-1} e_{m-j-1}(d^p)[x^2+(2m-2p)x]^j = \prod_{i\neq p}(x^2+(2m-2p)x+d_i^p)$$

$$= \prod_{i=1, i\neq p}^{m} (x+2m-p-i)(x+i-p),$$

$$\sum_{j=0}^{m-1} e_{m-j-1}(d^p)[x^2-(2m-2p)x]^j = \prod_{i\neq p}(x^2-(2m-2p)x+d_i^p)$$

$$= \prod_{i=1, i\neq p}^{m} (x-2m+p+i)(x-i+p),$$

and

$$\sum_{k}^{m-1} e_{m-k-1}(d^m)x^{2k} = \prod_{i\neq m}(x+d_i^m) = \prod_{i=1}^{m-1}(x+i)(x-i)\,.$$

We conclude that

$$\begin{aligned}\sigma(x) &= (-1)^m\binom{2m-2}{m-1}\prod_{i=1}^{m-1}(x+i)(x-i)\\ &\quad+\sum_{p=1}^{m-1}(-1)^p\binom{2m-2}{p-1}\left[\prod_{i=1,i\neq p}^{m}(x+2m-p-i)(x+i-p)\right.\\ &\quad\left.+\prod_{i=1,i\neq p}^{m}(x-2m+p+i)(x-i+p)\right].\end{aligned}$$

Therefore, by Theorem 3.2, $\sigma(x)$ is a constant:

$$\sigma(x) = -m(2m-2)!\,.$$

In particular, $\sigma_j = 0$ for $j > 0$, $\sigma_0 = -m(2m-2)!$, and

$$\frac{(2m-2)!}{4}\sigma = -m(2m-2)!\zeta'(0).$$

It follows that

$$\sigma = -4m\zeta'(0)\,. \tag{4.7}$$

From (4.6) and (4.7) we deduce the main result of this paper.

Theorem 4.2. *For $M = S^{2m-1}$ with the standard metric, the analytic torsion for M is*

$$T = \frac{1}{2^{m-1}(m-1)!}e^{-2m\zeta'(0)} = \frac{2\pi^m}{(m-1)!}\,.$$

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