Finite propagation operators and Hilbert bundles with end

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この講演は加藤毅氏（京都大）と岸本大祐氏（京都大）との共同研究による以下の 3 編の論文に基づく.

- Homotopy type of the space of finite propagation unitary operators on $\mathbb{Z}$ (arXiv:2007.06787),
- Homotopy type of the unitary group of the uniform Roe algebra on $\mathbb{Z}^n$ (arXiv:2102.00606, accepted by J. Topol. Anal.),
- Hilbert bundles with ends (arXiv:2105.02981, accepted by J. Topol. Anal.).

私（蔦谷）の紹介

- 主に空間の A_∞-構造などの高次ホモトピー構造に興味があります.
- 他にも代数トポロジーを使う問題を中心にいろいろ研究しています.

Contents

Hilbert bundle with end

Notation

- Let (I, d) be a countable metric space (with discrete topology).

$$\ell^2(I) = \{(v_i)_{i\in I} \mid v_i \in \mathbb{C}, \sum_{i\in I} |v_i|^2 < \infty\}$$

- Let $B(\mathcal{H})$ denote the space of bounded operators on a Hilbert space $\mathcal{H}$.
- For $T \in B(\ell^2(I))$, the matrix representation $T = (T_{ij})_{i,j\in I}$ with respect to the standard orthonormal basis.

Finite propagation operator

Definition

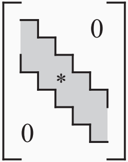
- For $T \in B(\ell^2(I))$, define prop $T \in \mathbb{Z}_{\geq 0}$ by

$$\operatorname{prop} T = \sup\{d(i,j) \mid T_{ij} \neq 0\}.$$

T is a finite propagation operator if prop $T < \infty$.

Example

- When I is bounded, all operators in $B(\ell^2(I))$ are of finite propagation.
- When $I = \mathbb{Z}$, the matrix of a finite propagation operator looks like:



Space of finite propagation operators

- $\mathcal{U}_L(I) \subset B(\ell^2(I))$
  the finite propagation unitary operators of prop $\leq L$. $\mathcal{U}_L(I)$ is equipped with the *norm topology*.
- Let
  $$\mathcal{U}_{\mathrm{fp}}(I) = \bigcup_{L \geq 0} \mathcal{U}_L(I)$$
  with *the inductive limit topology of the norm topology*. As a set, $\mathcal{U}_{\mathrm{fp}}(I)$ consists of the finite propagation operators on $\ell^2(I)$.

Hilbert space with end

Let $\mathcal{H}$ be a Hilbert space.

Definition (KKT)

- Two isometries $\phi, \phi' \colon \mathcal{H} \to \ell^2(I)$ are said to be equivalent if the composite $\phi' \circ \phi^{-1} \colon \ell^2(I) \to \ell^2(I)$ has finite propagation.
- An end of $\mathcal{H}$ modeled on I is an equivalence class of isomteries $\mathcal{H} \to \ell^2(I)$. We say an operator in the equivalence class is of finite propagation.

Example

- When I is bounded, then the end of a Hilbert space $\mathcal{H}$ is unique.
- When I is not bounded, there are many non-equivalent ends.

Hilbert bundle with end

Hilbert bundles in our work are as follows.

Definition

- Let $\pi\colon E \to X$ be a continuous map and each fiber $\pi^{-1}(x)$ be equipped with a structure of Hilbert space.
  We say E is a Hilbert bundle if it admits a local trivialization $\pi^{-1}(U) \cong U \times \mathcal{H}$ around each point $x \in X$.

Definition (KKT)

- An end of a Hilbert bundle $\pi\colon E \to X$ is the data consisting of local trivializations $\{\pi^{-1}(U_\lambda) \cong U_\lambda \times \ell^2(I)\}_\lambda$ over an open covering $\{U_\lambda\}_\lambda$ of X satisfying the following conditions
  - the transition functions have values in $\mathcal{U}_{\mathrm{fp}}(I)$,
  - the transition functions $U_\lambda \cap U_{\lambda'} \to \mathcal{U}_{\mathrm{fp}}(I)$ are continuous (with respect to the inductive limit of norm topology).

