Higher homotopy normalities in topological groups

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- Homotopy normalities in topological groups, arXiv:2111.15096.

Contents

Introduction to crossed module

Crossed module

- A normal subgroup $H \subset G$ of a topological group G is a subgroup stable under the inner automorphisms in G.
- Crossed module is an extension of normal subgroup to non-inclusions $H \to G$.

Definition (MacLane, J.H.C.Whitehead 1940s)

- A (topological) crossed module consists of homomorphisms $f\colon H \to G$ and $\rho\colon G \to \mathrm{Aut}(H)$ satisfying the conditions
  - $f(\rho(g)(x)) = gf(x)g^{-1}$ for any $x \in H$ and $g \in G$.
  - $\rho(f(h))(x) = hxh^{-1}$ for any $x, h \in H$,
- For example, the map $\pi_1(F) \to \pi_1(E)$ induced from the fiber inclusion of a fiber bundle $F \to E \to B$ is naturally a crossed module.

Homotopy quotient

- If a compact Lie group G acts freely on a manifold X, we have the fiber bundle

$$G \to X \to X/G. \tag{*}$$

  The quotient X/G is considered to be "homotopically good".
- But if the action is not free, (*) is no longer a fiber bundle. Then the quotient X/G is considered to be "homotopically bad".
- Any action of a topological group G on a space X can be made into a homotopically good action:

$$G \times (EG \times X) \to EG \times X, \quad (g, (u, x)) \mapsto (ug^{-1}, gx).$$

  EG is a principal G-bundle with contractible total space.
- $EG \times_G X = (EG \times X)/G$ is called the Borel construction, which is considered to be a "homotopy quotient".

Homotopy quotient of crossed module

- If $f\colon H \to G$ is a crossed module, the image $f(H) \subset G$ is a normal subgroup. But the quotient $G/f(H)$ is a "homotopically bad quotient" in general.
- The homotopy quotient should be the Borel construction $EH \times_H G$ under the action through $f\colon H \to G$.

Theorem (Farjoun–Segev 2010)

- If $f\colon H \to G$ is a crossed module, then the Borel construction $EH \times_H G$ (with some good model of EH) is naturally a topological group.
- Even if H and G are discrete, $EH \times_H G$ needs not be discrete.

Introduction to higher homotopy associativity

Projective space

- The classifying space BG of a topological group G has a natural filtration

$$* = B_0G \subset B_1G \subset \cdots \subset B_nG \subset \cdots \subset B_\infty G = BG.$$

- B_nG is called the *n-th projective space*.
- $B_{n+1}G$ is the mapping cone of some map $\Sigma^n G^{\wedge n+1} \to B_nG$.

Example

- $B_nS^0 = \mathbb{R}P^n$ ($S^0 = \mathrm{O}(1) = \mathbb{Z}/2\mathbb{Z}$).
- $B_nS^1 = \mathbb{C}P^n$ ($S^1 = \mathrm{U}(1) = \mathrm{SO}(2)$).
- $B_nS^3 = \mathbb{H}P^n$ ($S^3 = \mathrm{Sp}(1) = \mathrm{SU}(2)$).

A_n-map

- Let $f\colon H \to G$ be a map between topological groups.
- If $f(h_1 h_2) = f(h_1)f(h_2)$ holds for any h_1, h_2, then we have $f(h_1 \cdots h_n) = f(h_1) \cdots f(h_n)$ for any $n \geq 1$ and $h_1, \ldots, h_n$.
- If a homotopy $f_2\colon [0,1] \times H^2 \to G$ between the maps

$$(h_1, h_2) \mapsto f(h_1 h_2) \quad \text{and} \quad (h_1, h_2) \mapsto f(h_1)f(h_2)$$

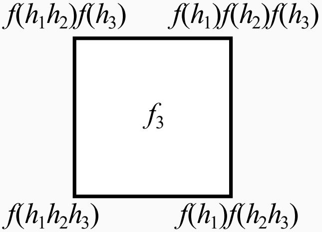
is given, we can construct the homotopies

$$f(h_1 h_2 h_3) \sim f(h_1)f(h_2 h_3) \sim f(h_1)f(h_2)f(h_3),$$
$$f(h_1 h_2 h_3) \sim f(h_1 h_2)f(h_3) \sim f(h_1)f(h_2)f(h_3).$$

These are in general different.

