An associative model of homotopy coherent functors and natural transformations

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Contents

1. Homotopy coherent functors and natural transformations

Motivation for homotopy coherent diagrams

- In homotopy theory, we often encounter with objects canonically related to higher homotopy data:
  - A_∞-structures (e.g. based loop space ΩX),
  - E_n-structures (e.g. iterated loop space $\Omega^n X$),
  - spaces obtained as homotopy (co)limits,
  - (weak) homotopy types of fiberwise and equivariant objects,
  - and so on.

Example

Principal G-bundles can be classified by the classifying space BG, which is the homotopy colimit of certain diagram derived from the group structure of G.

- We need to develop higher homotopy theoretic methods for theoretic and computational uses.

Homotopy diagram

Definition (Vogt, 1973)

A homotopy diagram $F\colon \mathcal{C} \to \mathcal{D}$ between topological cateogries consists of a correspondence $c \mapsto F(c)$ of objects and a family of continuous maps

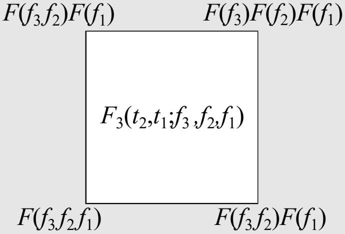
$$F_n\colon [0,1]^{n-1} \times \mathcal{C}(c_{n-1}, c_n) \times \cdots \mathcal{C}(c_0, c_1) \to \mathcal{D}(F(c_0), F(c_n))$$

for $n \geq 1$, and objects $c_0, \ldots, c_n$ of $\mathcal{C}$ satisfying the following condition:

$$F_n(t_{n-1}, \ldots, t_1; f_n, \ldots, f_1) = \begin{cases} F_{n-1}(t_{n-1}, \ldots, \widehat{t_k}, \ldots, t_1; f_n, \ldots, f_{k+1} \circ f_k, \ldots, f_1) & \text{for } t_k = 0, \\ F_{n-k}(t_{n-1}, \ldots, t_{k+1}; f_n, \ldots, f_{k+1}) \circ F_k(t_{k-1}, \ldots, t_1; f_k, \ldots, f_1) & \text{for } t_k = 1 \\ F_{n-1}(t_{n-1}, \ldots, t_2; f_n, \ldots, f_2) & \text{for } f_1 = \mathrm{id}, \\ F_{n-1}(t_{n-1}, \ldots, \max\{t_k, t_{k-1}\}, \ldots, t_1; f_n, \ldots, \widehat{f_k}, \ldots, f_1) & \text{for } f_k = \mathrm{id},\ 1 < k < n, \\ F_{n-1}(t_{n-2}, \ldots, t_1; f_{n-1}, \ldots, f_1) & \text{for } f_n = \mathrm{id}. \end{cases}$$

Homotopy diagram (continued)

- F_3 looks like:



- Boardman and Vogt extended this construction to operads and defined homotopy coherent algebras over an operad.

Homotopy homomorphism

- Let $[n] = \{0 < 1 < \cdots < n\}$.
- A "natural transformation" between homotopy diagrams is formulated as follows.

Definition (Vogt, 1973)

A homotopy homomorphism $\lambda\colon F \to G$ between homotopy diagrams $F, G\colon \mathcal{C} \to \mathcal{D}$ is a homotopy diagram $\lambda\colon [1] \times \mathcal{C} \to \mathcal{D}$ which restricts to F and G on $\{0\} \times \mathcal{C}$ and $\{1\} \times \mathcal{C}$, respectively.

- Boardman and Vogt (1973) formulated homotopy coherent morphisms between homotopy coherent algebras over an operad by this definition.
- In their formulation, it is difficult to specify the composite of two homotopy homomorphisms.

Homotopy homomorphism (continued)

- Let $\mathcal{C}$ and $\mathcal{D}$ be topological categories.
- Let $\mathcal{R}_n$ denote the class (or set) of homoropy diagrams $[n] \times \mathcal{C} \to \mathcal{D}$.
- The sequence of classes $\mathcal{R} = \{\mathcal{R}_n\}_n$ is a simplicial class of which the simplicial structure is derived from poset maps $[m] \to [n]$.
- Let $\Lambda^n_k \subset \Delta^n$ denote the k-th horn.

