Mapping spaces from projective spaces

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International Conference on Manifolds, Groups and Homotopy
Isle of Skye
21 June 2018

This talk is based on the paper

Tsutaya, Mapping spaces from projective spaces, Homology, Homotopy Appl. 18 (2016), 173–203.

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    - Higher homotopy commutativity
    - Homology of classifying spaces of gauge groups

Main results

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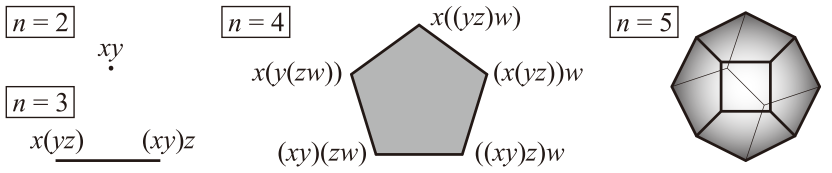
    - Main result 2

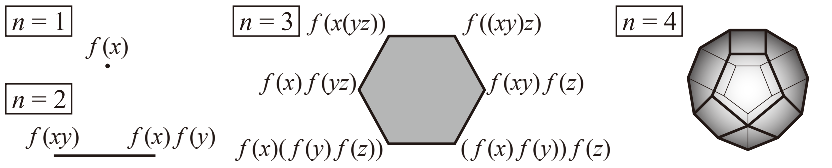
A_n-spaces

An H-space equipped with higher homotopy associativity data for the multiplication of n elements is called an A_n-space (Stasheff 1963).



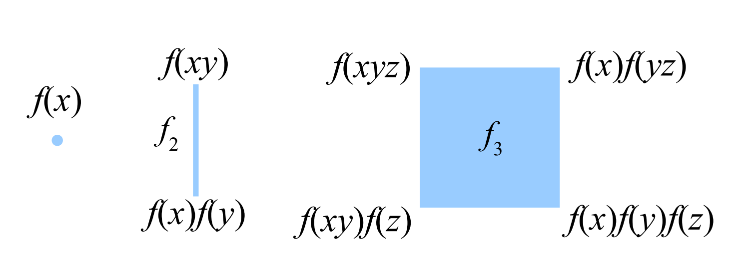
A_n-maps

A map between A_n-spaces preserving higher homotopy associativity data is called an A_n-map (Sugawara, Stasheff, Boardman–Vogt, Iwase).



A_n-maps between topological monoids

In this talk, we concentrate on A_n-maps between topological monoids (Sugawara, Stasheff).

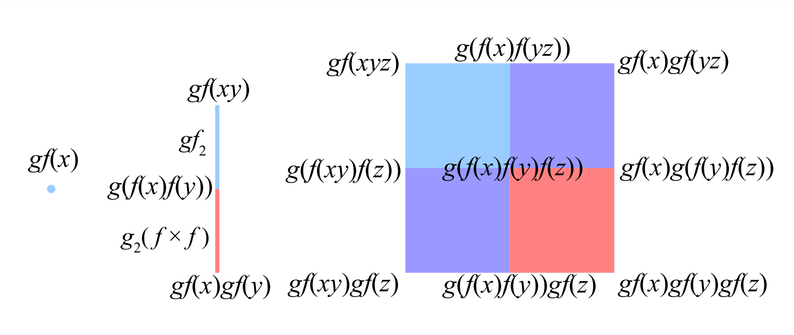


The i-th structure homotopy has the form

$$f_i\colon I^{\times(i-1)} \times G^{\times i} \to H.$$

Composition of A_n-maps

A_n-maps $(f, \{f_i\}_i): G \to H$ and $(g, \{g_i\}_i): H \to K$ can be composed.



This composition is associative if we apply the Moore path technique. Then we obtain the topological category $\mathcal{A}_n$ of topological monoids and A_n-maps with "size" $\in [0, \infty)$.

Projective spaces

The n-th projective space B_nG is the n-th stage of the bar construction

$$B_nG = B_n(*, G, *) = \left(\coprod_{0 \le i \le n} \Delta^i \times G^{\times i} \right) \Big/ \sim .$$

There are natural inclusions

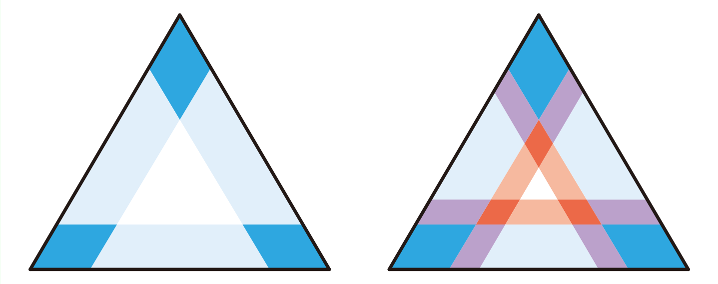
$$* = B_0G \subset B_1G \subset \cdots \subset B_\infty G = BG.$$

Example. $B_nS^0 = \mathbb{R}P^n$, $B_nS^1 = \mathbb{C}P^n$, $B_nS^3 = \mathbb{H}P^n$.

Projective spaces

B_n induces a continuous functor

$$B_n \colon \mathcal{A}_n \to \text{(based spaces)}.$$



Example. $B_1 G \cong \Sigma G$ naturally.

A_n-maps and projective spaces

Stasheff proved the converse.

Theorem (Stasheff 1963). A map $f\colon G \to H$ admits an A_n-map structure (A_n-form) if the composite

$$\Sigma G \xrightarrow{\Sigma f} \Sigma H \cong B_1 H \xrightarrow{\text{inclusion}} BH$$

extends over $B_n G$.

