A_n-maps and mapping spaces

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Tsutaya, Mapping spaces from projective spaces, Homology, Homotopy Appl. 18 (2016), 173–203.

Plan

1. A_n-maps
2. Main results
3. Applications

Main results

1. Main results
   - H-maps between topological monoids
   - A_n-maps
   - Projective spaces
   - A_n-maps and projective spaces

H-maps between topological monoids

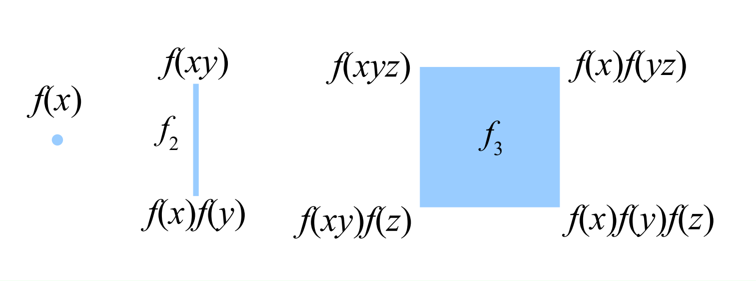
- A topological monoid is a pointed space G equipped with an associative multiplication $m\colon G \times G \to G$ such that the basepoint is the identity element.
- A pointed map $f\colon G \to G'$ between topological monoids is called an H-map if the following diagram commutes up to homotopy

$$\begin{array}{ccc} G \times G & \xrightarrow{\ m\ } & G \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ G' \times G' & \xrightarrow[\ m'\ ]{} & G'. \end{array}$$

A_n-maps (Sugawara, 1961; Stasheff, 1963)

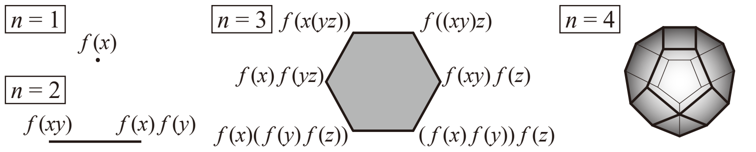
Let G and G' be topological monoids.

- A pair $(f, \{f_i\})$ of a pointed map $f\colon G \to G'$ and a family of maps $\{f_i\colon I^{\times(i-1)} \times G^{\times i} \to G'\}_{i=1}^{n}$ (called an A_n-form) satisfying appropriate conditions is called an A_n-map.
- The conditions for small n is depicted as follows.



Remarks on A_n-maps

- Stasheff (1963) introduced A_n-spaces as H-spaces equipped with higher homotopy associativity data.
- A_n-maps were generalized to morphisms between A_n-spaces (Boardman–Vogt, 1972; Iwase, 1983).



- The parameterizing spaces are called multiplihedra. They were constructed by Boardman–Vogt (using their tensor product and resolution of operads) and Iwase (realizing them as subsets of Euclidean spaces).

Projective spaces

- The n-th stage of the geometric realization

$$B_nG = \left(\coprod_{i=0}^{n} \Delta^i \times G^{\times i} \right) \Big/ \sim$$

  is called the n-th projective space of G. The full geometric realization $BG = B_\infty G$ is the classifying space of G.
- $B_0G =$ point, $B_1G = \Sigma G$ (reduced suspension).
- $B_0G \subset B_1G \subset \cdots \subset BG$.
- (Examples) $B_nS^0 = \mathbb{R}P^n$, $B_nS^1 = \mathbb{C}P^n$, $B_nS^3 = \mathbb{H}P^n$.

A_n-maps and projective spaces

- Sugawara (1961) constructed the induced map $B_n f : B_n G \to B_n G'$ of an A_n-map $f = (f, \{f_i\}) : G \to G'$. In particular, $B_1 f = \Sigma f$.
- Stasheff (1963) proved the converse in some sense. Suppose G' is grouplike (i.e. $\pi_0(G')$ is a group). Then a pointed map $f : G \to G'$ admits an A_n-form $\{f_i\}_{i=1}^{n}$ if there exists a map $B_n G \to BG'$ making the following diagram commutative:

$$\begin{array}{ccc} \Sigma G & \xrightarrow{\Sigma f} & \Sigma G' \\ \downarrow & & \downarrow \\ B_n G & \dashrightarrow & BG'. \end{array}$$

- Iwase (1983) extended these results to general A_n-spaces.

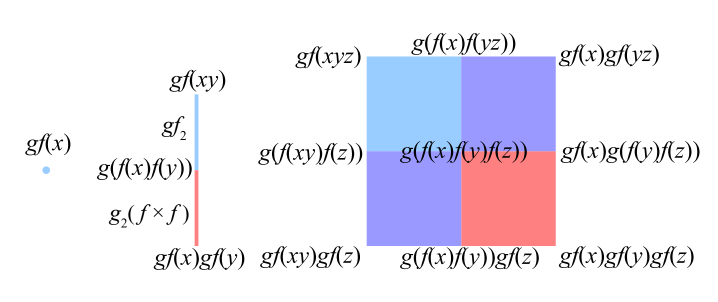
The main aim of this talk is to refine these results!

