Homotopy types of spaces of finite propagation unitary operators on $\mathbb{Z}$

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1. Introduction
    - Finite propagation operator
    - Uniform Roe algebra
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Finite propagation operator

$$H = \ell^2(\mathbb{Z}, \mathbb{C}) = \{(v_i)_{i \in \mathbb{Z}} \mid v_i \in \mathbb{C}, \sum_{i \in \mathbb{Z}} |v_i|^2 < \infty\}$$

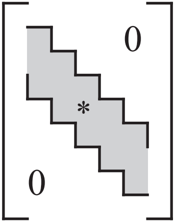
$$H_+ = \ell^2(\mathbb{Z}_{\geq 0}, \mathbb{C}), \quad H_- = \ell^2(\mathbb{Z}_{<0}, \mathbb{C}), \quad H = H_+ \oplus H_-.$$

$B(H)$ the space of bounded operators on H.
$T \in B(H) \Longrightarrow T = (T_{ij})_{ij}$
with respect to the standard orthonormal basis.

Definition

prop $T = \sup\{|i-j| \mid T_{ij} \neq 0\}$.

T is a finite propagation operator if prop $T < \infty$.



Uniform Roe algebra

Definition

$\mathbb{C}_u(\mathbb{Z}) = \{T \in B(H) \mid \text{prop } T < \infty\}$.
$C_u^*(\mathbb{Z})$ the norm closure of $\mathbb{C}_u(\mathbb{Z})$ in $B(H)$.
$C_u^*(\mathbb{Z})$ is called the unifrom Roe algebra on $\mathbb{Z}$.

- The uniform Roe algebras $C_u^*(X)$ are defined for other metric spaces X.
- The uniform Roe algebras are introduced by John Roe to study a generalization of the Atiyah–Singer index theorem.
- A uniform Roe algebra is a C^*-algebra.

Our problem 1

A $*$-algebra.

$$\mathrm{U}(A) = \{U \in A \mid UU^* = U^*U = \mathrm{id}\}.$$

$U \in \mathrm{U}(A)$ is called a unitary element in A.

Problem

Determine the homotopy type of the space $\mathrm{U}(\mathbb{C}_u(\mathbb{Z}))$ and $\mathrm{U}(C_u^*(\mathbb{Z}))$. For example, describe it by more familiar spaces.

- The space of invertible elements $\mathrm{GL}(A)$ has the same homotopy type as $\mathrm{U}(A)$ if A is a C^*-algebra.
- Once we solve this problem, we can compute various homotopy invariants such as homotopy groups and (co)homology groups of $\mathrm{U}(A)$.

Remark on K-theory

The K-groups of a C^*-algebra A is characterized as follows (i could be any non-negative integer by the Bott periodicity):

$$K_0(A) = \varinjlim_{n\to\infty} \pi_{2i+1}(\mathrm{U}_n(A)), \quad K_1(A) = \varinjlim_{n\to\infty} \pi_{2i}(\mathrm{U}_n(A)),$$

where $\mathrm{U}_n(A)$ denotes the n-th unitary group with coefficients in A and the inductive limits are taken along the inclusions

$$\mathrm{U}(A) = \mathrm{U}_1(A) \subset \mathrm{U}_2(A) \subset \mathrm{U}_3(A) \subset \cdots.$$

Then the K-groups can be regarded as stabilized homotopy invariants of A in some sense.

Remark

In general, the unstable (i.e. usual) homotopy group $\pi_i(\mathsf{U}(A))$ and its stabilization $\lim_{n\to\infty} \pi_i(\mathsf{U}_n(A))$ are different.
If $A = \ell^\infty(\mathbb{Z}, \mathbb{C})$ the space of bounded sequences, then

$$\begin{aligned}
&\pi_i(\mathsf{U}_n(\ell^\infty(\mathbb{Z},\mathbb{C}))) \\
&= \begin{cases} 0 & \text{for } 0 \le i < 2n \text{ even}, \\ \ell^\infty(\mathbb{Z},\mathbb{Z}) & \text{for } 1 \le i < 2n \text{ odd}, \\ \prod_{j\in\mathbb{Z}} \pi_i(\mathsf{U}(n)) & \text{for } i \ge 2n, \end{cases} \\
&\lim_{n\to\infty} \pi_i(\mathsf{U}_n(\ell^\infty(\mathbb{Z},\mathbb{C}))) \\
&= \begin{cases} 0 & \text{for } i \ge 0 \text{ even}, \\ \ell^\infty(\mathbb{Z},\mathbb{Z}) & \text{for } i \ge 1 \text{ odd}. \end{cases}
\end{aligned}$$

There are homotopically stable algebras as well.

Example

By Kuiper's theorem, $\mathrm{U}_n(B(H))$ is contractible for any n. Thus, $B(H)$ is homotopically stable.

- More generally, Schröder determined the homotopical stability of von Neuman algebras (1984).