A_n-maps and projective spaces

Question. Can we recover an A_n-form of an A_n-map $G \to H$ from the composite

$$B_nG \to B_nH \to BH?$$

Answer. Yes (up to homotopy).

Main result 1

Theorem (T 2016)

If G is a topological monoid (+ some cofibrantness condition) and H is a topological group, then the following composite is a weak homotopy equivalence:

$$\mathcal{A}_n(G,H) \xrightarrow{B_n} \mathbf{Map}_*(B_nG, B_nH) \to \mathbf{Map}_*(B_nG, BH).$$

Moreover, this map is H-equivariant.

The action of H on $\mathcal{A}_n(G,H)$ and $\mathbf{Map}_*(B_nG, BH)$ induced from the conjugation:

$$\mathbf{conj}(h)\colon H \to H, \qquad \mathbf{conj}(h)(x) = hxh^{-1},$$
$$B\,\mathbf{conj}(h)\colon BH \to BH.$$

Main result 2

From this theorem, there is a homotopy equivalence of fibre sequences

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H & \longrightarrow & \mathcal{A}_n(G,H) & \longrightarrow & X & \longrightarrow & BH \\
& \| & & \downarrow \simeq & & \downarrow \simeq & & \| \\
\cdots \longrightarrow & H & \longrightarrow & \mathrm{Map}_*(B_nG, BH) & \longrightarrow & \mathrm{Map}(B_nG, BH) & \longrightarrow & BH
\end{array}
$$

Then, when $G = H$, we obtain the following result.

Main result 2

Theorem (T 2016)

If G is a topological group (+ some cofibrantness condition), then there is a fibre sequence

$$\cdots \to \mathrm{Map}_*(B_nG, BG)_{i_n} \to \mathrm{Map}(B_nG, BG)_{i_n} \to BG \xrightarrow{B\,\mathrm{conj}} B\mathcal{A}_n(G,G)_{\mathrm{conj}}.$$

$\mathrm{Map}(B_nG, BG)_{i_n}$ = the path component containing the inclusion,

$\mathrm{Map}_*(B_nG, BG)_{i_n} = \mathrm{Map}_*(B_nG, BG) \cap \mathrm{Map}(B_nG, BG)_{i_n}$,

$\mathcal{A}_n(G,G)_{\mathrm{conj}}$ = the path components containing conjugations.

Main result 2

Remark.

1. When $n = \infty$, it is well known that this sequence extends one more step.

$$\cdots \to \mathrm{Map}_*(BG, BG)_{\mathrm{id}} \to \mathrm{Map}(BG, BG)_{\mathrm{id}} \to BG \xrightarrow{B\,\mathrm{conj}} B\,\mathrm{Map}_*(BG, BG)_{\mathrm{id}} \to B\,\mathrm{Map}(BG, BG)_{\mathrm{id}}.$$

2. When $1 \leq n < \infty$ and G is a compact connected simple Lie group, the sequence in the theorem no longer extends to the right (Hasui–Kishimoto–T).

2. Applications

- Higher homotopy commutativity
- Homology of classifying spaces of gauge groups

Homotopy commutativity

Recall. A discrete group Γ is commutative if and only if any inner automorphism $\Gamma \to \Gamma$ is the identity map.

Problem. Let G be a topological group. Characterize the following condition in terms of projective spaces: the homomorphism $\mathbf{conj}\colon G \to \mathcal{A}_\ell(G, G)$ is homotopic to the constant map at the identity as an A_k-map.

Homotopy commutativity

Answer. The previous condition is equivalent to the condition that the wedge sum of the inclusions

$$B_kG \vee B_\ell G \to BG$$

extends over the product $B_kG \times B_\ell G$ (equivalently, G is a $C(k,\ell)$-space (Kishimoto–Kono)).

Homotopy commutativity

Proof.

$$\mathbf{conj}\colon G \to \mathcal{A}_\ell(G,G) \text{ is null-homotopic as an } A_k\text{-map.}$$

$$\Leftrightarrow \quad B_kG \to BG \xrightarrow{B\,\mathbf{conj}} B\mathcal{A}_\ell(G,G)_{\mathbf{conj}} \text{ is null-homotopic.}$$

$$\Leftrightarrow \quad B_kG \to BG \text{ lifts to a map } B_kG \to \mathbf{Map}(B_\ell G, BG)_{i_\ell}.$$

$$\Leftrightarrow \quad B_kG \vee B_\ell G \to BG \text{ extends over } B_kG \times B_\ell G.$$

Homology of classifying spaces of gauge groups

Let G be a simple 1-connected Lie group, $f\colon S^4 \to BG$ the inclusion and p be a prime such that $G_{(p)}$ is a product of spheres.

Theorem(Kishimoto–Theriault). In the Serre p-local or mod p homology spectral sequence of fibration

$$\mathrm{Map}_*(S^4, BG)_f \to \mathrm{Map}(S^4, BG)_f \to BG,$$

the differentials are $H_*(\mathrm{Map}_*(S^4, BG)_f) \cong H_*(\Omega^3_0 G)$-linear.

Homology of classifying spaces of gauge groups

Remark. For the restriction $P = f^*EG$ of the universal bundle EG, there is a weak homotopy equivalence

$$B\mathcal{G}(P) \simeq \mathrm{Map}(S^4, BG)_f.$$

Proof. This follows from the fact that the above fibration is p-locally a retract of the principal fibration

$$\mathrm{Map}_*(\Sigma G, BG)_{i_1} \to \mathrm{Map}(\Sigma G, BG)_{i_1} \to BG.$$