De Rham cohomology of the weak stable foliation of the geodesic flow of a hyperbolic surface

Mitsunobu Tsutaya (Kyushu Univ.)
joint work with Hirokazu Maruhashi (Univ. Tokyo)

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Motivation: deformation theory and cohomology

M: a compact smooth manifold.
$U \subset \mathbb{R}^r$: an open neighborhood of 0.
Consider a smooth family (deformation) of some geometric structures on M parameterized by U.

Example

- complex structures $\{J_t \colon TM \to TM\}_{t \in U}$,
- foliations $\{\mathcal{F}_t\}_{t \in U}$,
- group actions $\{\rho_t \colon M \times G \to M\}_{t \in U}$, etc.

Motivation: deformation theory and cohomology

Question

Are there any non-trivial deformation?

Example

Any smooth family of complex structures on $\mathbb{C}P^n$ is trivial.

Example

Σ_g: closed oriented surface of genus $g \geq 2$. There exists a non-trivial family of complex structures on Σ_g parameterized by the Teichmüller space $\mathscr{T}_g$.

Motivation: deformation theory and cohomology

Question

How can we study the (non-)triviality of deformations?

Idea

$\mathscr{M} :=$ {geometric structures we consider}/isom.: moduli space.
For a family $\{\omega_t\}_{t\in U}$ (i.e. $\omega\colon U \to \mathscr{M}$), study the "differential"

$$(D\omega)_0\colon T_0U \to T_{\omega_0}\mathscr{M}.$$

The "tangent space" of $\mathscr{M}$ is considered as an appropriate cohomology in some cases.

Example (Kodaira–Spencer)

For complex structures on M, the "tangent space" is $H^1(M;TM)$ if $H^2(M;TM) = 0$.

Main object: leafwise cohomology

$\mathcal{F}$: a foliation on M,
E: a vector bundle over M equipped with a "flat $\mathcal{F}$-partial connection".
$\to \Omega^*(\mathcal{F};E)$: the leafwise de Rham complex of $\mathcal{F}$,
$H^*(\mathcal{F};E) = H^*(\Omega^*(\mathcal{F};E))$: the leafwise cohomology of $\mathcal{F}$.

Fact (Hamilton)

The "tangent space" of $\mathscr{M}$ at $\mathcal{F}$ is $H^1(\mathcal{F};TM/T\mathcal{F})$.

Result

$\mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$,
$\mathrm{PSO}(2) := \mathrm{SO}(2)/\{\pm I\} \subset \mathrm{PSL}(2,\mathbb{R})$,
$AN := \{\text{upper triangular matrices}\} \subset \mathrm{PSL}(2,\mathbb{R})$,
$\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$: a torsion-free cocompact lattice,
$\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})/\mathrm{PSO}(2) \cong \Sigma_g$ for some $g \geq 2$,
$M := \Gamma\backslash \mathrm{PSL}(2,\mathbb{R}) \cong$ the unit tangent bundle of Σ_g,
$\mathcal{F}$: the orbit foliation of the right action of AN on M.

Main result

We determined the cohomologies $H^*(\mathcal{F})$, $H^*(\mathcal{F}; TM/T\mathcal{F})$ etc.
This result can be applied to the deformations of the foliation $\mathcal{F}$ and the right action of AN on M.

2. Leafwise cohomology
   - Leafwise cohomology
   - Lie algebra cohomology
   - Locally free action

Leafwise cohomology

M: a smooth manifold,
$\mathcal{F}$: a foliation on M,
$T\mathcal{F} \subset TM$: the tangent bundle of $\mathcal{F}$.

Definition (de Rham complex)

$\Omega^*(M) = \Gamma(\bigwedge^* T^*M)$,

$$\begin{aligned}
&(d\omega)(X_1, \ldots, X_{i+1}) \\
&\quad := \sum_{j=1}^{i+1} (-1)^{j-1} X_j \omega(X_1, \ldots, \hat{X}_j, \ldots, X_{i+1}) \\
&\qquad + \sum_{j<k} (-1)^{j+k} \omega([X_j, X_k], X_1, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_{i+1}),
\end{aligned}$$

$H^*(M) = H^*(\Omega^*(M))$: de Rham cohomology.

Leafwise cohomology

M: a smooth manifold,
$\mathcal{F}$: a foliation on M,
$T\mathcal{F} \subset TM$: the tangent bundle of $\mathcal{F}$.