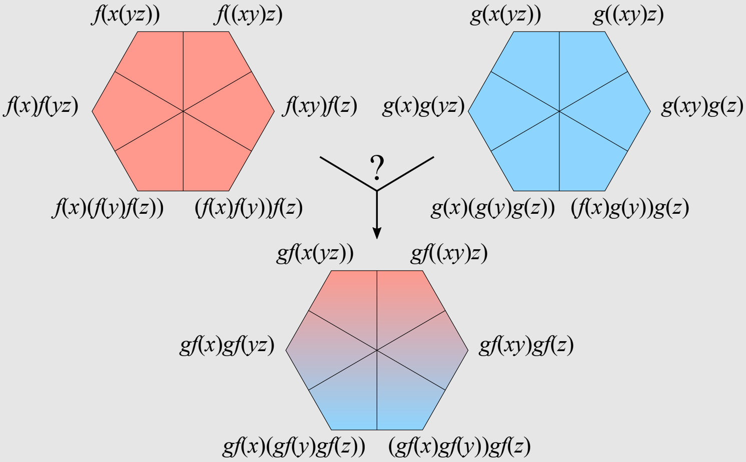
Theorem (Boardman–Vogt, 1973)

The simplicial class $\mathcal{R}$ satisfies the following extension property: any map $\Lambda^n_k \to \mathcal{R}$ extends over Δ^n for $n \geq 2$ and $0 < k < n$.

- A simplicial set satisfying this condition is now called a quasicategory.
- The theorem guarantees that the composition of homotopy homomorphisms can be defined and is unital and associative up to homotopy.

Goal of this talk

- In this talk, let us explore a way to define specified associative compositions of homotopy coherent functors and natural transformations (to draw pictures of them).



2. An associative model

Definition

An A_∞-functor $F\colon \mathcal{C} \to \mathcal{D}$ of length $\ell \geq 0$ between topological cateogries consists of a correspondence $c \mapsto F(c)$ of objects and a family of continuous maps

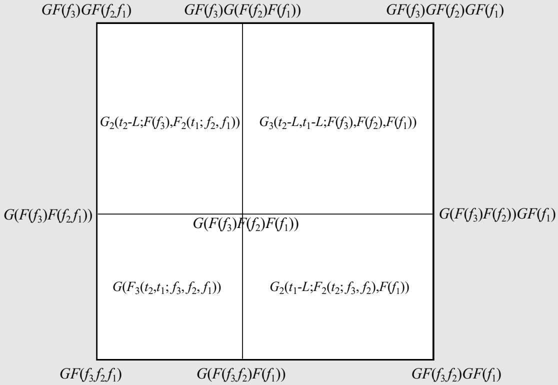
$$F_n\colon [0,\infty]^{n-1} \times \mathcal{C}(c_{n-1}, c_n) \times \cdots \times \mathcal{C}(c_0, c_1) \to \mathcal{D}(F(c_0), F(c_n))$$

for $n \geq 1$, and objects $c_0, \ldots, c_n$ of $\mathcal{C}$ satisfying the following condition:

$$\begin{aligned}
&F_n(t_{n-1}, \ldots, t_1; f_n, \ldots, f_1) \\
&= \begin{cases}
F_{n-1}(t_{n-1}, \ldots, \widehat{t_k}, \ldots, t_1; f_n, \ldots, f_{k+1} \circ f_k, \ldots, f_1) & \text{for } t_k = 0, \\
F_{n-k}(t_{n-1}, \ldots, t_{k+1}; f_n, \ldots, f_{k+1}) \circ F_k(t_{k-1}, \ldots, t_1; f_k, \ldots, f_1) & \text{for } t_k \geq \ell, \\
F_{n-1}(t_{n-1}, \ldots, t_2; f_n, \ldots, f_2) & \text{for } f_1 = \mathrm{id}, \\
F_{n-1}(t_{n-1}, \ldots, \max\{t_k, t_{k-1}\}, \ldots, t_1; f_n, \ldots, \widehat{f_k}, \ldots, f_1) & \text{for } f_k = \mathrm{id},\ 1 < k < n, \\
F_{n-1}(t_{n-2}, \ldots, t_1; f_{n-1}, \ldots, f_1) & \text{for } f_n = \mathrm{id}.
\end{cases}
\end{aligned}$$

Composition of A_∞-functors

- The composite of A_∞-functors F and G is depicted as follows:



- Mimicking the Moore path, the composition becomes unital and associative.