Remark on topology of $\mathcal{U}_{\mathrm{fp}}(I)$

- We assume the transition functions $U_\lambda \cap U_{\lambda'} \to \mathcal{U}_{\mathrm{fp}}(I)$ are continuous *with respect to the inductive limit of norm topology*.
- This assumption could be too restrictive. But we still have some interesting examples.
- By Kuiper's theorem, which states that $\mathrm{U}_1(B(\ell^2(I)))$ is contractible, any Hilbert bundle is trivial. Assigning an end makes a Hilbert bundle non-trivial.
- Choosing an end of a Hilbert bundle is equivalent to taking a reduction of the structure group from $\mathrm{U}_1(B(\ell^2(I)))$ to $\mathcal{U}_{\mathrm{fp}}(I)$.

Classification of Hilbert bundles with ends

Definition

- Two Hilbert bundles $E, E' \to X$ with end modeled on I are isomorphic if there exists an isomorphism of Hilbert bundles $E \to E'$ which takes continuous values in $\mathcal{U}_{\mathrm{fp}}(I)$ through local trivializations.

Proposition

- The following map is bijective:

$$[X, B\mathcal{U}_{\mathrm{fp}}(I)] \to \{\text{isom. classes of Hilb. bdl.s with ends modeled on } I\}$$
$$[f] \mapsto f^*\tilde{E},$$

where $[X, X']$ denotes the homotopy classes of maps $X \to X'$ and $\tilde{E}$ is the universal bundle over the classifying space $B\mathcal{U}_{\mathrm{fp}}(I)$.

Problems

- Give natural examples of Hilbert bundles with ends.
- Determine the cohomology groups of $B\mathcal{U}_{\mathrm{fp}}(I)$.
  $\rightarrow$ "characteristic classes"

Examples

Uniform vector bundle

Let X be a metric space.

Definition

- A uniform vector bundle (of finite rank) is the pair $(E, \{\phi_\lambda\}_\lambda)$ as follows:
  - $\pi\colon E \to X$ is a vector bundle of rank r,
  - $\{\phi_\lambda\colon \pi^{-1}(U_\lambda) \to U_\lambda \times \mathbb{C}^r\}_\lambda$ is local trivializations over an open covering $\{U_\lambda\}_\lambda$ such that the transition functions $U_\lambda \cap U_{\lambda'} \to \mathrm{U}_r(\mathbb{C})$ are uniformly equicontinuous.

Remark

- A sequence of functions $\{f_\mu\}_\mu$ is uniformly equicontinuous if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any μ and x, y with $d(x, y) < \delta$, the estimate $|f_\mu(x) - f_\mu(y)| < \epsilon$ holds.

Examples of uniform vector bundles

Let X be a compact metric space.

Example

- Suppose $p\colon Y \to X$ is a covering space and a local isometry. Then the pullback p^*E of a vector bundle $E \to X$ is uniform.
- Let $\mathcal{S}$ be a finite set of isomorphism classes of vector bundles of rank r over X and $\{E_i\}_{i \in I}$ be a sequence of vector bundles with $E_i \in \mathcal{S}$. Then
$$\coprod_{i \in I} E_i \to I \times X$$
is a uniform vector bundle. If $\mathcal{S}$ is not finite, then the resulting vector bundle is not uniform in general.

Example 1: pushforward

Let $p \colon Y \to X$ be a covering space and a local isometry to a compact connected+ metric space X.

Proposition (KKT)

- For a uniform vector bundle E of rank r over Y, let $p_* E$ denote the fiberwise completion of

$$\coprod_{x \in X} \bigoplus_{y \in p^{-1}(x)} E_y \to X$$

  equipped with an appropriate topology is a Hilbert bundle with end modelled on $p^{-1}(x_0) \times \{1, 2, \ldots, r\}$ for some $x_0 \in X$.

$p_* E$ is called the pushforward.