Definition (leafwise de Rham complex)

$\Omega^*(\mathcal{F}) = \Gamma(\bigwedge^* T^*\mathcal{F})$,

$$\begin{aligned}(d_{\mathcal{F}}\omega)&(X_1, \ldots, X_{i+1}) \\ &:= \sum_{j=1}^{i+1} (-1)^{j-1} X_j \omega(X_1, \ldots, \hat{X}_j, \ldots, X_{i+1}) \\ &\quad + \sum_{j<k} (-1)^{j+k} \omega([X_j, X_k], X_1, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_{i+1}),\end{aligned}$$

$H^*(\mathcal{F}) = H^*(\Omega^*(\mathcal{F}))$: leafwise cohomology.

Leafwise cohomology

$E \to M$: a vector bundle,
$\Omega^*(M;E) := \Gamma(\bigwedge^* T^*M \otimes E)$.

Definition (connection)

An $\mathbb{R}$-linear map $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ is a connection if

$$\nabla(f\eta) = (df)\eta + f\nabla\eta.$$

$d^\nabla\colon \Omega^*(M;E) \to \Omega^{*+1}(M;E)$: the covariant derivative.

Definition

(E,∇) is flat if $(d^\nabla)^2 = 0$.
(E,∇) is flat $\Rightarrow H^*(M;E) := (\Omega^*(M;E), d^\nabla)$.

Leafwise cohomology

$E \to M$: a vector bundle,
$\Omega^*(\mathcal{F}; E) := \Gamma(\bigwedge^* T^*\mathcal{F} \otimes E)$.

Definition ($\mathcal{F}$-partial connection)

An $\mathbb{R}$-linear map $\nabla\colon \Gamma(E) \to \Gamma(T^*\mathcal{F} \otimes E)$ is an $\mathcal{F}$-partial connection if

$$\nabla(f\eta) = (d_{\mathcal{F}} f)\eta + f\nabla\eta.$$

$d_{\mathcal{F}}^{\nabla}\colon \Omega^*(\mathcal{F}; E) \to \Omega^{*+1}(\mathcal{F}; E)$: the covariant derivative.

Definition

(E, ∇) is flat if $(d_{\mathcal{F}}^{\nabla})^2 = 0$.
(E, ∇) is flat $\Rightarrow H^*(\mathcal{F}; E) := (\Omega^*(\mathcal{F}; E), d_{\mathcal{F}}^{\nabla})$.

Lie algebra cohomology

$\mathfrak{h}$: a Lie algebra,
V: a representation of $\mathfrak{h}$.

Definition (Lie algebra cohomology)

$C^*(\mathfrak{h};V) := \mathrm{Hom}(\bigwedge^* \mathfrak{h}, V)$,

$$\begin{aligned}(d\omega)&(X_1,\dots,X_{i+1})\\ &:= \sum_{j=1}^{i+1}(-1)^{j-1}X_j\omega(X_1,\dots,\hat{X}_j,\dots,X_{i+1})\\ &\quad + \sum_{j<k}(-1)^{j+k}\omega([X_j,X_k],X_1,\dots,\hat{X}_j,\dots,\hat{X}_k,\dots,X_{i+1}),\end{aligned}$$

$H^*(\mathfrak{h};V) := H^*(C^*(\mathfrak{h};V))$: the cohomology of $\mathfrak{h}$ with coefficients in V.

Locally free action

H: a Lie group,
M: a smooth manifold.
An action $M \curvearrowleft H$ is locally free if the subgroup

$$H_x = \{h \in H \mid xh = x\}$$

is discrete for each $x \in M$.
This implies:

- the following map is injective for each $x \in M$:

$$A \in \mathfrak{h} \mapsto \left.\frac{d}{dt} x e^{tA}\right|_{t=0} \in T_x M,$$

- the orbits define a foliation (orbit foliation).

Locally free action

$M \curvearrowleft H$: a locally free action,
$\mathcal{F}$: the orbit foliation
V: a representation of H
$\Rightarrow M \times V$ is equipped with a canonical flat $\mathcal{F}$-partial connection.

Proposition

$$\Omega^*(\mathcal{F}; M \times V) \cong C^*(\mathfrak{h}; C^\infty(M) \otimes V).$$

3. Cohomology of $\mathcal{F}$ on $\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$
    - Setting
    - Main result

Setting

$\operatorname{PSL}(2,\mathbb{R}) := \operatorname{SL}(2,\mathbb{R})/\{\pm I\}$,
$\operatorname{PSO}(2) := \operatorname{SO}(2)/\{\pm I\} \subset \operatorname{PSL}(2,\mathbb{R})$,
$\Gamma \subset \operatorname{PSL}(2,\mathbb{R})$: a torsion-free cocompact lattice,

$\Rightarrow \Gamma\backslash \operatorname{PSL}(2,\mathbb{R})/\operatorname{PSO}(2) \cong \Sigma_g$

: a closed oriented surface of genus $g \geq 2$,

$M := \Gamma\backslash \operatorname{PSL}(2,\mathbb{R}) \cong$ the unit tangent bundle of Σ_g.