Main results

2. Main results
   - Composition of A_n-maps
   - Category of topological monoids and A_n-maps
   - Continuous functor B_n
   - Main theorem

Composition of A_n-maps

- The "composition" of A_n-maps can be considered as in the following figure (Stasheff, 1963):



- But this composition is neither associative nor unital. We can make this composition associative and unital mimicking the Moore loops.

Category of topological monoids and A_n-maps

Define a topological category $\mathcal{A}_n$ as follows:

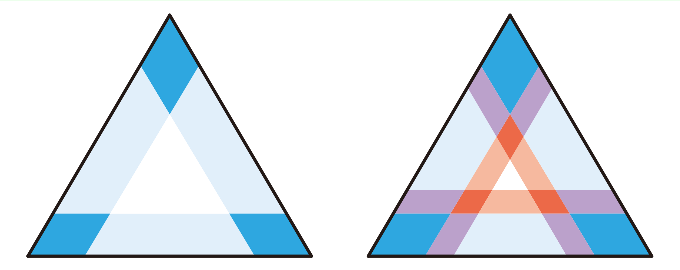
- Objects are topological monoids.
- Morphisms are triples $(f, \{f_i\}, \ell)$ of pointed maps, its Moore A_n-form
$$\{f_i\colon [0,\infty)^{\times(i-1)} \times G^{\times i} \to G'\}_{i=1}^n$$
with the size $\ell \in [0,\infty)$. If $\ell = 0$, we require $(f, \{f_i\})$ to be a homomorphism with the trivial A_n-form.
- A space of morphisms is topologized as a subspace
$$\mathcal{A}_n(G, G') \subset \prod_{i=1}^n \mathrm{Map}(I^{\times(i-1)} \times G^{\times i}, G') \times [0,\infty).$$

Continuous functor B_n

- The projective space functor is realized as a continuous functor

$$B_n \colon \mathcal{A}_n \to \text{(category of pointed spaces)}.$$

- The induced map and the compatibility with composition look like the following figure.



Main theorem

Theorem A (T, 2016)

Let G be a topological monoid (with some cofibrantness condition) and G' be a grouplike topological monoid. Then the following composite is a weak equivalence:

$$\mathcal{A}_n(G, G') \xrightarrow{B_n} \mathbf{Map}_*(B_nG, B_nG') \to \mathbf{Map}_*(B_nG, BG').$$

- This theorem also can be expressed as the following "adjunction" weak equivalence:

$$\mathcal{A}_n(G, \Omega^{\mathrm{M}}X) \simeq \mathbf{Map}_*(B_nG, X)$$

for the Moore loop space $\Omega^{\mathrm{M}}X$ of a pointed space X.

Remarks on the "adjoint pair" $(B_n, \Omega^{\mathrm{M}})$

- The associated "monad"

$$\Omega^{\mathrm{M}} B_1 = \Omega^{\mathrm{M}}\Sigma\colon \text{(pointed spaces)} \to \text{(topological monoids)}$$

  is equivalent to the James construction (James, 1955).

- The associated "comonad"

$$B_n \Omega^{\mathrm{M}}\colon \text{(pointed spaces)} \to \text{(pointed spaces)}$$

  is equivalent to the n-th Ganea construction (Iwase, 1998).

Applications

3. Applications
   - Extension of the evaluation fiber sequences
   - Homotopy commutativity
   - T_k^f-spaces

Extension of the evaluation fiber sequence

- Let $\mathrm{Map}(B_nG, BG)_{i_n}$ be the path-component of $\mathrm{Map}(B_nG, BG)$ containing the inclusion $i_n\colon B_nG \to BG$ and $\mathrm{Map}_*(B_nG, BG)_{i_n} = \mathrm{Map}_*(B_nG, BG) \cap \mathrm{Map}(B_nG, BG)_{i_n}$.
- For a topological group G, let $\mathcal{A}_n(G, G)_{\mathrm{conj}}$ be the union of path components containing conjugation homomorphisms.