A_n-map (continued)

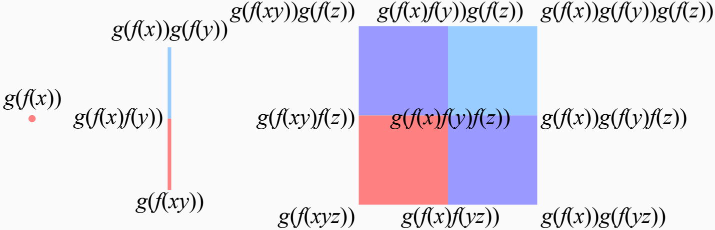
- These homotopies coincide up to homotopy of homotopies iff there exists a map $f_3\colon [0,1]^2 \times H^3 \to G$ depicted as follows.



- This idea is extended to A_n-map ($n = 1, 2, \ldots, \infty$). An A_n-map is a map $f\colon H \to G$ equipped with higher homotopies $\{f_i\colon [0,1]^{i-1} \times H^i \to G\}_{i=1}^{n}$ (Sugawara 1960, Stasheff 1963).

Category of topological groups and A_n-maps

- The composition of A_n-maps is defined as follows:

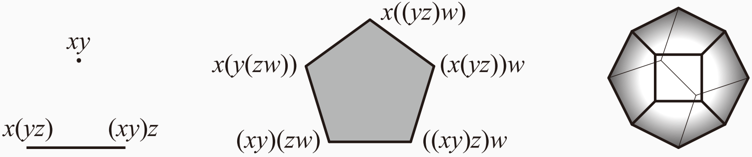


  This is unital and associative up to homotopy.
- Let $\mathcal{A}_n$ denote the (higher) category of topological groups and A_n-maps.
- The projective space construction extends to the functor $B_n\colon \mathcal{A}_n \to \mathbf{Top}_*$. This functor gives the one-to-one correspondence

$$\{\text{homotopy classes of } A_n\text{-maps } H \to G\}$$
$$\cong \{\text{homotopy classes of pointed maps } B_nH \to BG\}.$$

A_n-space

- Associativity of binary operation $x(yz) = (xy)z$ can also be loosened to "associativity up to higher homotopy".
- An A_n-space ($n = 2, 3, \dots, \infty$) is a space X equipped with a unital binary operation $m_2 \colon X \times X \to X$ and higher homotopies $\{m_i \colon \mathcal{K}_i \times X^i \to X\}_{i=2}^n$ parameterized over the associahedra $\mathcal{K}_i$ depicted below (Stasheff 1963).



- A_n-map between A_n-spaces is also defined (Stasheff 1970, Boardman–Vogt 1973, Iwase 1983).
- The i-th projective space B_iX of an A_n-space X is also defined for $i \leq n$.

Higher homotopy normality

Homotopy normal map (James)

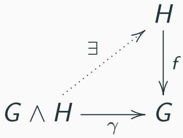
- In the rest of this talk, let H and G be topological groups of homotopy types of CW complexes.

Definition (James 1967)

- A homomorphism $f: H \to G$ is homotopy normal in the sense of James if the map

$$\gamma: G \wedge H \to G, \quad \gamma(g, x) = gxg^{-1}x^{-1}$$

lifts to H with respect to f up to homotopy.