Setting

$M = \Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$
$AN :=$ {upper triangular matrices} $\subset \mathrm{PSL}(2,\mathbb{R})$,

$\Rightarrow M \curvearrowleft AN$: a locally free action,
$\mathcal{F}$: the orbit foliation.

Setting

$$H = \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \ E = \frac{1}{2}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\ :\ \text{basis of } \mathfrak{an},\ \ [H,E] = \frac{1}{\sqrt{2}}E.$$

$\mathbb{R}_\lambda = \mathbb{R}1_\lambda$: the 1-dimensional representation of $\mathfrak{an}$ such that $H1_\lambda = \lambda 1_\lambda$, $E1_\lambda = 0$ ($\lambda \in \mathbb{R}$),

$$\Rightarrow \Omega^*(\mathcal{F}; M \times \mathbb{R}_\lambda) \cong C^*(\mathfrak{an}; C^\infty(M) \otimes \mathbb{R}_\lambda).$$

Example

- $\mathbb{R}_0$: the trivial representation,
- $\mathbb{R}_{\frac{1}{\sqrt{2}}} \cong \mathbb{R}E$ as representations of $\mathfrak{an}$,
- $TM/T\mathcal{F} \cong M \times \mathbb{R}_{-\frac{1}{\sqrt{2}}}$ as vector bundles with flat $\mathcal{F}$-partial connections.

Main result

Theorem (Maruhashi–T.)

Suppose $\lambda = \frac{n}{\sqrt{2}}$ for an integer n. Then, there exist maps

$$\delta\colon C^*(\mathfrak{an}; C^\infty(M)\otimes\mathbb{R}_\lambda) \to C^{*-1}(\mathfrak{an}; C^\infty(M)\otimes\mathbb{R}_\lambda)$$
$$p\colon C^*(\mathfrak{an}; C^\infty(M)\otimes\mathbb{R}_\lambda) \to C^*(\mathfrak{an}; C^\infty(M)\otimes\mathbb{R}_\lambda)$$

such that

$$d\delta + \delta d = \mathrm{id} - p, \quad p^2 = p, \quad \operatorname{im} p = \ker d \cap \ker \delta$$

and there exists a closed subspace decomposition

$$C^*(\mathfrak{an}; C^\infty(M)\otimes\mathbb{R}_\lambda) = \operatorname{im} d \oplus \operatorname{im}\delta \oplus \operatorname{im} p.$$

Main result

(Theorem continued)

Moreover, the following hold ($\lambda = \frac{n}{\sqrt{2}}$):

- if $n \geq 2$, $H^*(\mathfrak{an}; C^\infty(M) \otimes \mathbb{R}_\lambda) = 0$,
- if $n = 1$,

$$\dim H^i(\mathfrak{an}; C^\infty(M) \otimes \mathbb{R}_{\frac{1}{\sqrt{2}}}) = \begin{cases} 0 & i = 0, \\ 1 & i = 1, 2, \end{cases}$$

- if $n = 0$,

$$\dim H^i(\mathfrak{an}; C^\infty(M)) = \begin{cases} 1 & i = 0, \\ 2g + 1 & i = 1, \\ 2g & i = 2, \end{cases}$$

- if $n \leq -1$,

$$\dim H^i(\mathfrak{an}; C^\infty(M) \otimes \mathbb{R}_\lambda) = \begin{cases} 0 & i = 0, \\ 2(1 - 2n)(g - 1) & i = 1, 2. \end{cases}$$

Main result

Remark

- There exists a decomposition as representation of $\mathrm{PSL}(2,\mathbb{R})$
  $$L^2(M) \cong \mathbb{C} \oplus \bigoplus_{\mu>0} \mathcal{H}_\mu \oplus (\mathcal{D}_1^+ \oplus \mathcal{D}_1^-)^{\oplus 2g} \oplus \bigoplus_{m\geq 2} (\mathcal{D}_m^+ \oplus \mathcal{D}_m^-)^{\oplus (2m-1)(g-1)}.$$
- The irreducible unitary representations $\mathcal{D}_m^\pm$ are called the discrete series representations. The multiplicity is computed by the Riemann–Roch theorem.
- The direct summands contributing to the cohomology are as follows:
  - $\mathbb{C}$ for $n = 1$,
  - $\mathbb{C}$ and $\mathcal{D}_1^\pm$ for $n = 0$,
  - $\mathcal{D}_{1-n}^\pm$ for $n \leq -1$.
- The fact $\dim H^1(\mathcal{F}) = 2g + 1$ was shown by Matsumoto–Mitsumatsu (2003).

