Higher homotopy normalities in topological groups

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Classifying spaces in homotopy theory: in honour of Ran Levi's 60th Birthday
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1. Higher homotopy associativity and commutativity

- A map $f\colon H \to G$ between topological groups is said to be an H-map if

$$f \circ \mu \simeq \mu \circ (f \times f).$$

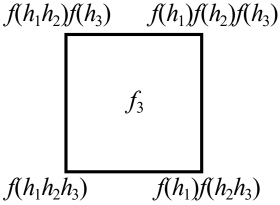
- H-map is much far from homomorphism. There exists an H-map $f\colon H \to G$ not H-equivalent to a homomorphisms as in the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ G' & \xrightarrow[\text{a homomorphism}]{} & H' \end{array}$$

- So, what is the obstruction to a map being a homomorphism up to H-equivalence?

A_∞-map

- Sugawara (1960) (with Milnor's result) answered this question.
  - An H-map $f\colon H \to G$ is homotopy equivalent to a homomorphism if and only if it admits an A_∞-form $\{f_i\colon I^{i-1} \times H^i \to G\}_{i\geq 1}$ which describes how the associativity is preserved through f.
  - f_2 is the homotopy between $f \circ \mu$ and $\mu \circ (f \times f)$.
  - $f_3\colon [0,1]^2 \times H^3 \to G$ is depicted as follows.



  - An H-map equipped with an A_∞-form is called an A_∞-map.

Projective space and A_n-map

- The previous result actually follows from the following result:
  - A pointed map $f\colon H \to G$ admits an A_∞-form if and only if the suspension $\Sigma f\colon \Sigma H \to \Sigma G$ extends to a map between the classifying space $BH \to BG$:

$$\begin{array}{ccc} \Sigma H & \xrightarrow{f} & \Sigma G \\ \downarrow & & \downarrow \\ BH & \underset{\exists}{\dashrightarrow} & BG \end{array}$$

- Stasheff (1963) generalized this result.
  - The classfying space has the natural filtration:

$$* = B_0G \subset \Sigma G = B_1G \subset B_2G \subset \cdots \subset B_nG \subset \cdots \subset BG.$$

  - B_nG is called the n-th projective space of G.
  - He proved that $\Sigma f\colon \Sigma H \to \Sigma G$ extends to $B_nH \to BG$ if and only if f admits an A_n-form $\{f_i\colon I^{i-1} \times H^i \to G\}_{1 \le i \le n}$.
  - This result is basic in the present work.

Homotopy commutativity

- A topological group G is said to be homotopy commutative if the Samelson product
$$G \wedge G \to G, \qquad (x, y) \mapsto xyx^{-1}y^{-1}$$
is null-homotopic.
- Through the isomorphism
$$[G \wedge G, G] \cong [G \wedge G, \Omega BG] \cong [\Sigma G \wedge G, BG],$$
the Samelson product corresponds to the Whitehead product $[\iota, \iota]$ of the inclusion $\iota \colon \Sigma G \to BG$.
  - So, G is homotopy commutative if and only if $[\iota, \iota] = 0$.

Higher homotopy commutativity

- C_n-space in the sense of Williams is defined by some higher homotopy condition using permutohedra P_n.
- An equivalent condition is as follows: the wedge sum of the inclusion

$$(\Sigma G)^{\vee n} \to BG$$

extends over the product

$$(\Sigma G)^{\times n} \to BG.$$

- This is just the vanishing of the higher Whitehead products.

Higher homotopy commutativity (continued)

- The higher homotopy commutativity of Lie groups and their p-localizations is well studied. Here is a typical argument to show the non-commutativity.

Example

Let $G = \mathrm{SU}(2) = S^3$ and p an odd prime. Suppose the wedge sum of the inclusion

$$(S^4)^{\vee \frac{p+1}{2}} \to B\,\mathrm{SU}(2) = \mathbb{H}P^\infty$$

extends to a map

$$f\colon (S^4)^{\times \frac{p+1}{2}} \to B\,\mathrm{SU}(2).$$

We know $\mathcal{P}^1 c_2 = a c_2^{\frac{p+1}{2}}$ with $a \neq 0$ in $H^{2p+2}(B\,\mathrm{SU}(2);\mathbb{F}_p)$. Then $f^*\mathcal{P}^1 c_2 \neq 0$. But $\mathcal{P}^1$ must be trivial in $H^*((S^4)^{\times \frac{p+1}{2}};\mathbb{F}_p)$ by the Cartan formula. This contradicts to $f^* c_2 \neq 0$. Therefore, $\mathrm{SU}(2)$ is not p-locally a $C_{\frac{p+1}{2}}$-space.

- What about "higher homotopy normality"?

