T_n^f-property of $BSU(2)$ and relation to fiberwise A_n-triviality

No. 1

§1 Background

X, A : based sp. $f : A \to X$: based

<u>Def</u> (Iwase-Mimura-Oda-Yoon, 2012)

X : T_n^f-space $\quad$ n-th proj. sp. of ΩX (or Ganea const. of X)

$$\overset{\text{def}}{\Leftrightarrow} \quad \begin{array}{ccc} A \vee B_n\Omega X & \xrightarrow{(f, i_n)} & X \\ \cap & \nearrow_{\exists F} & \\ A \times B_n\Omega X & & \end{array} \qquad (B_1\Omega X = \Sigma\Omega X)$$

$i_n : B_n\Omega X \to B\Omega X \simeq X$: incl. $\square$

<u>Rem.</u>

- C_n^f-sp. and T_n^f-sp. are equivalent (T. 2016). $\square$
- T_n^f-sp. $\Rightarrow$ T_{n-1}^f-sp.
- T_n^{id}-sp. is also called T_n-sp. (Aguadé 1987).

<u>Relation to htpy commutativity</u>

- G : top. gp.
  - G : htpy. comm. $\Leftrightarrow$ BG : $T_1^{i_1}$-sp.
  - G : htpy. comm., $\operatorname{cat} A \leq 1$ $\Rightarrow$ BG : T_1^f-sp. $\quad$ LS category

<u>Relation to adjoint bundles</u>

- G : top. gp.
  P : prin. G-bdl over A classified by $f : A \to BG$.
  - $\operatorname{ad} P := P \times G/\sim \quad ((u, x) \sim (ug, g^{-1}xg))$: adjoint bdl.
    - $\operatorname{ad} P$ is a fiberwise top. gp.
    - $\Gamma(\operatorname{ad} P) \cong$ gp. of automorph.s on P (gauge gp.)

<u>Prop.</u>

$\operatorname{ad} P \simeq A \times G$: as f.w. A_n-sp.
$\Leftrightarrow$ BG : T_n^f-sp. $\square$

<u>Rem.</u>

X : based, path-conn.
$\Rightarrow$ ($LX \simeq X \times \Omega X$: as f.w. sp. $\Leftrightarrow$ X : T_1^{id}-sp.) $\square$

§2 T^f_n- property of $BSU(2)$.

- $B_n SU(2) = B_n Sp(1) = \mathbb{H}P^n$
  $= S^4 \cup e^8 \cup \cdots \cup e^{4n}$.
- $k : S^4 \xrightarrow[\deg . k]{} S^4 = \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^\infty$ $\quad (k \in \mathbb{Z})$.

Problem

$$\begin{tikzcd} S^4 \vee \mathbb{H}P^n \arrow[r, "{(k, i_n)}"] \arrow[d, hook] & \mathbb{H}P^\infty \\ S^4 \times \mathbb{H}P^n \arrow[ur, dashed, "\exists F ?"'] & \end{tikzcd}$$

□

How to detect obstructions by K-theory.

- $K(\mathbb{H}P^\infty) = \mathbb{Z}[[a]]$, $\quad K(\mathbb{H}P^n) = \mathbb{Z}[a]/(a^{n+1})$
  $a = \mathbb{C}(\xi) - 2$, $\quad \xi$: canonical
  $K(S^4 \times \mathbb{H}P^n) = \mathbb{Z}[u, a]/(u^2, a^{n+1})$
- Suppose $\exists F$.

$$\Rightarrow \quad F^* a = 1 \times a + ku \times 1 + \sum_{j=1}^{n} \varepsilon_j(k)\, u \times a^j \quad (\varepsilon_j(k) \in \mathbb{Z}).$$

Prop. (T. 2012)

- $\exists\, \varepsilon_j \in \mathbb{Q}$ s.t. $\forall k \in \mathbb{Z}$, $\varepsilon_j(k) = k\,\varepsilon_j$.

$$\left.\begin{array}{l} \bullet\ \nu_2(k) \geq 2\left[\frac{n+1}{2}\right] \\ \bullet\ p: \text{odd} \Rightarrow \nu_p(k) \geq \left[\frac{2n}{p-1}\right] \end{array}\right\} (A) \quad (\Leftrightarrow \forall \varepsilon_j \in \mathbb{Z} \text{ for } j = 1, \ldots, n)$$

□

Is the condition (A) sufficient?

A Not known.

Lem.

- $\pi_{11}(\mathbb{H}P^\infty) \cong \pi_{10}(S^3) \cong \mathbb{Z}/15\mathbb{Z}$.
- $4 \cdot \pi_m(\mathbb{H}P^\infty)_{(2)} = 0$ for $m > 3$ (James 1957)
- $p \cdot \pi_m(\mathbb{H}P^\infty)_{(p)} = 0$ for p: odd, $m > 3$ (Selick 1978) □

T^f_n - property

$n=1$ (Crabb-Sutherland, 2010)

$\nu_2(k) \geq 2, \quad \nu_3(k) \geq 1$

$\therefore \quad BSU(2) : T^k_1\text{-sp.} \iff 12 \mid k. \qquad (12 = 2^2 3^1)$

$n=2$ (Crabb-Sutherland, 2010)

$\nu_2(k) \geq 2, \quad \nu_3(k) \geq 2, \quad \nu_5(k) \geq 1, \quad \pi_{11}(\mathbb{H}P^\infty) = \mathbb{Z}/15$

$\therefore \quad BSU(2) : T^k_2\text{-sp.} \iff 180 \mid k \qquad (180 = 2^2 3^2 5^1)$

$n=3$ (T.)

$\nu_2(k) \geq 4, \quad \nu_3(k) \geq 3, \quad \nu_5(k) \geq 1, \quad \nu_7(k) \geq 1$

$\therefore \quad BSU(2) : T^k_3 - \text{sp.} \iff 15120 \mid k \qquad (15120 = 2^4 3^3 5^1 7^1)$

Rem

If $BSU(2)$ is localized away from 2, we can obtain similar results for $n \leq 20$. $\square$

§3 Error in the previous work.

Localize everything at odd p.

Suppose

$$\begin{tikzcd} S^4 \vee \mathbb{H}P^n \arrow[r, "(k, in)"] \arrow[d, hook] & \mathbb{H}P^\infty \\ S^4 \times \mathbb{H}P^n \arrow[ur, "\exists F"'] & \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} S^4 \times \mathbb{H}P^n \cup \mathbb{H}P^{n+\frac{p-1}{2}} \arrow[r, "F \circ (p \times id)"] \arrow[d] & \mathbb{H}P^\infty \arrow[d] \\ S^4 \times \mathbb{H}P^{n+\frac{p-1}{2}} \arrow[r] & \mathbb{H}P^\infty \cup e^{4n+8} \cup \cdots \cup e^{4n+2(p-1)+4} \arrow[u, bend right, "\exists ?\ \text{retraction}"'] \end{tikzcd}$$

To do next

- Use BP cohomology?
- Other Lie groups (for large prime p)?