Approximation by finite propagation operators

Consider the space of unitary operators of propagation $\leq L$.

$$\mathcal{U}_L = \{U \in B(H) \mid UU^* = U^*U = \mathrm{id}, \mathrm{prop}\, T \leq L\}$$

We have

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots, \quad \bigcup_{L \geq 0} \mathcal{U}_L = \mathrm{U}(\mathbb{C}_u(\mathbb{Z})).$$

Definition

$\mathcal{U} = \lim_{L \to \infty} \mathcal{U}_L$ inductive limit as topological spaces.

Remark

The canonical map $\mathcal{U} \to \mathrm{U}(\mathbb{C}_u(\mathbb{Z}))$ is continuous and bijective but not a homeomorphism.

Main results

$\ell^\infty(\mathbb{Z},\mathbb{Z})$ $\mathbb{Z}$-valued bounded sequences.
$\ell^\infty(\mathbb{Z},\mathbb{Z})_{\mathrm{shift}} = \ell^\infty(\mathbb{Z},\mathbb{Z})/\{a - Sa \mid a \in \ell^\infty(\mathbb{Z},\mathbb{Z})\}$ the coinvariant by shift $S(a_j)_j = (a_{j+1})_j$.

Theorem A (Kato–Kishimoto–T.)

$$\pi_i(\mathcal{U}) = \begin{cases} \mathbb{Z} & \text{for } i \geq 0 \text{ even,} \\ \ell^\infty(\mathbb{Z},\mathbb{Z})_{\mathrm{shift}} & \text{for } i \geq 1 \text{ odd,} \end{cases}$$

Moreover, the abelian group $\ell^\infty(\mathbb{Z},\mathbb{Z})_{\mathrm{shift}}$ is divisible and torsion-free, or equivalently, is a $\mathbb{Q}$-vector space.

Also, we obtained an answer to our problem.

Theorem B (Kato–Kishimoto–T.)

The space of finite propagtion unitary operators $\mathcal{U}$ has the (weak) homotopy type of the product

$$\mathbb{Z} \times B\,\mathrm{U}(\infty) \times \prod_{i \geq 1} K(\ell^\infty(\mathbb{Z}, \mathbb{Z})_{\mathrm{shift}}, 2i-1)$$

where

- $B\,\mathrm{U}(\infty)$ is the classifying space of the infinite unitary group $\lim_{n\to\infty} \mathrm{U}(n)$,
- $K(\Gamma, m)$ is the Eilenberg–MacLane space characterized by

$$\pi_i(K(\Gamma, m)) \cong \begin{cases} \Gamma & \text{for } i = m, \\ 0 & \text{for } i \neq m. \end{cases}$$

Comment

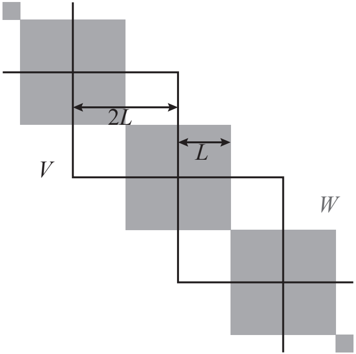
- Recently, we have determined the homotopy type of $\mathrm{U}(C_u^*(\mathbb{Z}))$: in fact, the inclusion $\mathcal{U} \to \mathrm{U}(C_u^*(\mathbb{Z}))$ is a homotopy equivalence.
- This result follows from combination of some classical results on unstable homotopy types of C^*-algebras.

2. Proof
   - Gross–Nesme–Vogts–Werner decomposition
   - Proof of Theorem A
   - Outline of proof of Theorem B

Gross–Nesme–Vogts–Werner decomposition

Lemma（Gross–Nesme–Vogts–Werner, 2012）

If $U \in \mathrm{U}(\mathbb{C}_u(\mathbb{Z}))$ belongs to the identity component and prop $U \leq L$, then there exist block diagonal unitary operators V, W with matrix forms as follows such that $U = VW$.



- $B_0(2L)$ the space of block diagonal operators like V,
- $B_{-L}(2L)$ the space of block diagonal operators like W,
- $B_0(L) = B_0(2L) \cap B_{-L}(2L)$.

Consider the quotient space

$$\mathcal{W}_L := [\mathrm{U}(B_0(2L)) \times \mathrm{U}(B_{-L}(2L))] / \mathrm{U}(B_0(L)) \subset \mathrm{U}(\mathbb{C}_u(\mathbb{Z}))$$

and the associated fiber bundle

$$\mathrm{U}(B_0(L)) \to \mathrm{U}(B_0(2L)) \times \mathrm{U}(B_{-L}(2L)) \to \mathcal{W}_L.$$

- By the previous lemma, it is not difficult to see that the identity component of $\mathcal{U} = \lim_{L\to\infty} \mathcal{U}_L$ is homeomorphic to the inductive limit

$$\lim_{L\to\infty} \mathcal{W}_L.$$

Thus we obtain the isomorphsim

$$\pi_i(\mathcal{U}) \cong \pi_i(\lim_{L\to\infty} \mathcal{W}_L) \cong \lim_{L\to\infty} \pi_i(\mathcal{W}_L)$$

for $i \geq 1$. The π_0 is known by Gross–Nesme–Vogts–Werner: $\pi_0(\mathcal{U}) \cong \mathbb{Z}$.

