Homotopy theory of A_n-spaces in Lie groups

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Outline

1. Background – A_n-space
   - H-space
   - Homotopical associativity
   - Higher homotopy associativity

G: pointed space
$m\colon G \times G \to G$ continuous map

Definition

(G, m) is a (homotopy unital) H-space
$\overset{\text{def}}{\iff} x \mapsto m(*, x)$ and $x \mapsto m(x, *)$ are homotopic to $\mathbf{id}\colon G \to G$.

Example

A topological group is an H-space.

Example

X: pointed space
$\Omega X = \{\ell\colon [0, 1] \to X \mid \ell(0) = \ell(1) = *\}$: based loop space
$\Longrightarrow \Omega X$ is an H-space with m: concatenation of loops.

$G = (G, m)$: H-space

Definition

G is homotopy associative
$\overset{\text{def}}{\iff}$ $(x, y, z) \mapsto x(yz)$ and $(x, y, z) \mapsto (xy)z$ are homotopic.

Example

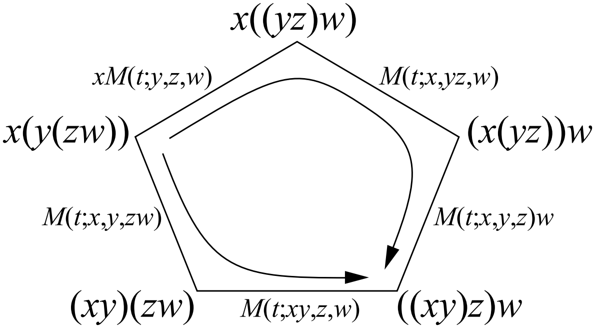
Topological groups and based loop spaces are homotopy associative.

Example

S^7 admits a structure of an H-space (cf. unit vectors in octonions), but no homotopy associative H-space.

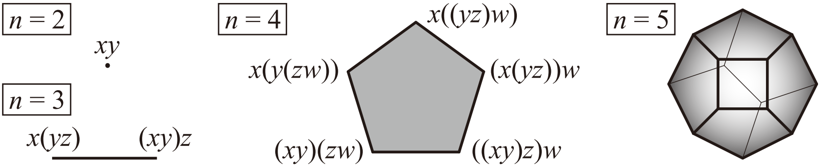
G: homotopy associative H-space
$M(t;x,y,z)$: "associating homotopy" such that

$$M(0;x,y,z) = x(yz), \qquad M(1;x,y,z) = (xy)z$$



There are two canonical way to associate $x(y(zw))$ and $((xy)z)w$. If these two homotoies are homotopic, we say G is an A_4-space.

→ Generalized to A_n-spaces ($n = 1, 2, \ldots, \infty$).

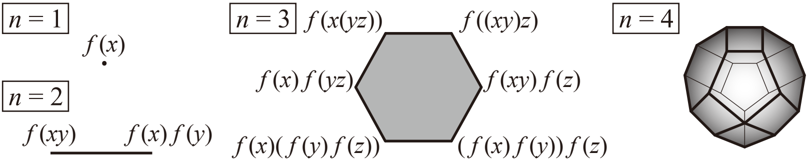


The parameter spaces are called associahedra $\mathcal{K}_n$ (Stasheff, 1963).

Example

Topological groups and based loop spaces are A_∞-spaces.

A_n-maps (morphisms between A_n-spaces) are higher homotopies parametrized by multiplihedra $\mathcal{J}_n$ (Stasheff, Boardman–Vogt, Iwase).



Example

Homomorphisms between topological groups are A_∞-maps.

Example

A topological group G and an A_∞-space ΩBG are A_∞-equivalent (i.e. $\exists G \to \Omega BG$: A_∞-map, homotopy equivalence).

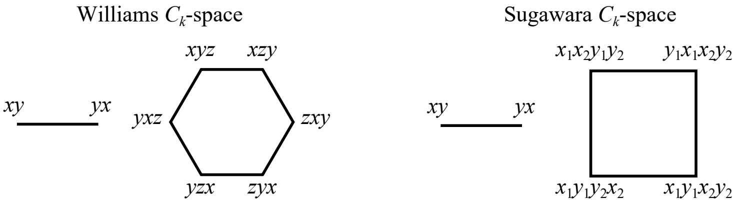
2. Higher homotopy commutativities
   - Homotopy commutativity
   - Higher homotopy commutativities
   - p-localizaton

G: H-space

Definition

G is homotopy commutative
$\overset{\text{def}}{\iff}$ two maps $(x, y) \mapsto xy$ and $(x, y) \mapsto yx$ are homotopic.