Example 2: fiberwise Fourier transform

Example

- By the identification on $[0,1] \times [0,1] \times \mathbb{C}$ generated by
$$(0, x, z) \sim (1, x, x^n z) \quad \text{and} \quad (w, 0, z) \sim (w, 1, z),$$
we obtain a line bundle $L_n \to Y = S^1 \times S^1$.
- Let $p\colon Y \to S^1$ be the first projection. Then applying the Fourier transform on each fiber of the bundle
$$E_n = \coprod_{w \in S^1} L^2(p^{-1}(w)) \to S^1,$$
we obtain a Hilbert bundle with end modeled on $\mathbb{Z}$.
- This can be obtained by the identification on $[0,1] \times \ell^2(\mathbb{Z})$ by
$$(w, (v_i)_i) \sim (w, (v_{i+n})_i).$$

Homotopy type of $B\mathcal{U}_{\mathrm{fp}}(\mathbb{Z})$

Uniform Roe algebra

Let $\mathrm{U}_1(A)$ denote the group of unitary elements in A.
To determine the homotopy type of $B\mathcal{U}_{\mathrm{fp}}(\mathbb{Z})$, it is sufficient to determine the homotopy type of $B\,\mathrm{U}_1(C_u^*(|\mathbb{Z}|))$ by the following theorem.

Theorem (KKT)

- The inclusion $\mathcal{U}_{\mathrm{fp}}(\mathbb{Z}) \to \mathrm{U}_1(C_u^*(|\mathbb{Z}|))$ is a homotopy equivalence.

Let I be a countable metric space (with discrete topology).

Definition

- The uniform Roe algebra $C_u^*(I)$ is the norm closure of finite propagation operators in $B(\ell^2(I))$.

Suppose $I = |\Gamma|$ is the underlying metric space of a finitely generated discrete group Γ.

- Then $C_u^*(|\Gamma|) \cong \ell^\infty(\Gamma) \rtimes \Gamma$ as C^*-algebras.

Homotopy groups of $\mathrm{U}_1(C_u^*(|\mathbb{Z}|))$

Theorem (KKT)

- Let $\ell^\infty(\mathbb{Z},\mathbb{Z})$ denote the $\mathbb{Z}$-valued bounded sequences over $\mathbb{Z}$ and $\ell^\infty(\mathbb{Z},\mathbb{Z})_S = \ell^\infty(\mathbb{Z},\mathbb{Z})/\{a - Sa \mid a \in \ell^\infty(\mathbb{Z},\mathbb{Z})\}$ where S is the shift. Then

$$\pi_n(\mathrm{U}_1(C_u^*(|\mathbb{Z}|))) \cong K_{n+1}(C_u^*(|\mathbb{Z}|)) \cong \begin{cases} \ell^\infty(\mathbb{Z},\mathbb{Z})_S & n \text{ is odd,} \\ \mathbb{Z} & n \text{ is even.} \end{cases}$$

- The inclusion induces the homotopy equivalence $B\,\mathrm{U}_1(C_u^*(|\mathbb{Z}|)) \simeq B\,\mathrm{U}_1(C_u^*(|\mathbb{Z}| \times \{1,\dots,r\}))$.

Proof.

- The stability follows from the stability for $A \rtimes \mathbb{Z}$ [Rieffel, 1987].
- Then it is sufficient to compute $K_n(C_u^*(|\mathbb{Z}|))$.
- The first statement follows from the Pimsner–Voiculescu exact sequence

$$\begin{tikzcd}
K_0(\ell^\infty(\mathbb{Z})) \arrow[r, "1-S"] & K_0(\ell^\infty(\mathbb{Z})) \arrow[r] & K_0(C_u^*(|\mathbb{Z}|)) \arrow[d] \\
K_1(C_u^*(|\mathbb{Z}|)) \arrow[u] & K_1(\ell^\infty(\mathbb{Z})) \arrow[l] & K_1(\ell^\infty(\mathbb{Z})) \arrow[l, "1-S"']
\end{tikzcd}$$

and

$$K_n(\ell^\infty(\mathbb{Z})) \cong \begin{cases} \ell^\infty(\mathbb{Z}, \mathbb{Z}) & n \text{ is even}, \\ 0 & n \text{ is odd}. \end{cases}$$

- The second statement is just the stability.