Definition (Vogt, 1973)

The topological category $\mathcal{WC}$ of a topological category $\mathcal{C}$ is defined as follows:

- an object of $\mathcal{WC}$ is an object of $\mathcal{C}$,
- the mapping space $\mathcal{WC}(c, c')$ is defined by

$$\mathcal{WC}(c, c') = \coprod_{c_1, \ldots, c_{n-1}} [0, \infty]^{n-1} \times \mathcal{C}(c_{n-1}, c') \times \cdots \times \mathcal{C}(c, c_1) \Big/ \sim$$

  with an appropriate identification,
- the composition is defined by

$$\begin{aligned}&[s_{m-1}, \ldots, s_1; g_m, \ldots, g_1] \circ [t_{n-1}, \ldots, t_1; f_n, \ldots, f_1] \\ &= [s_{m-1}, \ldots, s_1, \infty, t_{n-1}, \ldots, t_1; g_m, \ldots, g_1, f_n, \ldots, f_1].\end{aligned}$$

$\mathcal{W}$-construction (continued)

- An A_∞-functor $F \colon \mathcal{C} \to \mathcal{D}$ induces a continuous functor $\mathcal{W}F \colon \mathcal{W}\mathcal{C} \to \mathcal{W}\mathcal{D}$.
- We have the canonical functor $\epsilon \colon \mathcal{W}\mathcal{C} \to \mathcal{C}$: $\epsilon[t_{n-1}, \dots, t_1; f_n, \dots, f_1] = f_n \circ \cdots \circ f_1$, which induces homotopy equivalences on mapping spaces.
- The functor $\mathcal{W}$ and the forgetful functor are “almost” adjoint to each other.

A_∞-natural transformation

- Let $\Delta^n_\infty = \{(t_n, \dots, t_1) \in [0,\infty]^n \mid t_n \ge \cdots \ge t_1\}$.

Definition

An A_∞-natural transformation $\lambda\colon F \to G$ of length $\ell \ge 0$ between continuous functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of families of continuous maps $\lambda(c)\colon F(c) \to G(c)$ for objects c of $\mathcal{C}$ and

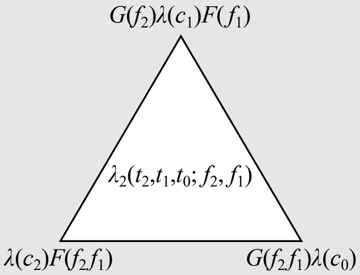
$$\lambda_n\colon \Delta^n_\infty \times \mathcal{C}(c_{n-1}, c_n) \times \cdots \times \mathcal{C}(c_0, c_1) \to \mathcal{D}(F(c_0), F(c_n))$$

for $n \ge 1$, and objects $c_0, \dots, c_n$ of $\mathcal{C}$ satisfying the following condition:

$$\begin{aligned} &\lambda_n(t_n, \dots, t_1; f_n, \dots, f_1) \\ &= \begin{cases} \lambda_{n-1}(t_n, \dots, t_2; f_n, \dots, f_2) \circ F(f_1) & \text{for } t_1 = 0, \\ \lambda_{n-1}(t_n, \dots, \widehat{t_k}, \dots, t_1; f_n, \dots, f_k \circ f_{k-1}, \dots, f_1) & \text{for } t_{k-1} = t_k, \\ G(f_n) \circ \lambda_{n-1}(t_{n-1}, \dots, t_1; f_{n-1}, \dots, f_1) & \text{for } t_n \ge \ell, \\ \lambda_{n-1}(t_{n-1}, \dots, \widehat{t_k}, \dots, t_1; f_n, \dots, \widehat{f_k}, \dots, f_1) & \text{for } f_k = \mathrm{id}. \end{cases} \end{aligned}$$

A_∞-natural transformation (continued)

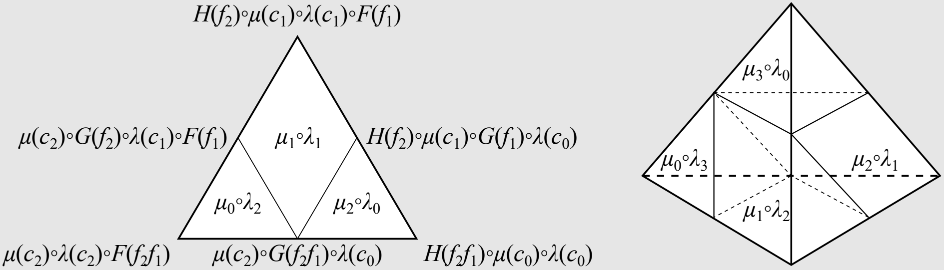
- λ_2 looks like:



- Our A_∞-natural transformation is essentially the same as Vogt's source- or target-reduced homotopy homomorphism while the latter is parametrized by $[0, 1]^n$.