4. Applications
   - Application to foliations
   - Application to locally free actions

Application to foliations

$M_0 := \Gamma_0 \backslash \mathrm{PSL}(2,\mathbb{R})$,
$\Gamma_0 \backslash \mathrm{PSL}(2,\mathbb{R}) / \mathrm{PSO}(2) \cong \Sigma_g$,
$\mathcal{F}_0$: the orbit foliation of $M_0 \curvearrowleft AN$,
$\{\mathcal{F}_t\}_{t \in U}$: a deformation of foliations parameterized by an open neighborhood $U \subset \mathbb{R}^r$ of 0,

$\Rightarrow \mathbb{R}^r \cong T_0 U \to H^1(\mathcal{F}; TM_0/T\mathcal{F}_0) \cong \mathbb{R}^{6g-6}$.

Application to foliations

$\mathscr{T}_g$: the Teichmüller space,
$\Gamma_t \subset \mathrm{PSL}(2,\mathbb{R})$: a smooth family of lattices corresponding to $t \in \mathscr{T}_g$,
$M_t := \Gamma_t \backslash \mathrm{PSL}(2,\mathbb{R}) \curvearrowleft AN$,
$\mathcal{F}_t$: the orbit foliation on $M_t \cong M_0$,

$\Rightarrow \{\mathcal{F}_t\}_{t \in \mathscr{T}_g}$: the deformation of foliation on M_0.

Theorem (Maruhashi–T.)

The induced map

$$T_t \mathscr{T}_g \to H^1(\mathcal{F}_t; TM_0/T\mathcal{F}_t)$$

is an isomorphism for each $t \in \mathscr{T}_g$.

Application to foliations

Remark

- In the proof, we observe that the following map is an isomorphism ($\mathfrak{sl}(2,\mathbb{R})/\mathfrak{an} \cong \mathbb{R}_{-\frac{1}{\sqrt{2}}}$):

$$H^1(\mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(2); C^\infty(M) \otimes \mathfrak{sl}(2,\mathbb{R})) \\ \to H^1(\mathfrak{an}; C^\infty(M) \otimes \mathfrak{sl}(2,\mathbb{R})/\mathfrak{an})$$

- Combining with the Nash–Moser implicit function theorem, $\mathscr{T}_g \to \mathscr{M}$ could be a local diffeomorphism.
- Ghys (1992) proved a similar result: a foliation $\mathcal{F}'$ sufficiently close to $\mathcal{F}_t$ ($t \in \mathscr{T}_g$) is diffeomorphic to $\mathcal{F}_{t'}$ for some $t' \in \mathscr{T}_g$.

Application to locally free actions

$\rho, \rho'\colon M \times H \to M$: smooth locally free actions with the same orbit foliation $\mathcal{F}$.

Definition

ρ and ρ' are parameter equivalent if there exists a diffeomorphism $\Phi\colon M \to M$ preserving each leaf of $\mathcal{F}$ such that

1. $\Phi(\rho(x,h)) = \rho'(\Phi(x),h)$ for any $x \in M$ and $h \in H$,
2. Φ is continuously homotopic to the identity through smooth maps preserving each leaf of $\mathcal{F}$.

Application to locally free actions

$M := \Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$,
$\Gamma\backslash \mathrm{PSL}(2,\mathbb{R}) / \mathrm{PSO}(2) \cong \Sigma_g$,
$\mathcal{F}$: the orbit foliation of $M \curvearrowleft AN$,

$\Rightarrow$ the "tangent space" of the moduli space of parameter equivalence classes of locally free actions at this action is $H^1(\mathcal{F}; M \times \mathfrak{an})$ ($\mathfrak{an}$ is the adjoint representation of AN).

Application to locally free actions

$H^1(\mathcal{F}; M \times \mathfrak{an}) \cong H^1(\mathfrak{an}; C^\infty(M) \otimes \mathfrak{an})$.

Theorem (Maruhashi–T.)

1. $$\dim H^i(\mathfrak{an}; C^\infty(M) \otimes \mathfrak{an}) = \begin{cases} 0 & i = 0, \\ 2g & i = 1, 2, \end{cases}$$
2. there exists a DGLA morphism from $C^*(\mathfrak{an}; C^\infty(M) \otimes \mathfrak{an})$ to a commutative DGLA which induces an isomorphism on cohomology,
3. we can construct a “universal formal deformation” of locally free actions at the given action $M \curvearrowleft AN$.

DGLA = differential graded Lie algebra.

Application to locally free actions

Remark

- We can also construct the operators δ and p for this cochain complex.
- Any non-trivial deformation of the given action $M \curvearrowleft AN$ has not been obtained.
- The parameter equivalence classes of locally free actions $M \curvearrowleft AN$ are completely classified by Asaoka (2012). Although he parameterized them by an open subset of $H^1(M) \cong \mathbb{R}^{2g}$, it is not known that such parametrization can be realized by a smooth deformation.