$\square$

The abelian group $\ell^\infty(\mathbb{Z}, \mathbb{Z})_S$

Proposition (KKT)

- The abelian group $\ell^\infty(\mathbb{Z}, \mathbb{Z})_S$ is a $\mathbb{Q}$-vector space.

The proof is straightforward.

Example

- If $(a_i)_i$ has finite support, then $[(a_i)_i] = 0$ in $\ell^\infty(\mathbb{Z}, \mathbb{Z})_S$.
  $\to \ell^\infty(\mathbb{Z}, \mathbb{Z})_S$ captures "asymptotic behavior".
- An embedding $\iota\colon \mathbb{Q} \to \ell^\infty(\mathbb{Z}, \mathbb{Z})_S$ is given by

$$\iota(1) = [\ldots, 1, 1, 1, \ldots].$$

Then we have $\iota(m) = [\ldots, m, m, m, \ldots]$ $(m \in \mathbb{Z})$,

$$\iota\left(\frac{1}{2}\right) = [\ldots, 1, 0, 1, 0, \ldots], \quad \iota\left(\frac{1}{3}\right) = [\ldots, 1, 0, 0, 1, 0, 0, \ldots].$$

Homotopy type of $\mathrm{U}_1(C_u^*(|\mathbb{Z}|))$

Roe algebra:

$$C^*(|\Gamma|) \cong \ell^\infty(\Gamma, \mathfrak{K}) \rtimes \Gamma,$$

where $\mathfrak{K}$ denotes the compact operators.

Theorem (KKT)

- The following (weak) homotopy equivalence holds:

$$B\,\mathrm{U}_1(C_u^*(|\mathbb{Z}|)) \simeq \mathrm{U}_\infty(\mathbb{C}) \times \prod_{n=1}^{\infty} K(\ell^\infty(\mathbb{Z}, \mathbb{Z})_S, 2n),$$

where $K(V, n)$ denotes the Eilenberg–MacLane space of type (V, n).

Proof.

- We can see
$$K_n(C^*(|\mathbb{Z}|)) \cong \begin{cases} 0 & i \text{ is even,} \\ \mathbb{Z} & i \text{ is odd.} \end{cases}$$
and $K_1(C_u^*(|\mathbb{Z}|)) \to K_1(C^*(|\mathbb{Z}|))$ is surjective.
- This implies the homotopy fibration
$$F \to B\,\mathrm{U}_\infty(C_u^*(|\mathbb{Z}|)) \to B\,\mathrm{U}_\infty(C^*(|\mathbb{Z}|))$$
admits a homotopy section. Thus
$$B\,\mathrm{U}_1(C_u^*(|\mathbb{Z}|)) \simeq B\,\mathrm{U}_\infty(C_u^*(|\mathbb{Z}|)) \simeq B\,\mathrm{U}_\infty(C^*(|\mathbb{Z}|)) \times F.$$
- Note that $B\,\mathrm{U}_\infty(C^*(|\mathbb{Z}|)) \simeq \mathrm{U}_\infty(\mathbb{C})$ and the homotopy groups
$$\pi_n(F) \cong \begin{cases} \ell^\infty(\mathbb{Z}, \mathbb{Z})_S & n \text{ is even,} \\ 0 & n \text{ is odd,} \end{cases}$$
are $\mathbb{Q}$-vector spaces. Then the theorem follows.

Characteristic classes

Cohomology groups

Let $V = \ell^\infty(\mathbb{Z}, \mathbb{Z})_S$ and

$$V^\vee = \mathrm{Hom}(V, \mathbb{Q}) = \{\text{shift invariant homomorphisms } \ell^\infty(\mathbb{Z}, \mathbb{Z}) \to \mathbb{Q}\}.$$

Take a basis $\mathcal{B}$ of $V^\vee$.

