ERRATUM TO "HOMOTOPY PULLBACK OF A_n-SPACES AND ITS APPLICATIONS TO A_n-TYPES OF GAUGE GROUPS" [TOPOLOGY APPL. 187 (1) (2015) 1–25]

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Abstract. In the article 'Homotopy pullback of A_n-spaces and its applications to A_n-types of gauge groups" [Topology Appl. 187 (1) (2015) 1–25], a classification of the A_n-types of gauge groups of principal SU(2)-bundles over S^4 is given when they are localized away from 2. The proof of this result contains some problems. We cannot say it is true or false at this point. But, instead, we improve the classification of the fiberwise A_3-types of the adjoint bundles.

1. Introduction

In Section 9 of [Tsu15], we studied the classification problem of the gauge groups of principal SU(2)-bundles over S^4. In the proof of Proposition 9.1 in [Tsu15], the author considered the map θ_r called the "relative Whitehead product". But, actually, it is not well-defined. From this failure, the proof does not work at all. He could not fix the proofs for Proposition 9.1, Corollary 9.2 and Theorem 1.2. The aim of this article is to prove a weaker version of Theorem 1.2 in [Tsu15] and to improve the result for the fiberwise A_3-types of adjoint bundles.

Let P_k be the principal SU(2)-bundle over S^4 such that $c_2(P_k)[S^4] = k \in \mathbb{Z}$. The following is a weaker version of Theorem 1.2 in [Tsu15], to which we only add the condition $n \leq 20$. We denote the largest integer less than or equal to t by $[t]$.

Theorem 1.1. *For a positive integer $n \leq 20$, the gauge groups $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are A_n-equivalent if $\min\{2n, v_2(k)\} = \min\{2n, v_2(k')\}$ and $\min\{[2n/(p-1)], v_p(k)\} = \min\{[2n/(p-1)], v_p(k')\}$ for any odd prime p. Moreover, if $v_2(k) \leq 1$, the converse is also true.*

Proof. To show the if part, it is sufficient to show that the wedge sum $(k, i)\colon S^4_{(p)} \vee \mathbb{H}P^n_{(p)} \to \mathbb{H}P^\infty_{(p)}$ extends over the product $S^4_{(p)} \times \mathbb{H}P^n_{(p)}$. The case when $p = 3$ has already been verified in [Tsu12, Section 5].

Suppose $p \geq 5$. By Toda's result [Tod66, Section 7], we have homotopy groups of $\mathbb{H}P^\infty_{(p)}$ as follows:

$$\pi_{4i+3}(\mathbb{H}P^\infty_{(p)}) \cong \pi_{4i+2}(S^3_{(p)}) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i \equiv 0 \bmod \frac{p-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i < \frac{p-1}{2}(2p+1)-1$, where $\frac{p-1}{2}(2p+1)-1 = 21$ if $p = 5$. This implies that, if $i < \frac{p-1}{2}(2p+1)-1$ and $i \not\equiv 0 \bmod \frac{p-1}{2}$, there is no obstruction to extending a map $S^4_{(p)} \times \mathbb{H}P^{i-1}_{(p)} \cup \mathbb{H}P^i_{(p)} \to \mathbb{H}P^\infty_{(p)}$ over $S^4_{(p)} \times \mathbb{H}P^i_{(p)}$. It also implies that, for $m < 2p+1$ and a map $f\colon S^4_{(p)} \times \mathbb{H}P^{\frac{p-1}{2}m-1}_{(p)} \cup \mathbb{H}P^{\frac{p-1}{2}m}_{(p)} \to \mathbb{H}P^\infty_{(p)}$, the composite $f \circ (p \times \mathrm{id})$ extends over $S^4_{(p)} \times \mathbb{H}P^{\frac{p-1}{2}m}_{(p)}$. Then we obtain the if part by induction and Theorem 1.1 in [Tsu15]. The proof of the converse in [Tsu15] correctly works for $n \leq 20$. □

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Remark 1.2. For $i = \frac{p-1}{2}(2p+1) - 1$ and $p \geq 5$, Toda's result [Tod66, Theorem 7.5] says

$$\pi_{4i+3}(\mathbb{H}P^\infty_{(p)}) \cong \mathbb{Z}/p\mathbb{Z}.$$

This is the first non-trivial homotopy group where the obstruction is not detected in our method.