Homotopy normal map (McCarty)

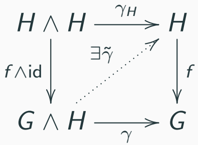
Definition (McCarty 1964)

- A homomorphism $f: H \to G$ is homotopy normal in the sense of McCarty if there exists a map $\tilde{\gamma}: G \wedge H \to H$ satisfying the following conditions:
  - $\tilde{\gamma}$ is a lift of the map

$$\gamma: G \wedge H \to G, \quad \gamma(g, x) = gxg^{-1}x^{-1}$$

    to H with respect to f up to homotopy,
  - $\tilde{\gamma} \circ (f \wedge \mathrm{id})$ is homotopic to the commutator map

$$H \wedge H \to H, \quad \gamma_H(h, x) = hxh^{-1}x^{-1}.$$



$N_1(1)$-map

Definition (T.)

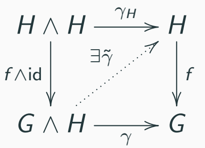
- A homomorphism $f\colon H \to G$ is an $N_1(1)$-map if there exists a map $\tilde{\gamma}\colon G \wedge H \to H$ satisfying the following conditions:
  - $\tilde{\gamma}$ is a lift of the map

    $$\gamma\colon G \wedge H \to G, \quad \gamma(g,x) = gxg^{-1}x^{-1}$$

    to H with respect to f up to homotopy,
  - $\tilde{\gamma} \circ f$ is homotopic to the commutator map

    $$\gamma_H\colon H \wedge H \to H, \quad \gamma_H(h,x) = hxh^{-1}x^{-1}.$$

  - the composite of these homotopies is homotopic to the stationary homotopy of the map $f \circ \gamma_H = \gamma \circ (f \wedge \mathrm{id})$.



$N_k(\ell)$-map

- $\mathcal{A}_\ell(H, G)$ denotes the space of A_ℓ-maps.
- $\mathrm{conj}_H \colon H \to \mathcal{A}_\ell(H, H)$ denotes the conjugation $\mathrm{conj}_H(h)(x) = hxh^{-1}$.

Definition (T.)

- A homomorphism $f \colon H \to G$ is an $N_k(\ell)$-map if an A_k-map $\rho \colon G \to \mathcal{A}_\ell(H, H)$ is given and the following conditions hold:
  - $\rho \circ f$ is homotopic to conj_H as an A_ℓ-map,
  - the map $* \to \mathcal{A}_\ell(H, G)$, $* \mapsto f$ is A_k-equivariant with respect to the action of G,
  - the composite of the previous A_k-equivariant map and f is homotopic to the trivial A_k-equivariant map $* \mapsto f$ with respect to the action of H (this is H-equivariant in the usual sense).
- If f is an $N_{k'}(\ell')$-map for $k' \geq k$ and $\ell' \geq \ell$, then f is an $N_k(\ell)$-map.

Fiberwise topological group

- A map $E \to B$ is said to be fiberwise topological group if each fiber is a topological group and the multiplication $E \times_B E \to E$, the unit $B \to E$ and the inversion $E \to E$ are continuous.
- The fiberwise classifying space $\mathscr{B}E$ and the n-th projective space $\mathscr{B}_n E$ are similarly defined.
- Fiberwise A_n-space is similarly defined.

Example

- The conjugation action of G on itself defines the fiberwise topological group $E_k G \times_G G \to B_k G$, where $E_k G$ is the pullback of EG by the inclusion $B_k G \to BG$.

Equivariant and fiberwise homotopy theory

- The Borel construction defines the correspondence

$$\text{a } G\text{-space } X \quad \mapsto \quad \text{a fiberwise space } EG \times_G X \to BG.$$

  This provides an "equivalence" between the equivariant homotopy theory and the fiberwise homotopy theory in an appropriate sense.
- The following main theorem is based on this equivalence.

Main theorem

Theorem (T.)

