Higher homotopy commutativity in localized Lie groups and gauge groups

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Homotopy Theory Symposium 2017
Takamatsu
24 Nov. 2017

This talk is based on the joint work with Sho Hasui (Osaka pref. Univ.) and Daisuke Kishimoto (Kyoto Univ.):
Higher homotopy commutativity in localized Lie groups and gauge groups, arXiv:1612.08816.

Outline

1. Results
   - Homotopy commutativity of Lie groups
   - Higher homotopy commutativity of Lie groups
   - Gauge groups
   - Extension of evaluation fiber sequence
   - Higher homotopy commutativity of gauge groups

Homotopy commutativity of Lie groups

- G: a topological group.

Definition

G is homotopy commutative

$\overset{\text{def}}{\Leftrightarrow}$ two maps $G \times G \to G$ given by $(x, y) \mapsto xy$ and $(x, y) \mapsto yx$ are homotopic

$\Leftrightarrow$ the commutator map $(x, y) \mapsto xyx^{-1}y^{-1}$ is null-homotopic.

Theorem (Araki–James–Thomas, 1960)

If a compact connected Lie group G is homotopy commutative, then G is a torus.

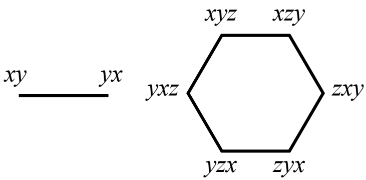
Higher homotopy commutativity of Lie groups

- Williams C_k-space is introduced by Williams (1969).
- Sugawara C_k-space is introduced by Sugawara ($k = \infty$, 1960) and McGibbon ($k < \infty$, 1989).

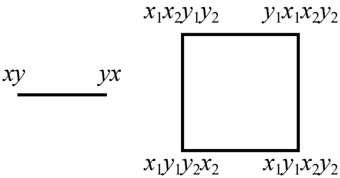
Proposition

- Sugawara C_k-space $\Rightarrow$ Williams C_k-space.
- Williams (or Sugawara) C_2-space $\Leftrightarrow$ homotopy commutative.
- G is a Sugawara C_∞-space $\Leftrightarrow$ BG is an H-space.

Williams C_k-space



Sugawara C_k-space



Higher homotopy commutativity of Lie groups

- G: a compact connected simple Lie group
  $G_{(0)} \simeq S^{2n_1-1}_{(0)} \times S^{2n_2-1}_{(0)} \times \cdots \times S^{2n_\ell-1}_{(0)}$ with $2 = n_1 \leq n_2 \leq \cdots \leq n_\ell$.
- p: an odd prime.

Proposition (McGibbon, 1982 and Saumell, 1995)

Unless (G, p, k) is in the exceptional cases, the following hold:

1. $p > kn_\ell \Rightarrow G_{(p)}$ is a Williams C_k-space,
2. $p < kn_\ell \Rightarrow G_{(p)}$ is not a Williams C_k-space.

Exceptional cases:

1. $(G, p, k) = (\mathrm{Sp}(2), 3, 2)$: $\mathrm{Sp}(2)_{(3)}$ is homotopy commutative.
2. $(G, p, k) = (\mathrm{G}_2, 5, k)$ for $k = 2, 3, 4$: $(\mathrm{G}_2)_{(5)}$ is homotopy commutative.

- The case when $(G, p, k) = (\mathrm{G}_2, 5, k)$ for $k = 3, 4$ had been open.

Higher homotopy commutativity of Lie groups

Theorem A (Hasui–Kishimoto–T)

Let G be a compact connected simple Lie group. Then, $G_{(p)}$ is a Sugawara C_k-space if $p > kn_\ell$.

Theorem (Hasui–Kishimoto–T)

$(\mathrm{G}_2)_{(5)}$ is not a Williams C_3-space.

- The latter follows from some computation of a higher Whitehead product. But the proof is omitted in these slides.

Gauge groups

- P: a principal G-bundle over B classified by $\alpha : B \to BG$.

Definition

The gauge group $\mathcal{G}(P)$ is the topological group of G-equivariant maps $P \to P$ covering the identity map $B \to B$.

- $\mathrm{ad}\, P := P \times G/\sim$ with $(ug, x) \sim (u, gxg^{-1})$: the adjoint bundle.

Proposition

- $\Gamma(\mathrm{ad}\, P) \cong \mathcal{G}(P)$ as topological groups.
- The path component $\mathrm{Map}(B, BG)_\alpha$ is homotopy equivalent to $B\mathcal{G}(P)$.
- $\mathrm{ad}\, EG \simeq LBG$ as fiberwise A_∞-spaces.