Suppose there exists an extension $f\colon S^4 \times \mathbb{H}P^{n-1} \cup \mathbb{H}P^n \to \mathbb{H}P^\infty$ of $(k, i)\colon S^4 \vee \mathbb{H}P^n \to \mathbb{H}P^\infty$, where $k \in \mathbb{Z} \cong \pi_4(\mathbb{H}P^\infty)$ and i is the inclusion $\mathbb{H}P^n \to \mathbb{H}P^\infty$. In the rest of this article, we compute the e-invariant [Ada66] of the obstruction to extending the map f over $S^4 \times \mathbb{H}P^n$. This obstruction is regarded as an element $h \in \pi_{4n+3}(\mathbb{H}P^\infty)$. The map h factors as the composite of the suspension map $\Sigma h'\colon S^{4n+3} \to S^4 = \mathbb{H}P^1$ and the inclusion $\mathbb{H}P^1 \to \mathbb{H}P^\infty$, where $h' \in \pi_{4n+2}(S^3)$ is the homotopy class corresponding to h under the isomorphism $\pi_{4n+3}(\mathbb{H}P^\infty) \cong \pi_{4n+2}(S^3)$. Consider the following maps among cofiber sequences:

$$\begin{array}{ccccccc}
S^{4n+3} & \longrightarrow & S^4 \times \mathbb{H}P^{n-1} \cup \mathbb{H}P^n & \longrightarrow & S^4 \times \mathbb{H}P^n & \longrightarrow & S^{4n+4} \\
\| & & \downarrow f & & \downarrow g & & \| \\
S^{4n+3} & \longrightarrow & \mathbb{H}P^\infty & \longrightarrow & X & \longrightarrow & S^{4n+4} \\
\| & & \uparrow & & \uparrow & & \| \\
S^{4n+3} & \xrightarrow{h} & S^4 & \longrightarrow & S^4 \cup_h e^{4n+4} & \longrightarrow & S^{4n+4}
\end{array}$$

As in [Tsu12], take the appropriate generator $a \in K(\mathbb{H}P^\infty)$ such that

$$K(\mathbb{H}P^\infty) = \mathbb{Z}[[a]].$$

Actually, one can take the generator a as the image of $\gamma - \mathbb{H}$ under the complexification map $\mathbf{c}\colon K\mathrm{Sp}(\mathbb{H}P^\infty) \to K(\mathbb{H}P^\infty)$ from the quaternionic K-theory, where γ denotes the canonical line bundle and $\mathbb{H}$ the 1-dimensional trivial quaternionic vector bundle. We denote the restriction of a on S^4 by $u \in K(S^4)$. Note that we can obtain the following by the Künneth theorem for K-theory:

$$K(S^4 \times \mathbb{H}P^{n-1} \cup \mathbb{H}P^n) = \mathbb{Z}[u \times 1, 1 \times a]/(u^2 \times 1, u \times a^n, 1 \times a^{n+1}),$$
$$K(S^4 \times \mathbb{H}P^n) = \mathbb{Z}[u \times 1, 1 \times a]/(u^2 \times 1, 1 \times a^{n+1}).$$

Lemma 1.3. *Let $0 \leq i \leq n$. Then the following holds.*

1. *(1) Suppose i is even. Then $u \times a^i$ is an image of the complexification map from the quaternionic K-theory.*
2. *(2) Suppose i is odd. Then $2u \times a^i$ is an image of the complexification map from the quaternionic K-theory, but $u \times a^i$ is not.*

Proof. Consider the following commutative diagram induced by the cofiber sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K\mathrm{Sp}(S^{4i+4}) & \longrightarrow & K\mathrm{Sp}(S^4 \times \mathbb{H}P^i) & \longrightarrow & K\mathrm{Sp}(S^4 \times \mathbb{H}P^{i-1} \cup \mathbb{H}P^i) & \longrightarrow & 0 \\
 & & \downarrow \mathbf{c} & & \downarrow \mathbf{c} & & \downarrow \mathbf{c} & & \\
0 & \longrightarrow & K(S^{4i+4}) & \longrightarrow & K(S^4 \times \mathbb{H}P^i) & \longrightarrow & K(S^4 \times \mathbb{H}P^{i-1} \cup \mathbb{H}P^i) & \longrightarrow & 0
\end{array}$$

Note that all the groups appearing in this diagram are free abelian. This implies the vertical maps are injective. As is well-known, the index of the image of the map $\mathbf{c}\colon K\mathrm{Sp}(S^{4i+4}) \to K(S^{4i+4})$ is 1 if i is even, and is 2 if i is odd. Now the lemma follows from the above diagram and the fact that the image of $K(S^{4i+4}) \to K(S^4 \times \mathbb{H}P^i)$ is generated by $u \times a^i$. □

Since $K\mathrm{Sp}(S^{4n+3}) = 0$, there is a lift $\tilde{a} \in K(X)$ of $a \in K(\mathbb{H}P^\infty)$ contained in the image of the complexification from $K\mathrm{Sp}(X)$. Denote the image of $\tilde{a}$ under the map $K(X) \to K(S^4 \cup_h e^{4n+4})$ by $\bar{a} \in K(S^4 \cup_h e^{4n+4})$. We take $s = \mathrm{ch}_2\, u \in H^4(S^4; \mathbb{Z})$, $b = \mathrm{ch}_2\, a \in H^4(\mathbb{H}P^\infty; \mathbb{Z})$ and $\tilde{b} = \mathrm{ch}_2\, \tilde{a} \in H^4(X; \mathbb{Z})$. We fix a generator of $H^{4n+4}(S^{4n+4}; \mathbb{Z})$ such that its image in $H^{4n+4}(S^4 \times \mathbb{H}P^n; \mathbb{Z})$ is $s \times b^n$ We denote its images by $\tilde{w} \in H^{4n+4}(X; \mathbb{Z})$ and $\bar{w} \in H^{4n+4}(S^4 \cup_h e^{4n+4}; \mathbb{Z})$.