Composition of A_∞-natural transformations

- The composite of $\lambda\colon F \to G$ and $\mu\colon G \to H$ is depicted as follows:



- Each piece in Δ^n_∞ is homeomorphic to $\Delta^p \times \Delta^q$ with $p+q=n$.
- Again, mimicking the Moore path, the composition becomes unital and associative.

3. Application to A_∞-spaces

Operad

Definition

A sequence of spaces $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ is said to be a non-symmetric operad if it is equipped with maps $\eta\colon * \to \mathcal{O}(1)$ and

$$\gamma\colon \mathcal{O}(s) \times (\mathcal{O}(r_1) \times \cdots \times \mathcal{O}(r_s)) \to \mathcal{O}(r_1 + \cdots + r_s)$$

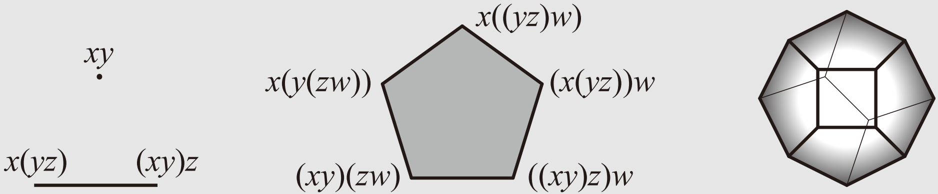
for $r_1, \ldots, r_s \geq 0$ making the following diagrams commute:

$$\begin{array}{ccc}
\mathcal{O}(t) \times \prod_i \mathcal{O}(s_i) \times \prod_{i,j} \mathcal{O}(r_{ij}) & \xrightarrow{\mathrm{id}\times\gamma} & \mathcal{O}(t) \times \prod_i \mathcal{O}(r_{i1} + \cdots + r_{is_i}) \\
{\scriptstyle \gamma\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \gamma} \\
\mathcal{O}(s_1 + \cdots + s_t) \times \prod_{i,j} \mathcal{O}(r_{ij}) & \xrightarrow[\gamma]{} & \mathcal{O}(\sum_{i,j} r_{ij})
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{O}(s) & \xrightarrow{\mathrm{id}\times\eta^{\times s}} & \mathcal{O}(s) \times \mathcal{O}(1)^s \\
{\scriptstyle \eta\times\mathrm{id}}\downarrow & \searrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \gamma} \\
\mathcal{O}(1) \times \mathcal{O}(s) & \xrightarrow[\gamma]{} & \mathcal{O}(s)
\end{array}$$

Stasheff operad

Example

Let $\mathcal{K}(n)$ be the n-th associahedron ($\mathcal{K}(0) = \mathcal{K}(1) = \mathcal{K}(2) = *$), which is considered to be the space of planar metric trees with n-leaves without bivalent vertex. The sequence $\mathcal{K} = \{\mathcal{K}(n)\}_{n \geq 0}$ is a non-symmetric operad called the Stasheff operad equipped with grafting operations.



- The Stasheff operad is obtained from the operad **Assoc** of the unital and associative binary operation applying the Boardman–Vogt (based) W-construction.

Associated category

- Let $\mathcal{O} = \{\mathcal{O}(n)\}_n$ be a non-symmetric operad.
- We can construct the topological category $\widetilde{\mathcal{O}}$ as follows:
  - the objects are $0, 1, 2 \ldots,$
  - the mapping space $\widetilde{\mathcal{O}}(m, n)$ is defined by $\widetilde{\mathcal{O}}(0, 0) = \{\mathrm{id}\}$ and

$$\widetilde{\mathcal{O}}(m, n) = \coprod_{0=i_0 \leq i_1 \leq \cdots \leq i_n = m} \mathcal{O}(i_1 - i_0) \times \cdots \times \mathcal{O}(i_n - i_{n-1}),$$

  - the composition is induced from γ in an obvious manner.
  - Example: $\widetilde{\mathcal{O}}(m, 0) = \emptyset$ (for $m > 0$), $\widetilde{\mathcal{O}}(m, 1) = \mathcal{O}(m)$.
- The category $\widetilde{\mathcal{O}}$ is equipped with a monoidal structure $\oplus$: $m \oplus m' = m + m'$ with an obvious map

$$\oplus\colon \widetilde{\mathcal{O}}(m, n) \times \widetilde{\mathcal{O}}(m', n') \to \widetilde{\mathcal{O}}(m + m', n + n').$$

Algebra over an operad

Definition

Let $\mathcal{O}$ be a non-symmetric operad and $\mathcal{C}$ be a topological monoidal category. An $\mathcal{O}$-algebra A in $\mathcal{C}$ is a monoidal functor $A\colon \widetilde{\mathcal{O}} \to \mathcal{C}$.

