Pontryagin–Thom construction in topological coincidence theory

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Plan

Introduction

1. Introduction
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Topological fixed point theory

A typical problem in topological fixed point theory is as follows:

- Determine if a given self map $f: X \to X$ can be deformed to a fixed point free map f' (i.e. $f'(x) \neq x$ for any $x \in X$).

Topological coincidence theory

A typical problem in topological coincidence theory is as follows:

- Determine if given maps $f_0, f_1 : X \to Y$ can be deformed to a coincidence free maps f_0', f_1' (i.e. $f_0'(x) \neq f_1'(x)$ for any $x \in X$).

This relates to the following problems:

- Fixed point problem.
- Root problem: can a given map $f : X \to X$ be deformed to a map f' of which the image does not contain a given point $a \in X$?
- Intersection problem: can given maps $f_0 : X \to Z$ and $f_1 : Y \to Z$ be deformed to maps with disjoint images?

Bordism invariant $\omega(f_0,f_1)$

Let $f_0,f_1 : M \to N$ be maps between smooth closed connected manifolds.

- Hatcher–Quinn (1974) and Koschorke (2006) introduced some obstruction bordism class

$$\omega(f_0,f_1) \in \Omega_{\dim M - \dim N}(\mathrm{Hoeq}(f_0,f_1); TN - TM)$$

  for the coincidence problem of f_0 and f_1, where $\mathrm{Hoeq}(f_0,f_1)$ is the homotopy equalizer of f_0 and f_1.
- If f_0 and f_1 can be deformed to coincidence free maps, then $\omega(f_0,f_1) = 0$.
- When $\dim M < 2\dim N - 2$, the converse also holds.

First aim

One of the main aim of this talk is to describe $\omega(f_0, f_1)$ by Pontryagin–Thom construction. This enables us to compute $\omega(f_0, f_1)$ by Serre spectral sequence in some cases.

Decomposition of $\omega(f_0, f_1)$

- Denote the path component corresponding to $\alpha \in \pi_0(\mathrm{Hoeq}(f_0, f_1))$ by $\mathrm{Hoeq}(f_0, f_1)_\alpha$.
- There is a decomposition

$$\begin{aligned}&\Omega_*(\mathrm{Hoeq}(f_0, f_1); TN - TM)\\ &= \bigoplus_{\alpha \in \pi_0(\mathrm{Hoeq}(f_0, f_1))} \Omega_*(\mathrm{Hoeq}(f_0, f_1)_\alpha; TN - TM).\end{aligned}$$

- We define $\omega(f_0, f_1)_\alpha \in \Omega_*(\mathrm{Hoeq}(f_0, f_1)_\alpha; TN - TM)$ as the corresponding component of $\omega(f_0, f_1)$.

Jiang invariance of $\omega(f_0, f_1)$

- We call the following subgroup the Jiang subgroup of f_0:

$$J(f_0) = \{\alpha \in \pi_1(N) \mid (\alpha, f_0)\colon S^1 \vee M \to N \text{ extends over } S^1 \times M\}.$$

- It is known that $J(f_0)$ acts on $\pi_0(\mathrm{Hoeq}(f_0, f_1))$.
- The Jiang invariance (Crabb (2010)) is stated as follows: for $\alpha \in \pi_0(\mathrm{Hoeq}(f_0, f_1))$ and $\beta \in J(f_0)$, $\omega(f_0, f_1)_\alpha = 0$ if and only if $\omega(f_0, f_1)_{\beta\alpha} = 0$.

Second aim

The second aim is to realize the Jiang invariance by some action of string topology spectra using the Pontryagin–Thom description of $\omega(f_0, f_1)$.

Background: Lefschetz and Reidemeister traces

2. Background: Lefschetz and Reidemeister traces
   - Lefschetz trace
   - Lefschetz fixed point theorem
   - Local description
   - Reidemeister class
   - Reidemeister trace
   - Reidemeister's fixed point theorem

Lefschetz trace

Let $f : M \to M$ be a self map on a smooth closed connected manifold M with $\dim M = m$.