Extension of evaluation fiber sequence

- $\mathcal{A}_n(G,G)_{\mathrm{inn}}$: the space of A_n-maps homotopic to the inner automorphisms on a topological group G.
- $i_n\colon B_nG \to BG$: the inclusion of the n-th projective space into the classifying space.
- $\mathrm{Map}_*(B_nG, BG)_{i_n} := \mathrm{Map}_*(B_nG, BG) \cap \mathrm{Map}(B_nG, BG)_{i_n}$.

Theorem (T, 2016)

- $\mathcal{A}_n(G,G)_{\mathrm{inn}}$ is a grouplike A_∞-space,
- $\mathcal{A}_n(G,G)_{\mathrm{inn}} \simeq \mathrm{Map}_*(B_nG, BG)_{i_n}$,
- the following is a homotopy fiber sequence:
  $$\mathrm{Map}_*(B_nG, BG)_{i_n} \to \mathrm{Map}(B_nG, BG)_{i_n} \xrightarrow{\mathrm{ev}} BG \to B\mathcal{A}_n(G,G)_{\mathrm{inn}}.$$

Question

Does this homotopy fiber sequence no longer extend to the right for $n < \infty$?

Higher homotopy commutativity of gauge groups

Theorem B (Hasui–Kishimoto–T)

Let G be a compact connected simple Lie group. Then, the following hold.

1. If $p > (k+n)n_\ell$, then $\mathcal{G}(E_nG)_{(p)}$ is a Sugawara C_k-space.
2. If $(n+1)n_\ell < p < (k+n)n_\ell$, then $\mathcal{G}(E_nG)_{(p)}$ is not a Williams C_k-space.

Corollary (Hasui–Kishimoto–T)

If G is as above, then $\mathrm{Map}(B_nG, BG)_{i_n} \simeq B\mathcal{G}(E_nG)$ does not deloop for $n < \infty$.

2. Proof of Theorem A

- A_n-spaces
- Projective spaces
- Projective spaces of product A_n-space
- Sugawara C_k-space
- p-regularity of Lie groups as A_k-spaces
- Proof of Theorem A

A_n-spaces

"Definition"

- A based space G is an A_n-space

  $\overset{\text{def}}{\Leftrightarrow}$ G is equipped with a unital binary operation and higher homotopy associativity up to multiplication of n elements (A_n-form).

- A based map $f: G \to H$ between A_n-spaces is an A_n-map

  $\overset{\text{def}}{\Leftrightarrow}$ f is equipped with certain higher homotopy (A_n-form) compatible with A_n-forms of G and H.

- An A_n-map is an A_n-equivalence if it is a homotopy equivalence.

Example

- G is a topological group $\Rightarrow$ G is a grouplike A_∞-space.
- G and H are A_n-spaces $\Rightarrow$ $G \times H$ is an A_n-space equipped with a canonical A_n-form.

Projective spaces

- If G is an A_n-space, then there exists a functorial construction of spaces (projective spaces)
  $* = B_0G \subset \Sigma G = B_1G \subset B_2G \subset \cdots \subset B_nG$.
- $n = \infty \Rightarrow BG := \mathbf{colim}_n\, B_nG$: the classifying space of G.

Proposition (Stasheff, 1963)

- G: an A_n-space.
- H: a grouplike A_∞-space.
- $f : G \to H$: a based map.

Then, f admits an A_n-form

$\Leftrightarrow$ the composite $\Sigma G \xrightarrow{\Sigma f} \Sigma H \to BH$ extends over B_nG.

Projective spaces of product A_n-space

Lemma (Iwase, 1998 and Hasui–Kishimoto–T)

- $G_1, \ldots, G_\ell$: A_k-spaces.
- H: a grouplike A_∞-space.
- $f : G_1 \times \cdots \times G_\ell \to H$: a based map.

Then, f admits an A_k-form

⇔ the composite $\Sigma G_1 \vee \cdots \vee \Sigma G_\ell \xrightarrow{(\Sigma f_1, \ldots, \Sigma f_\ell)} \Sigma H \to BH$ extends over $\bigcup_{k_1+\cdots+k_\ell=k} B_{k_1}G_1 \times \cdots \times B_{k_\ell}G_\ell$, where $f_i = f|_{G_i}$.

Sugawara C_k-space

Definition (Sugawara, 1960 and McGibbon, 1984)

An A_k-space G is a Sugawara C_k-space

$\overset{\text{def}}{\Leftrightarrow}$ the multiplication $G \times G \to G$ is an A_k-map.

Proposition (Hemmi–Kawamoto, 2011)

A topological group G is a Sugawara C_k-space

$\Leftrightarrow$ $B_kG \vee B_kG \xrightarrow{(i_k, i_k)} BG$ extends over $\bigcup_{k_1+k_2=k} B_{k_1}G \times B_{k_2}G$.

- G is a Williams C_k-space
  $\Leftrightarrow$ $(\Sigma G)^{\vee k} \xrightarrow{(i_1, \dots, i_1)} BG$ extends over $(\Sigma G)^{\times k}$.

p-regularity of Lie groups as A_k-spaces

- G: a compact connected simple Lie group
- p: an odd prime.