- Let $f\colon H \to G$ be a homomorphism and $F\colon B_kH \times_H H \to B_kG \times_G G$ denote the induced map of f. Then f is an $N_k(\ell)$-map if and only if there exists a fiberwise A_ℓ-space $E \to B_kG$ and fiberwise A_n-maps

$$\phi\colon E_kH \times_H H \to (B_kf)^*E \quad \text{and} \quad \psi\colon E \to E_kG \times_G G$$

  such that the following conditions hold:
  - ϕ is a fiberwise A_ℓ-equivalence,
  - the restriction of $\psi \circ \phi$ to the fiber over the basepoint is homotopic to f as an A_ℓ-map,
  - the composite $\psi \circ \phi$ is homotopic to F as a map covering $B_kf\colon B_kH \to B_kG$.
- The key to the proof is observing the classifying maps of fiberwise A_ℓ-spaces (Crabb–Sutherland 2000, T. 2012).

H-structure on homotopy quotient

H-space

- An H-space is nothing but an A_2-space (i.e. the space with continuous unital binary operation).

Theorem (T.)

- The homomorphism $H \to *$ is an $N_k(\ell)$-map if and only if H is a $C(k,\ell)$-space.

- $C(k,\ell)$-space is some higher homotopy commutativity (Kishimoto–Kono 2010). In particular, H is an $C(\infty,\infty)$-space iff BH is an H-space.
- This theorem immediately follows from the main theorem.
- This theorem is analogous to the fact that $H \to *$ is a crossed module iff H is commutative.

H-structure on homotopy quotient

- Due to the previous theorem, we cannot expect that the homotopy quotient of an $N_k(\ell)$-map naturally inherits any H-structure with higher homotopy associativity in cotrast to the result of Farjoun–Segev.

Theorem (T.)

- Let $f\colon H \to G$ be a homomorphism. Then the Borel construction $X = EH \times_H G$ is an H-space if f is an $N_k(k)$-map and $\operatorname{cat} X \leq k$.

Example

- (Even the rationalization of) the inclusion $\mathrm{SO}(2n) \to \mathrm{SO}(2n+1)$ is not an $N_1(1)$-map since $S^{2n} = \mathrm{SO}(2n+1)/\mathrm{SO}(2n)$ is not an H-space. But one can see that it is rationally homotopy normal in the sense of McCarty.

Higher homotopy normality of $\mathrm{SU}(m) \to \mathrm{SU}(n)$

Preceding results

- There have been many non-normality results.
- (James 1967)
  The inclusion $\mathrm{U}(m) \to \mathrm{U}(n)$ is not (2-locally) homotopy normal in the sense of James for $1 \leq m < n$. Similar results hold for $\mathrm{O}(m) \to \mathrm{O}(n)$ $(2 \leq m < n)$ and $\mathrm{Sp}(m) \to \mathrm{Sp}(n)$ for $1 \leq m < n$.
- Other results include: McCarty (1964), James (1971), Kachi (1982), Furukawa (1985), Furukawa (1987), Furukawa (1995), Kudou–Yagita (1998), Kudou–Yagita (2003), Kono–Nishimura (2003), Nishimura (2006), Kishimoto–T. (2018).
- These results suggest that $H \to G$ tends to fail to be p-locally homotopy normal for small prime p.

Higher homotopy commutativity and normality

- In contrast to the fact that any subgroup of an abelian group is normal, homotopy commutativity does not imply homotopy normality in general (e.g. the rationalization of $\mathrm{SO}(2n) \to \mathrm{SO}(2n+1)$).

Theorem (T.)

- Let $f\colon H \to G$ be a homomorphism between semisimple compact connected Lie groups. Suppose that the mod p cohomologies are

$$H^*(H;\mathbb{F}_p) = \Lambda(x_1,\ldots,x_m), \quad H^*(G;\mathbb{F}_p) = \Lambda(y_1,\ldots,y_n)$$

with $m \le n$ and $f^*(y_i) = x_i$ for $i = 1,\ldots,m$. If $p \ge kn + \ell m$, then f is p-locally an $N_k(\ell)$-map.