- A homotopy coherent algebra over an operad $\mathcal{O}$ is understood to be a $\mathcal{WO}$-algebra over the Boardman–Vogt W-construction $\mathcal{WO}$ of $\mathcal{O}$.
- In particular, an algebra over the Stasheff operad in the category of topological spaces is called an A_∞-space.

Homotopy $\mathcal{O}$-map

Definition

Let $A, B\colon \widetilde{\mathcal{O}} \to \mathcal{C}$ be $\mathcal{O}$-algebras in a topological monoidal category $\mathcal{C}$. A homotopy $\mathcal{O}$-map $f\colon A \to B$ is defined to be an A_∞-natural transformation $f\colon A \to B$ satisfying the conditions

$$f(m) = f(1)^{\otimes m}$$

and

$$f_i(t_i, \ldots, t_1; \phi_i \oplus \psi_i, \ldots, \phi_1 \oplus \psi_i) = f_i(t_i, \ldots, t_1; \phi_i, \ldots, \phi_1) \otimes f_i(t_i, \ldots, t_1; \psi_i, \ldots, \psi_i)$$

for any sequences of composable morphisms $\phi_i, \ldots, \phi_1$ and $\psi_i, \ldots, \psi_1$ in $\widetilde{\mathcal{O}}$.

- It is obvious that the composite of homotopy $\mathcal{O}$-maps is again a homotopy $\mathcal{O}$-map.
- The composition of homotopy $\mathcal{O}$-maps is unital and associative. Then we obtain the topological category of $\mathcal{O}$-algebras and homotopy $\mathcal{O}$-maps.

Strictly unital homotopy $\mathcal{O}$-map

- A non-symmetric operad $\mathcal{O}$ is said to be reduced if $\mathcal{O}(0) = *$.
- Let $\sigma_j = \eta^{\oplus j-1} \oplus * \oplus \eta^{\oplus n-j} \in \widetilde{\mathcal{O}}(n-1, n)$.

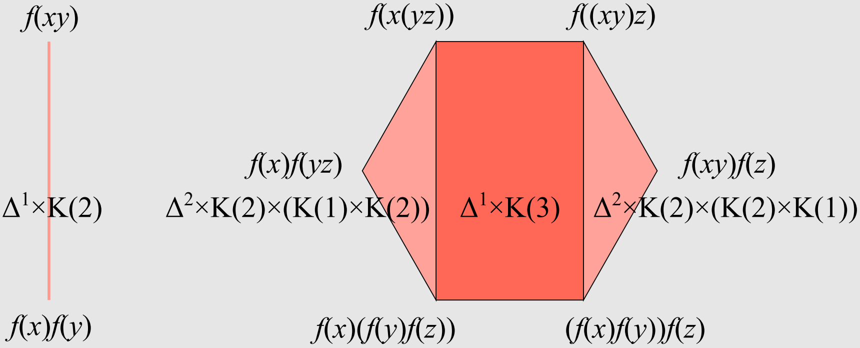
Definition

Let $A, B \colon \widetilde{\mathcal{O}} \to \mathcal{C}$ be algebras over a reduced non-symmetric operad $\mathcal{O}$ in a topological monoidal category $\mathcal{C}$. A homotopy $\mathcal{O}$-map $f \colon A \to B$ is said to be strictly unital if the following condition is satisfied:

$$\begin{aligned} &f_i(t_i, \ldots, t_1; \phi_i, \ldots, \phi_{k+1}, \sigma_j, \phi_{k-1}, \ldots, \phi_1) \\ &= f_{i-1}(t_i, \ldots, \widehat{t_k}, \ldots, t_1; \phi_i, \ldots, \phi_{k+1} \circ \sigma_j, \phi_{k-1}, \ldots, \phi_1). \end{aligned}$$