- For any fiber bundle $E \to B$ with fiber F, we have the homotopy long exact sequence

$$\cdots \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \cdots .$$

Combining these facts, we can compute the homotopy groups $\pi_i(\mathcal{U})$.

Proof of Theorem A

We obtain the following exact sequence for $1 \leq i \leq L$ (called stable range) by the previous fiber bundle:

$$0 \to \pi_{2i}(\mathcal{W}_L) \to \pi_{2i-1}(\mathrm{U}(B_0(L))) \\ \to \pi_{2i-1}(\mathrm{U}(B_0(2L)) \times \mathrm{U}(B_{-L}(2L))) \to \pi_{2i-1}(\mathcal{W}_L) \to 0.$$

Since we know the homotopy groups of $\mathrm{U}(B_k(L)) \cong \mathrm{U}_L(\ell^\infty(\mathbb{Z}, \mathbb{C}))$, we can compute as follows:

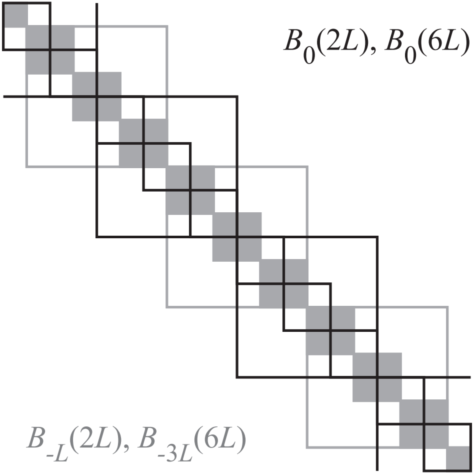
Lemma

$$\pi_i(\mathcal{W}_L) = \begin{cases} \mathbb{Z} & \text{for } 2 \leq i \leq 2L \text{ even,} \\ \ell^\infty(\mathbb{Z}, \mathbb{Z})_{\mathrm{shift}} & \text{for } 1 \leq i \leq 2L-1 \text{ odd.} \end{cases}$$

To observe the inductive limit $\lim_{L\to\infty} \mathcal{W}_L$, we consider the following inclusion between fiber bundles:

$$\begin{array}{ccccc}
\mathrm{U}(B_0(L)) & \longrightarrow & \mathrm{U}(B_0(2L)) \times \mathrm{U}(B_{-L}(2L)) & \longrightarrow & \mathcal{W}_L \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{U}(B_0(3L)) & \longrightarrow & \mathrm{U}(B_0(6L)) \times \mathrm{U}(B_{-3L}(6L)) & \longrightarrow & \mathcal{W}_{3L}
\end{array}$$

This is depicted as follows:



This inclusion induces the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \pi_{2i}(\mathcal{W}_L) & \to & \pi_{2i-1}(\mathsf{U}(B_0(L))) & \to & \pi_{2i-1}(\mathsf{U}(B_0(2L)) \times \mathsf{U}(B_{-L}(2L))) & \to & \pi_{2i-1}(\mathcal{W}_L) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \pi_{2i}(\mathcal{W}_{3L}) & \to & \pi_{2i-1}(\mathsf{U}(B_0(3L))) & \to & \pi_{2i-1}(\mathsf{U}(B_0(6L)) \times \mathsf{U}(B_{-3L}(6L))) & \to & \pi_{2i-1}(\mathcal{W}_{3L}) & \to & 0
\end{array}$$

Computing homomorphisms in this diagram, we obtain:

Lemma

$$\pi_i(\lim_{n\to\infty} \mathcal{W}_{3^n L}) = \begin{cases} \mathbb{Z} & \text{for } i \geq 2 \text{ even,} \\ \ell^\infty(\mathbb{Z}, \mathbb{Z})_{\mathrm{shift}} & \text{for } i \geq 1 \text{ odd.} \end{cases}$$

We can see by a purely algebraic argument that $\ell^\infty(\mathbb{Z}, \mathbb{Z})_{\mathrm{shift}}$ is a $\mathbb{Q}$-vector space.

This completes the proof of Theorem A.

Outline of proof of Theorem B

The proof of Theorem B proceeds as follows.

- We can construct some nice principal fibration

$$F \to \mathcal{U} \to \mathbb{Z} \times B\,\mathrm{U}(\infty).$$

  We proved that this fibration admits a section by some topological argument using cohomology groups.
- It is well-known that if a principal fibration admits a section, then it is trivial. Then we have a homotopy equivalence

$$\mathcal{U} \simeq \mathbb{Z} \times B\,\mathrm{U}(\infty) \times F.$$

- We also proved that the homotopy groups $\pi_i(F)$ are as follows:

$$\pi_i(F) \cong \begin{cases} 0 & \text{for } i \geq 0 \text{ even,} \\ \ell^\infty(\mathbb{Z}, \mathbb{Z})_{\mathrm{shift}} & \text{for } i \geq 1 \text{ odd.} \end{cases}$$

  These groups are $\mathbb{Q}$-vector spaces.
- In such a situation, F is known to have the homotopy type of a product of Eilenberg–MacLane spaces (cf. rational homotopy theory).
- Thus Theorem B follows.