- The number

$$\operatorname{trace}(f) = \sum_{i=0}^{m} (-1)^i \operatorname{trace} H_i(f; \mathbb{Q})$$

  is called the Lefschetz trace of f.

Lefschetz fixed point theorem

Lefschetz fixed point theorem

If trace(f) $\neq 0$, then f cannot be deformed to a fixed point free map.

- If $\dim M \geq 3$ and M is simply connected, then the converse also holds.

Local description

- If f has a finite number of fixed points $x_1, \ldots, x_k$, then there is an equality

$$\mathrm{trace}(f) = \sum_{j=1}^{k} \mathrm{ind}(f; x_j),$$

where $\mathrm{ind}(f; x_j)$ is called the fixed point index of f at x_j.
- This equality implies the Lefschetz fixed point theorem.

Reidemeister class

- The homotopy fixed point space $\mathrm{Hofix}(f)$ is defined by the pullback square (compare with the usual fixed point set)

$$\begin{array}{ccc} \mathrm{Hofix}(f) & \longrightarrow & M^{[0,1]} \\ \downarrow & & \downarrow (ev_0, ev_1) \\ M & \xrightarrow[(\mathrm{id}, f)]{} & M \times M. \end{array}$$

- Each fixed point $x \in M$ naturally lifts to a point in $\mathrm{Hofix}(f)$. The path component contains this lift is called the corresponding Reidemeister class (or Nielsen class).

Reidemeister trace

- Suppose f has a finite number of fixed points $x_1, \ldots, x_k$. The following formal sum $\rho(f)$ is called the Reidemeister trace:

$$\rho(f) = \sum_{j=1}^{k} \mathrm{ind}(f; x_j)\alpha_{x_j} \in \mathbb{Z}[\pi_0(\mathrm{Hofix}(f))],$$

where $\alpha_{x_j} \in \pi_0(\mathrm{Hofix}(f))$ is the Reidemeister class corresponding to x_j.

Reidemeister's fixed point theorem

Theorem (Reidemeister (1936))

If $\rho(f) \neq 0$, then f cannot be deformed to a fixed point free map.

- If $\dim M \geq 3$, then the converse also holds.
- The invariant $\omega(\mathrm{id}, f)$ coincides with $\rho(f)$ (Koschorke (2006)).

Description of $\omega(f_0, f_1)$ by Pontryagin–Thom construction

3. Description of $\omega(f_0, f_1)$ by Pontryagin–Thom construction
    - Spectrum
    - Thom spectrum
    - Pontryagin–Thom construction
    - Lifting of Pontryagin–Thom construction
    - Homotopy equalizer
    - $\omega(f_0, f_1)$ by Pontryagin–Thom construction
    - Remarks on $\omega(f_0, f_1)$

Spectrum

- A spectrum $E = (\{E_k, \epsilon_k\}_{k\geq 0})$ consists of sequences of based spaces $\{E_k\}_{k\geq 0}$ and maps $\{\epsilon_k : \Sigma E_k \to E_{k+1}\}_{k\geq 0}$.
- The definition of general maps between spectra is slightly complicated. In this talk, we only consider maps such as $E \to E'$ realized by $E_k \to E'_k$ for some sufficiently large k. This is sufficient if E is a finite CW spectrum.

Thom spectrum

Let X be a space and ξ a vector bundle over X. Denote the associated disk and sphere bundles by $D(\xi)$ and $S(\xi)$, respectively.

- The based space $\mathrm{Thom}(\xi) = D(\xi)/S(\xi)$ is called the Thom space of ξ.
- Consider the Whitney sum $\epsilon_X^1 \oplus \xi$ with the trivial line bundle ϵ_X^1. Then $D(\epsilon_X^1 \oplus \xi)/S(\epsilon_X^1 \oplus \xi) \cong \Sigma D(\xi)/S(\xi)$.
- $X^\xi = \{\mathrm{Thom}(\epsilon_X^k \oplus \xi)\}_k$ is called the Thom spectrum of ξ.
- This is generalised to a stable vector bundle ξ. For example, M^{-TM} for a smooth closed manifold M is the Thom spectrum of the stable normal bundle $-TM$.
- There is a natural isomorphism

$$\pi_i(X^\xi) \cong \Omega_{i-\mathrm{rank}\,\xi}(X;\xi)$$

where π_i denotes the i-th stable homotopy group.