2. Higher homotopy normality

Crossed module

- In the rest of this talk, let H and G be topological groups of homotopy types of CW complexes.
- A normal subgroup $H \subset G$ is a subgroup stable under the inner automorphisms.
- Crossed module is a generalization of normal subgroup to non-inclusions.

Definition (MacLane, J.H.C.Whitehead 1940s)

A (topological) crossed module consists of homomorphisms $f\colon H \to G$ and $\rho\colon G \to \mathrm{Aut}(H)$ satisfying the conditions

- $\rho(f(h))(x) = hxh^{-1}$ for any $x, h \in H$,
- $f(\rho(g)(x)) = gf(x)g^{-1}$ for any $x \in H$ and $g \in G$.

- Farjoun and Segev (2010) proved that the Borel construction $EH \times_H G$ of a crossed module $f\colon H \to G$ naturally inherits a group structure. This should be considered as "the homotopy quotient group of a homotopically normal subgroup".

$N_k(\ell)$-map

- $\mathcal{A}_\ell(H, G)$ denotes the space of A_ℓ-maps.
- $\mathrm{conj}_H \colon H \to \mathcal{A}_\ell(H, H)$ denotes the conjugation $\mathrm{conj}_H(h)(x) = hxh^{-1}$.

Definition (T.)

A homomorphism $f \colon H \to G$ is an $N_k(\ell)$-map if an A_k-map $\rho \colon G \to \mathcal{A}_\ell(H, H)$ is given and the following conditions hold:

- $\rho \circ f$ is homotopic to conj_H as an A_ℓ-map,
- the map $* \to \mathcal{A}_\ell(H, G)$, $* \mapsto f$ is A_k-equivariant with respect to the action of G,
- the higher homotopy in the second condition coincides with the first condition on H.

- This is a higher homotopy analogue of crossed module.
- $N_1(1)$-map is equivalent to homotopy normal map introduced by McCarty (1964).
- James (1967) defined another homotopy normality which is slightly weaker than McCarty's.

$N_1(1)$-map

Definition (McCarty 1964)

A homomorphism $f\colon H \to G$ is homotopy normal (an $N_1(1)$-map) if there exists a map $\tilde{\gamma}\colon G \wedge H \to H$ making the diagram

$$\begin{array}{ccc} H \wedge H & \xrightarrow{\gamma_H} & H \\ {\scriptstyle f\wedge \mathrm{id}}\downarrow & \overset{\exists\tilde{\gamma}}{\nearrow} & \downarrow{\scriptstyle f} \\ G \wedge H & \xrightarrow[\gamma]{} & G \end{array}$$

commute up to homotopy and the homotopies comapatible with the stationary homotopy of the outer square.

Immediate consequences

- If $f\colon H \to G$ is an $N_k(\ell)$-map and $k \geq k'$ and $\ell \geq \ell'$, then f is an $N_{k'}(\ell')$-map.
- If $f\colon H \to G$ is a crossed module, then f is an $N_\infty(\infty)$-map.
- The homomorphism $f\colon H \to *$ is an $N_k(\ell)$-map if and only if $\mathrm{conj}_H\colon H \to \mathcal{A}_\ell(H, H)$ is homotopic to the constant map as an A_k-map.
  - The latter condition is equivalent to being a $C(k,\ell)$-space (T. 2016), which is a higher homotopy commutativity introduced by Kishimoto and Kono (2010).
  - $C(\infty,\infty)$-space and Sugawara C_∞-space are known to be equivalent. Then we conclude that $H \to *$ is an $N_\infty(\infty)$-map if and only if BH is an H-space.
  - This is analogous to the fact that $H \to *$ is a crossed module if and only if H is commutative.

3. Results

Equivariant and fiberwise homotopy theory

- The Borel construction defines the correspondence

$$\text{a } G\text{-space } X \quad \mapsto \quad \text{a fiberwise space } EG \times_G X \to BG.$$

  This provides an "equivalence" between the equivariant homotopy theory and the fiberwise homotopy theory in an appropriate sense.
- The idea of the main theorem is based on this equivalence.

Main theorem

Theorem (T.)

Let $f\colon H \to G$ be a homomorphism and $F\colon E_kH \times_H H \to E_kG \times_G G$ denote the induced map of f. Then f is an $N_k(\ell)$-map if and only if there exists a fiberwise A_ℓ-space $\mathcal{E} \to B_kG$ and F factors as

$$E_kH \times_H H \xrightarrow{\phi} \mathcal{E} \xrightarrow{\psi} E_kG \times_G G$$

up to homotopy over $B_kf\colon B_kH \to B_kG$ such that the following conditions hold:

- ϕ covers B_kf and ψ covers the identity on B_kG,
- ϕ and ψ restrict to A_ℓ-maps on each fiber,
- ϕ is a weak homotopy equivalence on each fiber,
- the restriction of $\psi \circ \phi$ to the fiber over the basepoint is homotopic to f as an A_ℓ-map.