Higher homotopy commutativities:



The higher homotopies for Williams C_k-space are parametrized by permutohedra P_k (convex hull of $(\sigma(1), \ldots, \sigma(k)) \in \mathbb{R}^k$, σ: permutation).

G: A_n-space

Definition

G is a Sugawara C_k-space $\overset{\text{def}}{\Longleftrightarrow}$ the binary operation $G \times G \to G$ is an A_k-map.

cf. Γ: group, $\Gamma \times \Gamma \to \Gamma$: homomorphism $\Rightarrow$ Γ: commutative

Fact:
Sugawara C_2-space $\Leftrightarrow$ Williams C_2-space $\Leftrightarrow$ homotopy commutative
Fact:
Sugawara C_k-space $\Longrightarrow$ Williams C_k-space.

Question

What can we say about (higher) homotopy commutativity of Lie groups?

Remark

A connected Lie group G is A_∞-equivalent to the maximal compact subgroup K by the inclusion $K \to G$.

G: compact connected Lie group

Theorem (Araki–James–Thomas, 1960)

G is homotopy commutative
$\Rightarrow G$ is a torus.

Theorem (Bott, 1960)

$\mathbf{SU}(s)$ and $\mathbf{SU}(t)$ are not homotopy commutative in $\mathbf{SU}(s+t-1)$.

In these examples, there is no difference between the usual commutativity and the homotopy commutativity.

p: prime number or $0 \in \mathbb{Z}_{\geq 0}$
$\exists L_p$: Spaces $\to$ Spaces: p-localizaton functor
$\exists \eta : X \to L_p X$: natural map (We denote $X_{(p)} = L_p X$.)

- X: simply connected
  $\Rightarrow \pi_n(X_{(p)}) \cong \pi_n(X)_{(p)} = \pi_n(X) \otimes \mathbb{Z}_{(p)}$,
  $\eta_* : \pi_n(X) \to \pi_n(X_{(p)})$ is the canonical homomorphism of the p-localization.

Some “obstructions” vanish after localization.

Fact

G: A_∞-space
$\Rightarrow G_{(p)}$: A_∞-space.

$\to$ What can we say about (higher) homotopy commutativity of p-localized Lie groups?

3. Full higher commutativity of Lie groups
   - Homotopy commutativity of p-localized Lie groups
   - Method to prove

G: compact connected simple Lie group,

$$H^*(G;\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_1,\ldots,x_\ell)\ |x_i| = 2n_i - 1, n_1 \le \cdots \le n_\ell$$

Theorem (McGibbon 1984 (C_2), Saumell 1995 (Williams C_k ($k > 2$), $G \neq \mathrm{G}_2$), Hasui–Kishimoto–T 2019 (Sugawara C_k, $G = \mathrm{G}_2$))

1. If $p > kn_\ell$, then $G_{(p)}$ is a Sugawara C_k-space.
2. If $p < kn_\ell$, then $G_{(p)}$ is not a Williams C_k-space except in the case $(G,p) = (\mathrm{Sp}(2),3), (\mathrm{G}_2,5)$.
3. $\mathrm{Sp}(2)_{(3)}$ $(n_\ell = 4)$ is homotopy commutative.
4. $(\mathrm{G}_2)_{(5)}$ $(n_\ell = 6)$ is homotopy commutative but not a Williams C_3-space.

G: A_∞-space
$\Rightarrow * = B_0G \subset B_1G \subset \cdots \subset BG$
BG: classifying space of G
B_nG: n-th projective space of G

Example

$G = \mathbb{Z}/2\mathbb{Z}, S^1, S^3 (= \mathrm{SU}(2) = \mathrm{Sp}(1))$
$\Rightarrow B_nG = \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n$

X, Y: pointed spaces
$X \vee Y = \{(x, y) \in X \times Y \mid x = * \text{ or } y = *\} \subset X \times Y$
$X^{\vee n} = \{(x_1, \ldots, x_n) \in X^{\times n} \mid \text{if } x_j \neq *, \text{then } x_i = * \text{ for } i \neq j\} \subset X^{\times n}$

Fact

G is a Williams C_k-space
$\Leftrightarrow (B_1G)^{\vee k} \to BG$, $(*, \ldots, x, \ldots, *) \mapsto x$ extends over $(B_1G)^{\times k}$.

Fact

G is a Sugawara C_k-space
$\Leftrightarrow B_kG \vee B_kG \to BG$, $(*, x) \mapsto x$, $(x, *) \mapsto x$ extends over $B_kG \times B_kG$.