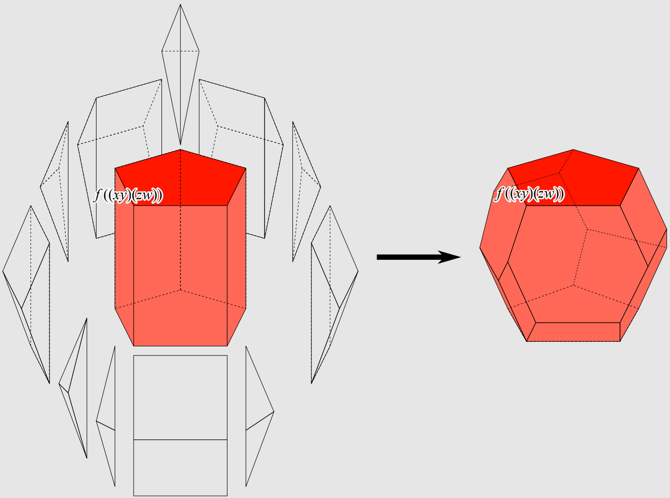
- Similarly, we obtain the topological category of $\mathcal{O}$-algebras and strictly unital homotopy $\mathcal{O}$-maps.

- We call a strictly unital homotopy $\mathcal{K}$-map ($\mathcal{K}$ is the Stasheff operad) an A_∞-map.
- The parameter spaces (multiplihedra) are depicted as follows.

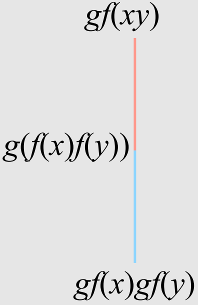


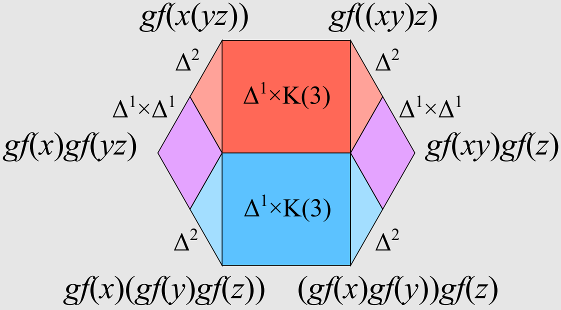
How to draw the multiplihedra (continued)

- The parameter space of f_4.

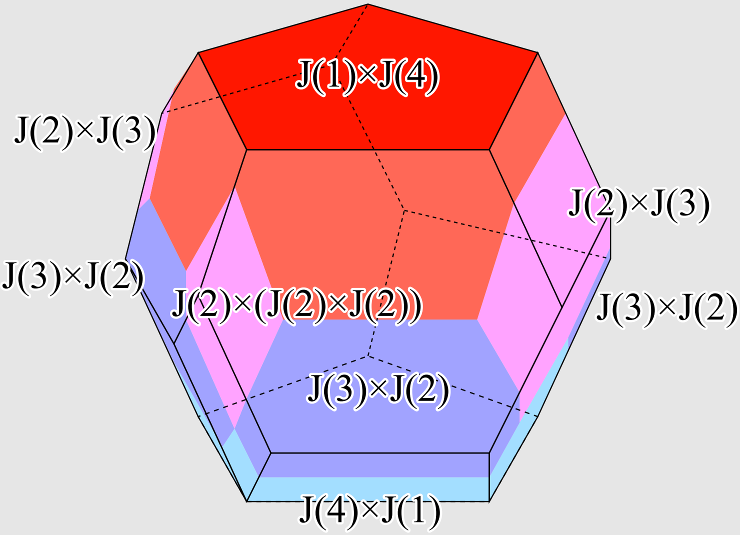


- The composition of A_∞-maps is depicted as follows, where each piece is obtained from the composition of A_∞-natural transformations.





How to draw the composition of A_∞-maps (continued)



Summary

- A_∞-functor $F\colon \mathcal{C} \to \mathcal{D}$ with higher homotopy
$$F_n\colon [0,\infty]^{n-1} \times \mathcal{C}(c_{n-1}, c_n) \times \cdots \times \mathcal{C}(c_0, c_1) \to \mathcal{D}(F(c_0), F(c_n)).$$
- A_∞-natural transformation $\lambda\colon F \to G$ with higher homotopy
$$\lambda_n\colon \Delta^n_\infty \times \mathcal{C}(c_{n-1}, c_n) \times \cdots \times \mathcal{C}(c_0, c_1) \to \mathcal{D}(F(c_0), F(c_n)).$$
- Their compositions defined by mimicking the Moore path are unital and associative.
- A_∞-natural transformation can be applied to define homotopy $\mathcal{O}$-maps between $\mathcal{O}$-algebras.
- We can draw pictures of their higher homotopies!

Thank you!