- This result says that $H \to G$ tends to be a p-local $N_k(\ell)$-map trivially for large prime p.

Higher homotopy normality of $\mathrm{SU}(m) \to \mathrm{SU}(n)$

Theorem (T.)

- If $\max\{kn-2, (k-1)n+2\} < p \leq kn+2(\ell-1)$ for some $n \geq 3$ and $k, \ell \geq 1$, then the inclusion $\mathrm{SU}(2) \to \mathrm{SU}(n)$ is not a p-local $N_k(\ell)$-map.
- If $\max\{kn-m, (k-1)n+2\} < p \leq kn+(\ell-2)m$ for some $2 \leq m < n$ and $k, \ell \geq 1$, then the inclusion $\mathrm{SU}(m) \to \mathrm{SU}(n)$ is not a p-local $N_k(\ell)$-map.

- This result is not very sharp. For example, the normality is undetermined for $kn+(\ell-2)m < p < kn+\ell m$.
- A similar result is obtained for $\mathrm{SO}(2m+1) \to \mathrm{SO}(2n+1)$.

Proof of the theorem

- To illustrate the proof with the easiest case, we see that the inclusion $\mathrm{SU}(2) \to \mathrm{SU}(3)$ is not 3-locally an $N_1(1)$-map.
- Suppose that it is 3-locally an $N_1(1)$-map. Then there exist a fiberwise pointed space $\mathcal{E} \to B_1\,\mathrm{SU}(3) = \Sigma\,\mathrm{SU}(3)$ and fiberwise pointed maps

$$\Phi\colon E_1\,\mathrm{SU}(2) \times_{\mathrm{SU}(2)} \Sigma\,\mathrm{SU}(2) \to \mathcal{E}|_{\Sigma\,\mathrm{SU}(2)},$$
$$\Psi\colon \mathcal{E} \to E_1\,\mathrm{SU}(3) \times_{\mathrm{SU}(3)} B\,\mathrm{SU}(3)$$

such that Φ is a homotopy equivalence and $\Psi \circ \Phi$ is fiberwise pointed homotopic to the inclusion (functoriality of fiberwise projective spaces).

Proof of the theorem (continued)

- We have the mod p cohomologies

$$H^*(E_1\,\mathrm{SU}(3)\times_{\mathrm{SU}(3)} B\,\mathrm{SU}(3)) = (\mathbb{F}_p\{c_2^B, c_3^B\}\oplus S)\otimes \mathbb{F}_p[c_2^F, c_3^F],$$
$$H^*(\mathcal{E}) = (\mathbb{F}_p\{c_2^B, c_3^B\}\oplus S)\otimes \mathbb{F}_p\{c_2^F\}$$

for some 1-dimensional submodule S.

- We find $\Psi^*(c_3^F) = c_3^B$ since Ψ is fiberwise pointed.
- Applying the Steenrod operation $\mathcal{P}^1$, we have

$$\Psi^*(\mathcal{P}^1 c_3^F) = \Psi^*(\pm c_2^F c_3^F) = \pm c_2^F c_3^B,$$
$$\mathcal{P}^1\Psi^*(c_3^F) = \mathcal{P}^1 c_3^B = \pm c_2^B c_3^B.$$

These computations contradict each other, completing the proof.

Summary

- $N_k(\ell)$-map is a higher homotopical analogue of crossed module.
- $N_1(1)$-map $\Rightarrow$ McCarty's homotopy normal map $\Rightarrow$ James' homotopy normal map.
- An $N_k(\ell)$-map is characterized by fiberwise A_ℓ-maps over the k-th projective spaces.
- The homotopy quotient $EH \times_H G$ of an $N_k(k)$-map is an H-space if $\operatorname{cat} EH \times_H G \leq k$ holds.
- Fiberwise pointed maps between fiberwise projective spaces is applied to detect obstructions to being $N_k(\ell)$-maps.

Thank you!