Extendability follows from the results of homotopy groups of spheres. Non-extendability follows from the computations of certain Steenrod operations and Chern characters.

4. Samelson products in Lie groups
    - Samelson product
    - Samelson products in Lie groups
    - Higher Samelson product and another partial higher homotopy commutativity
    - A_n-triviality of adjoint bundles

G: topological group
$\alpha\colon A \to G$, $\beta\colon B \to G$

Definition

$\langle \alpha, \beta \rangle\colon A \wedge B \to G$, $(a, b) \mapsto aba^{-1}b^{-1}$, is called the Samelson product.

G is homotopy commutative
$\Leftrightarrow$ all Samelson products are trivial.

$\epsilon_r \in \pi_{2r+1}(\mathrm{SU}(n)) \cong \mathbb{Z}$: generator ($r = 1, 2, \ldots, n-1$)

Theorem (Bott, 1960)

If $r + s = n + 1$, then the order of $\langle \epsilon_r, \epsilon_s \rangle \in \pi_{2n}(\mathrm{SU}(n)) \cong \mathbb{Z}/n!\mathbb{Z}$ is $\frac{n!}{r!s!}$.

Determined: The Samelson products of free parts of homotopy groups in $G_{(p)}$ for a compact connected simple Lie group G for $p > n_\ell$ and "quasi-p-regular" case (Bott, Mahowald, Hamanaka–Kono, Hasui–Kishimoto–Ohsita, Kishimoto–T, Hasui–Kishimoto–Miyauchi–Ohsita).

For $p > n_\ell$, all such Samelson products are detected by the Steenrod operation $\mathcal{P}^1$ in $BG_{(p)}$ (Kishimoto–T, 2018).

$\alpha\colon S^{n-1} \to G$
$\langle \alpha, \mathrm{id}_G \rangle$ appears as ∂ in the evaluation fiber sequence

$$\mathcal{G}(P_\alpha) \to G \xrightarrow{\partial} \mathrm{Map}_*(S^n, BG)_\alpha \to \mathrm{Map}(S^n, BG)_\alpha \to BG.$$

$\mathcal{G}(P_\alpha)$: the gauge group of the principal G-bundle over S^n classified by α.
$\to$ The homotopy type of $\mathcal{G}(P_\alpha)$ is determined by $\langle \alpha, \mathrm{id}_G \rangle$.
Such Samelson products are studied in many cases.

∃ higher versions of Samelson product.
Little is known about higher Samelson products in Lie groups....

Remark

All Samelson products of order k in G are trivial
⇔ G is a Williams C_k-space.

Another kind of "partial higher homotopy commutativity":
for $\alpha\colon S^n \to BG$, does the map

$$(\alpha, \mathrm{incl})\colon S^n \vee B_kG \to BG$$

extend over $S^n \times B_kG$?
(cf. T_k^α-space by Iwase–Mimura–Oda–Yoon)

Remark

G is a Sugawara C_k-space
$\Rightarrow$ it extends.

P_α: principal G-bundle over S^n classified by $\alpha\colon S^n \to BG$
$\rightsquigarrow \mathbf{ad}\, P_\alpha = P_\alpha \times_G G \quad ((ug, x) \sim (u, gxg^{-1}))$: group bundle
$\Gamma(\mathbf{ad}\, P_\alpha) \cong \mathcal{G}(P_\alpha)$ as topological groups

Proposition

$(\alpha, \mathrm{incl})\colon S^n \vee B_kG \to BG$ extends over $S^n \times B_kG$
$\Leftrightarrow \mathbf{ad}\, P_\alpha$ is trivial as a fiberwise A_k-space.

$G = \mathrm{SU}(2)$
$\pi_4(B\,\mathrm{SU}(2)) \cong \pi_3(\mathrm{SU}(2)) \cong \mathbb{Z}$
a_k: characterized by the following condition:
 $\mathrm{ad}\,P_\ell$ and $\mathrm{ad}\,P_{\ell'}$ are equivalent as fiberwise A_k-spaces
 if and only if $(a_k, \ell) = (a_k, \ell')$.

Theorem

$a_1 = 12 = 2^2 3^1$, $a_2 = 180 = 2^2 3^2 5^1$ (Crabb–Sutherland, 2000),
$a_3 = 15120 = 2^4 3^3 5^1 7^1$ (T, 2018),
$\nu_3(a_k) = k$ (T, 2012).

The growth of the number of the fiberwise A_k-equivalence classes of $\mathrm{ad}\,P_\ell$ ($\ell \in \pi_4(B\,\mathrm{SU}(2))$) is at least $e^{\frac{k}{\log k}}$ (T, 2012).