- The last for conditions correspond to the compatibility required in the definition of $N_k(\ell)$-map.

Remark on main theorem

- Roughly, this theorem states that $f\colon H \to G$ is an $N_k(\ell)$-map if and only if the following "unusual" factorization of $F\colon E_kH \times_H H \to E_kG \times_G G$ exists:

$$\begin{array}{ccccc}
H & = & H & \xrightarrow{f} & G \\
\downarrow & & \downarrow & & \downarrow \\
E_kH \times_H H & \longrightarrow & \mathcal{E} & \longrightarrow & E_kG \times_G G \\
\downarrow & & \downarrow & & \downarrow \\
B_kH & \xrightarrow[B_kf]{} & B_kG & = & B_kG
\end{array}$$

Remark on main theorem (continued)

- The "usual" factorization is as follows. The middle column is induced from the conjugation action of H on G through f.

$$\begin{array}{ccccc}
H & \xrightarrow{f} & G & = & G \\
\downarrow & & \downarrow & & \downarrow \\
E_kH \times_H H & \longrightarrow & E_kH \times_H G & \longrightarrow & E_kG \times_G G \\
\downarrow & & \downarrow & & \downarrow \\
B_kH & = & B_kH & \xrightarrow[B_kf]{} & B_kG
\end{array}$$

- This factorization is possible for any homomorphism f.

H-structure on Borel construction

Theorem (T.)

- Let $f\colon H \to G$ be a homomorphism. Then the Borel construction $X = EH \times_H G$ is an H-space if f is an $N_k(k)$-map and $\operatorname{cat} X \le k$.

Example

- Let $H = K(\mathbb{Q}, 2n-1)$ and $G = K(\mathbb{Q}, 4n-1)$. Consider the homomorphism $f\colon H \to G$ with classifying map $Bf\colon K(\mathbb{Q}, 2n) \to K(\mathbb{Q}, 4n)$ corresponding to $u^2 \in H^{4n}(K(\mathbb{Q}, 2n); \mathbb{Q})$. Then the Borel construction is

$$EH \times_H G \simeq \operatorname{hofib}(Bf) \simeq S^{2n}_{(0)}.$$

Since $S^{2n}_{(0)}$ does not admit an H-structure and $\operatorname{cat} S^{2n}_{(0)} = 1$, f is not an $N_1(1)$-map.

Preceding results on examples

- There have been many results on homotopy normality of Lie groups.
- (James 1967)
  The inclusion $\mathrm{U}(m) \to \mathrm{U}(n)$ is not (2-locally) homotopy normal in the sense of James for $1 \leq m < n$. Similar results hold for $\mathrm{O}(m) \to \mathrm{O}(n)$ ($2 \leq m < n$) and $\mathrm{Sp}(m) \to \mathrm{Sp}(n)$ for $1 \leq m < n$.
- Other results include: McCarty (1964), James (1971), Kachi (1982), Furukawa (1985), Furukawa (1987), Furukawa (1995), Kudou–Yagita (1998), Kudou–Yagita (2003), Kono–Nishimura (2003), Nishimura (2006), Kishimoto–T. (2018).
- These results suggest that $H \to G$ tends to fail to be p-locally homotopy normal for small prime p.

Higher homotopy normality of $SU(m) \to SU(n)$

- Main theorem provides an obstruction theory for $N_k(\ell)$-map applying the fiberwise projective space functor.
- Then, by a typical argument using Steenrod operations as mentioned before, we can obtain the following result.

Theorem (T.)

- If $\max\{kn-2,(k-1)n+2\} < p \le kn+2(\ell-1)$ for some $n \ge 3$ and $k,\ell \ge 1$, then the inclusion $SU(2) \to SU(n)$ is not a p-local $N_k(\ell)$-map.
- If $\max\{kn-m,(k-1)n+2\} < p \le kn+(\ell-2)m$ for some $2 \le m < n$ and $k,\ell \ge 1$, then the inclusion $SU(m) \to SU(n)$ is not a p-local $N_k(\ell)$-map.

- This result is not very sharp. For example, the normality is not determined when $kn+(\ell-2)m < p < kn+\ell m$.
- A similar result is obtained for $SO(2m+1) \to SO(2n+1)$.

Summary

- $N_k(\ell)$-map is a higher homotopical analogue of crossed module.
- $N_k(\ell)$-map is characterized by fiberwise A_ℓ-maps over the k-th projective spaces.
- The Borel construction $EH \times_H G$ of an $N_k(k)$-map $f\colon H \to G$ is an H-space if $\operatorname{cat} EH \times_H G \leq k$ holds.
- Fiberwise projective space provides a method to detect obstructions to being $N_k(\ell)$-maps.

Thank you!