As in [Ada66, Section 7], the e-invariant λ of the map $S^{4n+3} \to S^4$ is characterized by

$$\mathrm{ch}\, \bar{a} = s + \lambda \bar{w}$$

in $H^*(S^4 \cup_h e^{4n+4}; \mathbb{Q})$, where λ is well-defined as a residue class in $\mathbb{Q}/\mathbb{Z}$ if n is odd, and in $\mathbb{Q}/2\mathbb{Z}$ if n is even. If the map $S^{4n+3} \to S^4$ is null-homotopic, then λ is 0 as the corresponding residue class.

By the result of [Tsu12], we have

$$f^* a = 1 \times a + k \sum_{j=0}^{n-1} \epsilon_j u \times a^j,$$

where $\epsilon_0 = 1$ and $\epsilon_1, \ldots, \epsilon_{n-1} \in \mathbb{Q}$ are inductively defined by the equations

$$\frac{1}{(2\ell+1)!} = \sum_{i=1}^{\ell} \sum_{\substack{j_1+\cdots+j_i=\ell \\ j_1,\ldots,j_i \geq 1}} \frac{2^i \epsilon_i}{(2j_1)! \cdots (2j_i)!}.$$

Since a is in the image of the complexification from $K\mathrm{Sp}(\mathbb{H}P^\infty)$, we have $k\epsilon_i \in \mathbb{Z}$ for even $j < n$ and $k\epsilon_i \in 2\mathbb{Z}$ for odd $j < n$ by Lemma 1.3. Combining with [Tsu12, Propositions 4.2 and 4.4], we have the following proposition.

Lemma 1.4. *The following hold.*

(1) For $p = 2$, $\nu_2(k) \geq 2[\frac{n}{2}]$.
(2) For an odd prime p, $\nu_p(k) \geq [\frac{2(n-1)}{p-1}]$.

There exists $\beta \in \mathbb{Z}$ such that the following holds:

$$g^* \tilde{a} = 1 \times a + k \sum_{j=0}^{n-1} \epsilon_j u \times a^j + \beta u \times a^n.$$

Again by Lemma 1.3, $\beta \in 2\mathbb{Z}$ if n is odd. Note that the Chern characters $\mathrm{ch}\, a$ and $\mathrm{ch}\, \tilde{a}$ are computed as

$$\mathrm{ch}\, a = \sum_{j=1}^{\infty} \frac{2b^j}{(2j)!}, \qquad \mathrm{ch}\, \tilde{a} = \sum_{j=1}^{\infty} \frac{2\tilde{b}^j}{(2j)!} + \lambda \tilde{w}.$$

Then, by computing $\mathrm{ch}_{2n+2}\, g^* \tilde{a}$ by two ways as in [Tsu12, Section 2], we obtain

$$\frac{k}{(2n+1)!} + \lambda = \sum_{i=1}^{n-1} \sum_{\substack{j_1+\cdots+j_i=\ell \\ j_1,\ldots,j_i \geq 1}} \frac{2^i \epsilon_i k}{(2j_1)! \cdots (2j_i)!} + \beta.$$

By the definition of ϵ_n, we get

$$\beta = \lambda + \epsilon_n k.$$

Then we have the following proposition from the e-invariant λ and Lemma 1.4.

Proposition 1.5. *If f extends over $S^4 \times \mathbb{H}P^n$, then the following hold.*

(1) For $p = 2$, $\nu_2(k) \geq 2[\frac{n+1}{2}]$.
(2) For an odd prime p, $\nu_p(k) \geq [\frac{2n}{p-1}]$.

Actually, nothing is improved by this proposition for odd p. But, for $p = 2$, we obtain the new result since the torsion part of $\pi_*(\mathbb{H}P^\infty_{(2)})$ is annihilated by 4 [Jam57, Corollary (1.22)].

Theorem 1.6. *The adjoint bundle* $\mathrm{ad}\, P_k$ *is trivial as a fiberwise* A_3*-space if and only if k is divisible by* $15120 = 2^4 3^3 5^1 7^1$.

From this result, one may expect that we can derive the classification of 2-local A_3-types of the gauge groups. But, to distinguish between $\mathcal{G}(P_4)_{(2)}$ and $\mathcal{G}(P_8)_{(2)}$ as A_3-spaces, we need some new technique. So, we leave this problem for now.

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