Applications of Stasheff's A_∞-theory to Lie groups

Mitsunobu Tsutaya (Kyushu Univ.)

MSJ Spring Meeting 2017
Tokyo Metropolitan University
24 Mar. 2017

Outline

1. Background
   - Homotopical properties of binary operation
   - Higher homotopy structures on binary operation
2. Stasheff's A_∞-theory
   - A_n-maps
   - Classifying space and projective spaces
3. Applications to Lie groups
   - Gauge groups and mapping spaces
   - Finiteness of gauge groups
   - Extension of evaluation fiber sequence
   - Higher homotopy commutativity of Lie groups

Background

1. Background
   - Homotopical properties of binary operation
   - Higher homotopy structures on binary operation

Homotopical properties of binary operation

G a pointed space, $m\colon G \times G \to G$ a (continuous) map.
Suppose $m(*, x) = m(x, *) = x$. We denote as $xy = m(x, y)$.
The pair $G = (G, m)$ is called an H-space.

Definition

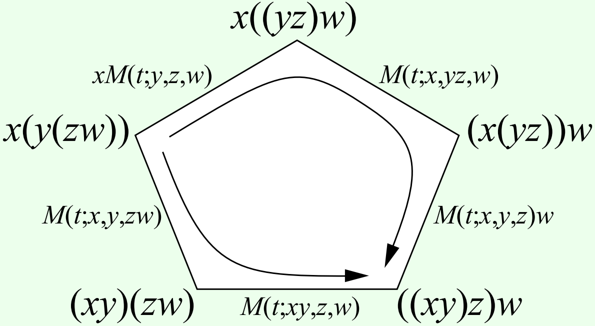
1. G is homotopy associative if two maps $G \times G \times G \to G$ given by $(x, y, z) \mapsto x(yz)$ and $(x, y, z) \mapsto (xy)z$ are homotopic,
2. G is homotopy invertible if there exists a map $\iota\colon G \to G$ such that two maps $G \to G$ given by $x \mapsto x\iota(x)$ and $x \mapsto \iota(x)x$ are homotopic to the constant map,
3. G is homotopy commutative if two maps $G \times G \to G$ given by $(x, y) \mapsto xy$ and $(x, y) \mapsto yx$ are homotopic.

Example

If G is a topological group, then G is a homotopy associative, homotopy invertible H-space.

Higher homotopy structures on binary operation

G a homotopy associative H-space. Fix an associating homotopy $M(t;x,y,z)$ such that $M(0;x,y,z) = x(yz)$ and $M(1;x,y,z) = (xy)z$. With respect to this homotopy, the following two homotopies are not homotopic in general.



The existence of a homotopy between these homotopies is the first higher homotopy associativity.

Higher homotopy structures on binary operation

Similarly, there are more higher homotopy associativities and higher homotopy commutativities.

	structures	parameterizations
associativity	A_n-space	associahedra
homomorphism	A_n-map	multiplihedra, cubes
commutativity	Williams C_k-space	permutohedra
	Sugawara C_k-space	cubes, resultohedra
	k-fold loop space	little k-cubes

Stasheff's A_∞-theory

2. Stasheff's A_∞-theory
   - A_n-maps
   - Classifying space and projective spaces

A_n-maps

G, H topological monoids, $f\colon G \to H$ a basepoint preserving map.

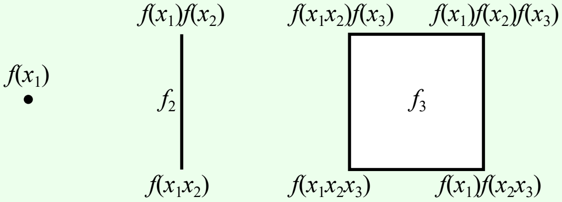
Definition (Sugawara 1960 ($n = \infty$), Stasheff 1963)

A family of maps $\{f_i\colon [0,1]^{\times(i-1)} \times G^{\times i} \to H\}$ is called an A_n-form on f if the following conditions are satisfied:

1. $f_1 = f$,
2. $f_i(t_1,\dots,t_{i-1};x_1,\dots,x_i)$
$$= \begin{cases} f_{i-1}(t_1,\dots,\hat{t}_k,\dots,t_{i-1};x_1,\dots,x_kx_{k+1},\dots,x_i) & t_k = 0 \\ f_k(t_1,\dots,t_{k-1};x_1,\dots,x_k)f_{i-k}(t_{k+1},\dots,t_i;x_{k+1},\dots,x_i) & t_k = 1, \end{cases}$$
3. "unit conditions".

A pair $(f,\{f_i\})$ is called an A_n-map. We denote the space of A_n-maps by $\mathcal{A}_n(G,H)$. If the underlying map f is a homotopy equivalence, $(f,\{f_i\})$ is said to be an A_n-equivalence. Two topological monoids are said to be A_n-equivalent if there is an A_n-equivalence between them.