Lemma (Stasheff, 1963)

$S^{2n-1}_{(p)}$ is an A_{p-1}-space such that $B_{p-1}S^{2n-1}_{(p)} \simeq S^{2n}_{(p)} \cup e^{4n}_{(p)} \cup \cdots \cup e^{2(p-1)n}_{(p)}$.

Lemma (Hasui–Kishimoto–T)

$p > kn_\ell \Rightarrow G_{(p)} \simeq S^{2n_1-1}_{(p)} \times \cdots \times S^{2n_\ell-1}_{(p)}$ as A_k-spaces.

Proof

- $\bigcup_{k_1+\cdots+k_\ell=k} B_{k_1}S^{2n_1-1}_{(p)} \times \cdots \times B_{k_\ell}S^{2n_\ell-1}_{(p)}$ consists of cells of even dimensions $\leq 2kn_\ell$.
- $\pi_i(BG_{(p)}) = 0$ for odd $i < 2p+1$.

Proof of Theorem A

Theorem A (Hasui–Kishimoto–T)

Let G be a compact connected simple Lie group. Then, $G_{(p)}$ is a Sugawara C_k-space if $p > kn_\ell$.

Proof

- There is no obstruction to extending the map $(S^{2n_1}_{(p)} \vee \cdots \vee S^{2n_\ell})^{\vee 2} \to BG_{(p)}$ over $\bigcup_{k_1+\cdots+k_\ell+k'_1+\cdots+k'_\ell=k} B_{k_1}S^{2n_1-1} \times \cdots \times B_{k_\ell}S^{2n_\ell-1} \times B_{k'_1}S^{2n_1-1} \times \cdots \times B_{k'_\ell}S^{2n_\ell-1}$.

Proof of Theorem B

3. Proof of Theorem B
   - Higher homotopy commutativity of adjoint bundles
   - Higher homotopy commutativity of gauge groups

Higher homotopy commutativity of adjoint bundles

- P: a principal G-bundle over B classified by $\alpha\colon B \to BG$.
- $E_nG := EG|_{B_kG}$: the universal bundle restricted over B_nG.

Proposition (Kishimoto–Kono, 2010)

$\mathbf{ad}\,P$ is trivial as a fiberwise A_k-space

⇔ the map $(\alpha, i_k)\colon B \vee B_kG \to BG$ extends over $B \times B_kG$.

Theorem (Hasui–Kishimoto–T)

G is a Sugawara C_{n+k}-space

⇒ $\mathbf{ad}\,E_nG$ is trivial as a fiberwise A_k-space.

Proof

- G is a Sugawara C_{n+k}-space ⇒ $B_nG \vee B_kG \to BG$ extends over $B_nG \times B_kG$.

Higher homotopy commutativity of gauge groups

- G: a compact connected simple Lie group.

Theorem B (Hasui–Kishimoto–T)

1. If $p > (k+n)n_\ell$, then $\mathcal{G}(E_nG)_{(p)}$ is a Sugawara C_k-space.
2. If $(n+1)n_\ell < p < (k+n)n_\ell$, then $\mathcal{G}(E_nG)_{(p)}$ is not a Williams C_k-space.

Corollary (Hasui–Kishimoto–T)

Suppose $p > (k+n)n_\ell$. Then the p-localized gauge group $\mathcal{G}(P)_{(p)}$ of a principal G-bundle P over B is a Sugawara C_k-space if $\mathbf{cat}\, B \leq n$.

Proof

- $\mathbf{cat}\, B \leq n \Rightarrow$ the classifying map $B \to BG$ of P factors through B_nG up to homotopy.

Problems related to (higher) homotopy commutativity of Lie groups:

- Homotopy nilpotency of Lie groups (Kaji–Kishimoto, 2010) and gauge groups (Crabb–Sutherland–Zhang, 1999).
- (Higher) Samelson products in G.
  - Spheres to $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ (Bott, 1960).
  - Spheres to exceptional groups (Hamanaka–Kono, 2010 and Hasui–Kishimoto–Ohsita, 2014).
  - Spheres to $\mathrm{SO}(2n)$ (Kishimoto–T).
  - Sphere bundles over spheres to quasi-p-regular Lie groups (Hasui–Kishimoto–Miyauchi–Ohsita).
  - Classifications of homotopy types of gauge groups.
- A_n-triviality of adjoint bundles of principal $\mathrm{SU}(2)$-bundles (T).
- McGibbon's $N(G) \in \mathbb{Z}_{>0}$ (1984):
  $x \mapsto x^n$ is an H-map $\Leftrightarrow n^2 - n \equiv 0 \bmod N(G)$.
  - Arkowitz–Curjel, 1967, McGibbon, 1984, Theriault, 2013 and Russhard, 2015.