Theorem B (T, 2016)

Let G be a topological group (with some cofibrantness condition). Then the map $G \to \mathcal{A}_n(G, G)_{\mathrm{conj}}$ assigning conjugations deloops and the following sequence of maps is a fiber sequence:

$$G \to \mathrm{Map}_*(B_nG, BG)_{i_n} \to \mathrm{Map}(B_nG, BG)_{i_n} \xrightarrow{\text{evaluation}} BG \to B\mathcal{A}_n(G, G)_{\mathrm{conj}}.$$

Proof of Theorem B

- The conjugation in G induces the actions on the mapping spaces $\mathcal{A}_n(G,G)$, $\mathbf{Map}_*(B_nG, BG)$.
- This action on $\mathbf{Map}_*(B_nG, BG)$ coincides with the induced action of the evaluation fiber sequence

$$G \to \mathbf{Map}_*(B_nG, BG)_{i_n} \to \mathbf{Map}(B_nG, BG)_{i_n} \to BG.$$

- The map assigning conjugations $G \to \mathcal{A}_n(G,G)_{\mathrm{conj}}$ is a homomorphism.
- The weak equivalence in Theorem A is G-equivariant.
- Thus the above evaluation fiber sequence is equivalent to the left 4 terms of the following fiber sequence:

$$G \to \mathcal{A}_n(G,G)_{\mathrm{conj}} \to EG \times_G \mathcal{A}_n(G,G)_{\mathrm{conj}} \to BG \to B\mathcal{A}_n(G,G)_{\mathrm{conj}}.$$

Remarks on Theorem B

- When $n = 0$, the evaluation fiber sequence is trivial:

$$G \to \mathrm{Map}_*(B_0G, BG)_{i_0} = * \to \mathrm{Map}(B_0G, BG)_{i_0} = BG \xrightarrow{=} BG.$$

- When $n = \infty$, the extension of the evaluation fiber sequence is well-known:

$$\begin{aligned} G \to \mathrm{Map}_*(BG, BG)_{\mathrm{id}} \to \mathrm{Map}(BG, BG)_{\mathrm{id}} &\xrightarrow{\text{evaluation}} BG \\ &\to B\,\mathrm{Map}_*(BG, BG)_{\mathrm{id}} \to B\,\mathrm{Map}(BG, BG)_{\mathrm{id}}. \end{aligned}$$

- When $1 \leq n < \infty$, the evaluation fiber sequence no longer extends to the right if G is a compact connected simple Lie group since $\mathrm{Map}(B_nG, BG)_{i_n}$ is not a loop space (Hasui–Kishimoto–T).

Homotopy commutativity

- A discrete group Γ is commutative if and only if any inner automorphism $\Gamma \to \Gamma$ is the identity map. Consider an A_n-version of this property.

Theorem C (T, 2016)

Let G be a topological group (with some cofibrantness condition). The homomorphism $\mathbf{conj}\colon G \to \mathcal{A}_\ell(G,G)$ is homotopic to the constant map at the identity as an A_k-map if and only if the wedge sum of the inclusions

$$B_kG \vee B_\ell G \to BG$$

extends over the product $B_kG \times B_\ell G$.

- The latter condition is equivalent to the condition that G is a $C(k,\ell)$-space (Kishimoto–Kono, 2010).

Proof of Theorem C

$\mathbf{conj}\colon G \to \mathcal{A}_\ell(G,G)$ is null-homotopic as an A_k-map.

Theorem A $\Leftrightarrow$ $B_kG \to BG \xrightarrow{B\,\mathbf{conj}} B\mathcal{A}_\ell(G,G)_{\mathbf{conj}}$ is null-homotopic.

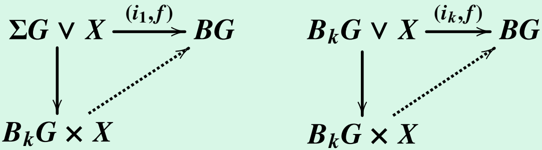
Theorem B $\Leftrightarrow$ $B_kG \to BG$ lifts to a map $B_kG \to \mathrm{Map}(B_\ell G, BG)_{i_\ell}$.

adjunction $\Leftrightarrow$ $B_kG \vee B_\ell G \to BG$ extends over $B_kG \times B_\ell G$.

T_k^f-spaces

Theorem D

Let G be a topological group (with some cofibrantness condition) and $f : X \to BG$ a map. Then the existence of the dotted arrows in these two diagrams are equivalent:



- These conditions are the definitions of a T_k^f-space and a C_k^f-space for BG by Iwase–Mimura–Oda–Yoon (2014).
- A T_k^{id}-space is just a T_k-space by Aguadé (1987).

Proof of Theorem D

- Suppose there exists a map $F\colon B_kG \times X \to BG$ making the left diagram commutative.
- Taking the adjoint, F corresponds to a map $X \to \mathcal{A}_k(G, G)$ by Theorem A.
- Let $h \in \mathcal{A}_k(G, G)$ be a homotopy inverse of $F(*)$. This exists because the underlying map of $F(*)$ is a homotopy equivalence.
- The map $B_kG \times X \to BG$ corresponding to the composite

$$X \to \mathcal{A}_k(G, G) \xrightarrow{\text{composing } h} \mathcal{A}_k(G, G)$$

is the desired dotted arrow in the right diagram.

Other applications

- Applications to fiberwise A_n-types of adjoint bundles.
- Applications to computations of the homology groups of the classifying spaces of gauge groups (Kishimoto–Theriault).