A_n-maps



A map $f : G \to H$ is a...

homomorphism $\Rightarrow$ A_∞-map $\Rightarrow \cdots$

$\cdots \Rightarrow$ A_n-map $\Rightarrow$ A_{n-1}-map $\Rightarrow \cdots$

$\cdots \Rightarrow$ A_2-map $\Rightarrow$ basepoint preserving map.

Topological monoids G and H are...

isomomorphic $\Rightarrow$ A_∞-equivalent $\Rightarrow \cdots$

$\cdots \Rightarrow$ A_n-equivalent $\Rightarrow$ A_{n-1}-equivalent $\Rightarrow \cdots$

$\cdots \Rightarrow$ A_2-equivalent $\Rightarrow$ homotopy equivalent.

Classifying space and projective spaces

G a topological monoid. There exists a sequence of spaces

$$* = B_0G \subset \Sigma G = B_1G \subset B_2G \subset \cdots \subset BG$$

such that if G is homotopy invertible, ΩBG is A_∞-equivalent to G.
B_nG is called the n-th projective space of G.
BG is called the classifying space of G.
They are given by the bar construction or the Dold–Lashof construction.

Proposition (Sugawara 1960 ($n = \infty$), Stasheff 1963)

There exists a map $B_n\colon \mathcal{A}_n(G,H) \to \mathrm{Map}_0(B_nG, B_nH)$ for each n making the following diagram commute:

$$\begin{array}{ccccccc}
B_0G & \longrightarrow & B_1G & \longrightarrow & \cdots & \longrightarrow & B_nG \\
\downarrow & & \downarrow{\scriptstyle B_1(f,\{f_i\})=\Sigma f} & & & & \downarrow{\scriptstyle B_n(f,\{f_i\})} \\
B_0H & \longrightarrow & B_1H & \longrightarrow & \cdots & \longrightarrow & B_nG
\end{array}$$

Classifying space and projective spaces

The converse is also known, which is convenient to study the obstruction to the existence of A_n-forms.

Proposition (Stasheff 1963)

Suppose H is a homotopy invertible (=grouplike) topological monoid. A basepoint preserving map $f : G \to H$ admits an A_n-form if there exists a dashed arrow such that the resulting square is commutative:

$$\begin{array}{ccc} \Sigma G & \xrightarrow{\Sigma f} & \Sigma H \\ \downarrow & & \downarrow \\ B_n G & \dashrightarrow & BH \end{array}$$

Corollary

Homotopy invertible topological monoids G and H are A_∞-equivalent if and only if their classifying spaces BG and BH are homotopy equivalent.

Applications to Lie groups

3. Applications to Lie groups
   - Gauge groups and mapping spaces
   - Finiteness of gauge groups
   - Extension of evaluation fiber sequence
   - Higher commutativity of Lie groups

Gauge groups and mapping spaces

G a topological group, $P \to B$ a principal G-bundle.

Definition

A map $f\colon P \to P$ is called an automorphism if f is G-equivariant and covers the identity on B.
The gauge group $\mathcal{G}(P)$ is the space of automorphisms on P.

P is a pullback of the universal bundle EG along a map $\alpha\colon B \to BG$.
Such a map α is called the classifying map of P.

$$\begin{array}{ccc} P & \longrightarrow & EG \\ \downarrow & & \downarrow \\ B & \xrightarrow{\alpha} & BG \end{array}$$

Proposition (Gottlieb 1972)

Let α be a classifying map of P. Then $B\mathcal{G}(P)$ is homotopy equivalent to the path component $\mathbf{Map}(B, BG)_\alpha$ containing α.

Finiteness of gauge groups

G a compact connected Lie group, B a finite CW complex.

Problem (Crabb–Sutherland 2000)

Consider all principal G-bundles P over B: is the number of homotopy types of $\mathcal{G}(P)$, or of $B\mathcal{G}(P)$, finite?

Theorem (Crabb–Sutherland 2000 ($n \leq 2$), T 2012 ($2 < n < \infty$))

Let n be a positive integer. Then, as P ranges over all principal G-bundles over B, the number of A_n-equivalence types of $\mathcal{G}(P)$ is finite.

Theorem (Kishimoto–T 2016)

Let G be a compact connected simple Lie group. Then, as P ranges over all principal G-bundles over the d-dimensional sphere S^d, the number of A_∞-equivalence types of $\mathcal{G}(P)$ is infinite if and only if $\#\pi_d(BG) = \infty$.