Pontryagin–Thom construction

Let M and N be smooth closed manifolds and ξ a stable vector bundle over N.

- For a map $f : M \to N$, the Pontryagin–Thom construction is a map $f^! : N^\xi \to M^{f^*(\xi+TN)-TM}$.
- When $\xi = -TN$, $f^! : N^{-TN} \to M^{-TM}$ is the Spanier–Whitehead dual of f.

Lifting of Pontryagin–Thom construction

Consider the pullback of Hurewicz fibrations:

$$\begin{array}{ccc} X_M & \xrightarrow{\tilde{f}} & X_N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[f]{} & N, \end{array}$$

where M and N are smooth closed manifolds. Let ξ be a stable vector bundle over X_N.

- Since $\tilde{f}$ has "finite codimension", we can give the Pontryagin–Thom construction

$$\tilde{f}^! : X_N^{\xi} \to X_M^{\tilde{f}^*\xi + \pi_M^*(f^*TN - TM)}.$$

Homotopy equalizer

The homotopy equalizer $\mathrm{Hoeq}(f_0, f_1)$ of maps $f_0, f_1 : M \to N$ is defined by the pullback square

$$\begin{array}{ccc} \mathrm{Hoeq}(f_0, f_1) & \xrightarrow{F} & N^{[0,1]} \\ \downarrow & & \downarrow \\ M & \xrightarrow[(f_0, f_1)]{} & N \times N \end{array}$$

- The map $F^! : N^{-TN} \to \mathrm{Hoeq}(f_0, f_1)^{TN-TM}$ is induced.
- Transposing this square, we also have the map $\tilde{\Delta}^! : M^{-TM} \to \mathrm{Hoeq}(f_0, f_1)^{TN-TM}$.

$\omega(f_0, f_1)$ by Pontryagin–Thom construction

Proposition

The following composite coincides with $\omega(f_0, f_1) \in \pi_0(\mathrm{Hoeq}(f_0, f_1)^{TN-TM})$:

$$S^0 \xrightarrow{\eta} N^{-TN} \xrightarrow{F^!} \mathrm{Hoeq}(f_0, f_1)^{TN-TM},$$

where η is the unit map, that is, the Spanier–Whitehead dual of the map $N \to$ point.

- The following composite also represents $\omega(f_0, f_1)$:

$$S^0 \xrightarrow{\eta} M^{-TM} \xrightarrow{\tilde{\Delta}^!} \mathrm{Hoeq}(f_0, f_1)^{TN-TM}.$$

Remarks on $\omega(f_0, f_1)$

- The original definition by Hatcher–Quinn and Koschorke is more geometric.
- More fibrewise homotopy theoretic descriptions are given by Klein–Williams (2007) and Crabb (2010).
- Suppose $\dim M = \dim N$. Then there is an isomorphism

$$\begin{aligned}&\pi_0(\mathrm{Hoeq}(f_0, f_1)^{TN-TM})\\ &\cong \mathbb{Z}[\pi_0(\mathrm{Hoeq}(f_0, f_1)_{\mathrm{ori}})] \oplus \mathbb{Z}/2\mathbb{Z}[\pi_0(\mathrm{Hoeq}(f_0, f_1)_{\text{non-ori}})],\end{aligned}$$

  where $\mathrm{Hoeq}(f_0, f_1)_{\mathrm{ori}}$ is the union of the path components on which $TN - TM$ is orientable and $\mathrm{Hoeq}(f_0, f_1)_{\text{non-ori}}$ is the union of the path components on which $TN - TM$ is not orientable (Koschorke (2006)).

Serre spectral sequence

4. Serre spectral sequence
   - Relative Serre spectral sequence
   - Serre spectral sequence for Thom spectrum

Relative Serre spectral sequence

Let h_* be a homology theory satisfying the wedge and weak homotopy equivalence axioms.