Theorem (KKT)

- $H^*(B\mathcal{U}_{\mathrm{fp}}(\mathbb{Z}); \mathbb{Q}) \cong \mathbb{Q}[\alpha_n(b) \mid b \in \mathcal{B}, n \geq 1] \otimes \Lambda_{\mathbb{Q}}(\beta_n \mid n \geq 1)$, where $\alpha_n(b) \in H^{2n}$ and $\beta_n \in H^{2n-1}$.

This immediately follows from the previous theorem.

Definition (KKT)

- For a Hilbert bundle with end E over X classified by $f : X \to B\mathcal{U}_{\mathrm{fp}}(\mathbb{Z})$, let

$$\alpha_n(E; b) = f^*\alpha_n(b) \quad \text{and} \quad \beta_n(E) = f^*\beta_n.$$

Example 1: pushforward along a trivial covering

- Let $\{E_i\}_{i\in\mathbb{Z}}$ be vector bundles of rank n over S^{2n} with the Chern number $a_i = c_n(E_i)[S^{2n}]$ such that

$$\sup\{|a_i| \mid i \in \mathbb{Z}\} < \infty.$$

- Consider the pushforward p_*E of

$$E = \coprod_{i\in I} E_i \to I \times S^{2n}$$

along the trivial covering $I \times S^{2n} \to S^{2n}$.

Proposition

- The equality $\alpha_n(p_*E; b)[S^{2n}] = b((a_i)_i)$ holds for any $b \in \mathcal{B}$.

Example 1: pushforward along a non-trivial covering

- Let L be the trivial bundle $\mathbb{R} \times \mathbb{C}$ over $\mathbb{R}$.
- The pushforward $p_* L$ along the universal covering $\mathbb{R} \to S^1$ is a Hilbert bundle with end modeled on $\mathbb{Z}$.

Proposition

- The equality $\beta_1(p_* L)[S^1] = 1$ holds.

Proof.
Let $\{e_i\}_{i \in \mathbb{Z}}$ be the standard basis of $\ell^2(\mathbb{Z})$. One can observe that e_i is mapped to e_{i+1} by the circular parallel transport. $\square$

- The non-triviality of β_1 can be extended to any non-trivial $\mathbb{Z}$-covering since $\mathbb{Z}$-coverings are classified by $H^1(X; \mathbb{Z})$.

Example 2: fiberwise Fourier transform

- We constructed the Hilbert bundle with end

$$E_n = \coprod_{w \in S^1} L^2(p^{-1}(x)) \to S^1,$$

which is also obtained by the identification on $[0,1] \times \ell^2(\mathbb{Z})$ generated by

$$(w, (v_i)_i) \sim (w, (v_{i+n})_i).$$

Proposition

- The equality $\beta_1(L_n)[S^1] = n$ holds.

Further problems

Other choices of I

- In general, it seems difficult to compute the cohomology groups of $B\mathcal{U}_{\mathrm{fp}}(I)$.

Theorem (KKT)

- The following (weak) homotopy equivalence holds:

$$
\begin{aligned}
&B\,\mathrm{U}_1(C_u^*(|\mathbb{Z}^2|)) \\
&\simeq \mathbb{Z} \times B\,\mathrm{U}_\infty(\mathbb{C}) \times V_0 \times \prod_{n=1}^{\infty}(K(V_0, 2n) \times K(V_1, 2n-1))
\end{aligned}
$$

for some (huge) $\mathbb{Q}$-vector spaces V_0 and V_1.

- In fact, the inclusion $\mathrm{U}_1(C_u^*(|\mathbb{Z}^n|)) \to C^*(|\mathbb{Z}^n|)$ admits a homotopy section.
- What about for other infinite I...?

Operations and applications

- Tensor product of Hilbert bundles with ends modeled on I_1 and I_2 coincides with the operation

$$C_u^*(I_1) \otimes C_u^*(I_2) \to C_u^*(I_1 \times I_2).$$

  How does it work on cohomologies?
- What does the geometric meaning of the characteristic class $\beta_n \in H^{2n-1}(B\mathcal{U}_{\mathrm{fp}}(\mathbb{Z}); \mathbb{Q})$ $(n \geq 2)$?
- Are there any good applications of our bundles and their characteristic classes?

Thank you!