Extension of evaluation fiber sequence

Any map $f\colon X \to Y$ can be replaced by a fibration up to homotopy equivalence:

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & \tilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}\text{ fibration}} \\ Y & = & Y \end{array}$$

The fiber F of $\tilde{f}$ is called the homotopy fiber of f. Taking the homotopy fiber repeatedly, we obtain the homotopy fiber sequence:

$$\cdots \to \Omega F \to \Omega X \xrightarrow{\Omega f} \Omega Y \to F \to X \xrightarrow{f} Y.$$

In general, this sequence does not extends to the right.
For a basepoint preserving map $f\colon X \to Y$, we have the evaluation fiber sequence

$$\cdots \to \Omega\,\mathrm{Map}(X,Y)_f \to \Omega Y \to \mathrm{Map}_0(X,Y)_f \to \mathrm{Map}(X,Y)_f \xrightarrow{\text{evaluation}} Y.$$

Extension of evaluation fiber sequence

Theorem (T 2016)

Let G, H be topological monoids. If H is homotopy invertible, then the following composite is a homotopy equivalence:

$$\mathcal{A}_n(G,H) \xrightarrow{B_n} \mathbf{Map}_0(B_nG, B_nH) \to \mathbf{Map}_0(B_nG, BH).$$

Corollary (T 2016)

Let G be a topological monoid and X a path connected pointed space. Then there is a natural homotopy equivalence

$$\mathcal{A}_n(G,\Omega X) \simeq \mathbf{Map}_0(B_nG, X).$$

Extension of evaluation fiber sequence

Let $i_n\colon B_nG \to BG$ be the inclusion and $\mathcal{A}_n(G,G)_{\mathrm{inn}}$ be the union of path components containing inner automorphisms.

Corollary (T 2016)

Let G be a topological group. Then there is a homotopy fiber sequence

$$\mathrm{Map}_0(B_nG, BG)_{i_n} \to \mathrm{Map}(B_nG, BG)_{i_n} \xrightarrow{\text{evaluation}} BG \to B\mathcal{A}_n(G,G)_{\mathrm{inn}}.$$

Let E_nG be the restriction of the universal bundle EG over B_nG. Then $B\mathcal{G}(E_nG) \simeq \mathrm{Map}(B_nG, BG)_{i_n}$.

Conjecture (T 2016)

If G is a compact connected simple Lie group, then $\mathcal{G}(E_nG)$ is not a double loop space when $1 \leq n < \infty$.

This implies that the above evaluation fiber sequence no longer extends for such G and n.

Higher homotopy commutativity of Lie groups

Theorem (Hubbuck 1969)

Let G be a homotopy commutative H-space having the homotopy type of a finite connected CW complex. Then G is homotopy equivalent to a torus.

Definition (Williams 1970)

A topological monoid G is said to be a Williams C_k-space if it admits a "C_k-form".

G a compact connected simple Lie group,
$H^*(G;\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_1,\dots,x_\ell)$, $x_i \in H^{2n_i-1}$, $n_1 \le \cdots \le n_\ell$.

Theorem (McGibbon 1984 ($k = 2$), Saumell 1995 ($k > 2$))

1. If $p > kn_\ell$, then $G_{(p)}$ is a Williams C_k-space.
2. If $p < kn_\ell$, then $G_{(p)}$ is not a Williams C_k-space except in the case $(G,p) = (\mathrm{Sp}(2),3), (\mathrm{G}_2,5)$.

Higher homotopy commutativity of Lie groups

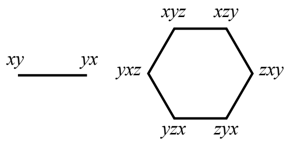
Definition (Sugawara 1960 ($k = \infty$), McGibbon 1989 ($k < \infty$))

A topological monoid G is said to be a Sugawara C_k-space if the multiplication $G \times G \to G$ admits an A_k-form.

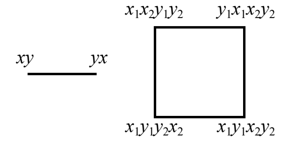
Proposition

G is a Sugawara C_k-space $\Rightarrow$ G is a Williams C_k-space.

Williams C_k-space



Sugawara C_k-space



Theorem (Hasui–Kishimoto–T)

Let G be a compact connected simple Lie group. Then, $G_{(p)}$ is a Sugawara C_k-space if $p > kn_\ell$.

Higher homotopy commutativity of Lie groups

Corollary (Hasui–Kishimoto–T)

Let G be a compact connected simple Lie group. Then, the following hold.

1. If $p > (k+n)n_\ell$, then $\mathcal{G}(E_nG)_{(p)}$ is a Sugawara C_k-space.
2. If $(n+1)n_\ell < p < (k+n)n_\ell$, then $\mathcal{G}(E_nG)_{(p)}$ is not a Williams C_k-space.

This implies that $\mathcal{G}(E_nG)_{(p)}$ is not a double loop space if $n < \infty$.

Outline of the proof of Theorem

G is a Sugawara C_k-space if and only if the wedge sum of the inclusions $B_kG \vee B_kG \to BG$ extends over the union $\bigcup_{i+j=k} B_iG \times B_jG$. It is sufficient to observe that the obstruction of this extension problem vanishes.

Related works:

- Finiteness of A_n-equivalence types of gauge groups, J. London Math. Soc. 85 (2012), 142-164.
- (with D. Kishimoto) Infiniteness of A_∞-types of gauge groups, J. Topol. 9 (2016), 181-191.
- Mapping spaces from projective spaces, Homology, Homotopy Appl. 18 (2016), 173-203.
- (with S. Hasui and D. Kishimoto) Higher homotopy commutativity in localized Lie groups and gauge groups, preprint, arXiv:1612.08816.