- Let $F \to E \xrightarrow{\pi} B$ be a Hurewicz fibration, $E' \to E$ a fibrewise closed cofibration and $A \subset B$ a closed cofibration. Then we have the relative Serre spectral sequence

$$E^2_{p,q} = \tilde{H}_p(B/A; \underline{h_q(F/F')}) \Longrightarrow h_{p+q}(E/(E' \cup \pi^{-1}(A))).$$

Serre spectral sequence for Thom spectrum

Let $F \to E \xrightarrow{\pi} B$ be a Hurewicz fibration and $\bar{\xi}$ and ξ be stable vector bundles over B and E, respectively.

- Applying the relative Serre spectral sequence, we obtain the spectral sequence

$$E^2_{p,q} = \tilde{H}_p(B^{\bar{\xi}}; \underline{h_q(F^{\xi|_F})}) \Longrightarrow h_{p+q}(E^{\pi^*\bar{\xi}+\xi}).$$

- A Pontryagin–Thom construction map induces a morphism of spectral sequences compatible with the natural maps on E^2 and E^∞-terms:

$$\tilde{H}_p(N^{\bar{\xi}}; \underline{h_q(F^{\xi|_F})}) \to \tilde{H}_p(M^{f^*(\bar{\xi}+TN)-TM}; \underline{h_q(F^{\xi|_F})}),$$

$$h_{p+q}(X_N^{\pi^*\bar{\xi}+\xi}) \to h_{p+q}(X_M^{\tilde{f}^*(\pi_N^*\bar{\xi}+\xi)+\pi_M^*(f^*TN-TM)}).$$

- A similar spectral sequence for a string topology spectrum is studied by Cohen–Jones–Yan (2004).

Jiang invariance by string topology spectrum

5. Jiang invariance by string topology spectrum
   - Generalized string topology spectrum
   - String topology spectrum and homotopy equalizer
   - Jiang invariance by string topology spectrum

Generalized string topology spectrum

Let $X \to M$ be a fibrewise topological monoid over a smooth closed manifold M. The following are due to Gruher–Salvatore (2008).

- By the Pontryagin–Thom construction associated to the diagonal map $M \to M \times M$ and the fibrewise multiplication, X^{-TM} is a ring spectrum.
- Similarly, if $Y \to M$ is a fibrewise module over X and ξ a stable vector bundle over Y, then Y^{ξ} is a module over X^{-TM}.

String topology spectrum and homotopy equalizer

$$f_0^* LN = \{(x, \ell) \in M \times N^{[0,1]} \mid \ell(0) = \ell(1) = f_0(x)\}$$
$$\mathrm{Hoeq}(f_0, f_1) = \{(x, \ell) \in M \times N^{[0,1]} \mid \ell(0) = f_0(x), \ell(1) = f_1(x)\}$$

- By the obvious concatenation of paths, $(f_0^* LN)^{-TM}$ is a ring spectrum and $\mathrm{Hoeq}(f_0, f_1)^{TN-TM}$ is a module over $(f_0^* LN)^{-TM}$.
- A section $s \colon M \to f_0^* LN$ (= a cyclic homotopy of f_0) defines an element $[s]_* \in \pi_0((f_0^* LN)^{-TM})$ by the composite

$$S^0 \xrightarrow{\eta} M^{-TM} \xrightarrow{s} (f_0^* LN)^{-TM}.$$

Jiang invariance by string topology spectrum

Theorem

The map $\tilde{\Delta}^! : M^{-TM} \to \mathrm{Hoeq}(f_0, f_1)^{TN-TM}$ is fixed under the action of $[s]_* \in \pi_0((f_0^* LN)^{-TM})$ for any section $s : M \to f_0^* LN$.

- This implies the Jiang invariance mentioned before.
- Similarly, there are an action of $(f_1^* LN)^{-TM}$ on $\mathrm{Hoeq}(f_0, f_1)^{TN-TM}$ and the corresponding Jiang invariance.