2017.08.04.Fri

Tsutaya

Homotopy theoretic classifications of gauge groups — No. 1

(Young Researchers ... 2017 @ Kyoto).

## §1 Gauge groups

History — James 1963, Gottlieb 1972

Homotopy theory of bundle maps

$\hookrightarrow$ Applications to gauge theory

Atiyah, Jones, Bott, ...

$\downarrow$

Masbaum 1991, Kono 1991

Homotopytypes of $\mathcal{G}(P)$ and $B\mathcal{G}(P)$

for $SU(2)$-bdl.s $/S^4$.

$\downarrow$

Various classifications 1992 ~

Kono, Sutherland, Tsukuda, Crabb, ....

$P \to B$ : prin. $G$-bdl.

$$\mathcal{G}(P) = \left\{ f: P \to P \text{ over } B : G\text{-equivar.} \right\} : \text{gauge gp.}$$

evaluation fiber seq.

$$\mathcal{G}(P) \to G \to \mathrm{Map}_0(B, BG)_\alpha \to \underbrace{\mathrm{Map}(B, BG)_\alpha}_{\text{path comp.}} \to BG$$

$\alpha : B \to BG$ : classifying map of $P$.

## §2 Classification of homotopy types.

$\varepsilon : S^d \to BG$ : fix. (not divisible by $\geq 2$).

Estimate than above: $P_k \to S^d$ : classified by $k\varepsilon$ $(k \in \mathbb{Z})$.

Prop. (Whitehead, 1946).

$$G \xrightarrow{\partial_{k\varepsilon}} \underbrace{\mathrm{Map}_0(S^d, BG)_\alpha}_{\text{iterated loop sp.}} \xrightarrow{\simeq} \Omega^d BG \simeq \Omega^{d-1} G$$

is homotopic to the adjoint of the Samelson prod.

$$\langle k\varepsilon, id_G \rangle : S^{d-1} \wedge G \to G.$$

$\square$

continued

**Prop.** (Hamanaka-Kono, Theriault)

Suppose $N\langle k\varepsilon, id_G\rangle = 0$ in $[S^{d-1}\wedge G, G]$.

Then, $\underbrace{(N,k)}_{\text{GCD}} = (N,\ell) \Rightarrow \mathcal{G}(P_k) \simeq \mathcal{G}(P_\ell)$. $\square$

**Estimate from below**

Find appropriate htpy invariants.

**Ex.**

(Kono, 1991)

htpy exact seq. and $\partial_{k\varepsilon}$

$G = SU(2)$, $B = S^4 \Rightarrow \pi_2(\mathcal{G}(P_k)) \cong \mathbb{Z}/(12,k)\mathbb{Z}$.

(Hamanaka-Kono, 2006).

unstable K-theory and Chern character

$G = SU(n)$, $B = S^4$:

$\Rightarrow \# [\Sigma^{2n-6}\mathbb{C}P^2, \mathcal{G}(P_k)] = \dfrac{(n-2)!\,(n(n^2-1),k)\,(n-1)!\,(n,k)}{2}$.

(Kishimoto-Theriault-T, 2017)

technique of unstable homotopy theory

$G = G_2$, $B = S^4$, $C_k$ : "6-skeleton" of $\mathcal{G}(P_k)$

They studied $H^*(C_k)$. $\square$

## §3 Classification of $A_n$-types.

$G, G'$ : top. gp.s

**Def.** (Sugawara 1960, Stasheff 1963)

$f : G \to G'$ : ptd. map

is an <u>$A_n$-map</u> if $f$ admits an <u>$A_n$-form</u>

$\{f_i : I^{\times(i-1)} \times G^{\times i} \to G'\}_{i=1}^n$

s.t. (i) $f_1 = f$.

(ii) $f_i(t_1,\dots,t_{i-1}; g_1,\dots,g_i) = \begin{cases} f_{i-1}(t_1,\dots,t_{k-1},t_{k+1},\dots,t_{i-1}; g_1,\dots,g_kg_{k+1},\dots,g_i) & t_k=0 \\ f_k(t_1,\dots,t_{k-1}; g_1,\dots,g_k)f_{i-k}(t_{k+1},\dots,t_{i-1}; g_{k+1},\dots,g_i) & t_k=1 \end{cases}$

If an $A_n$-map $f$ is a htpy eq., $f$ is called an <u>$A_n$-equivalence</u>. $A_n$-eq. is an equivalence relation on top. gp.s. $\square$

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<u>Rem.</u>

$A_1$-eq. $\Leftrightarrow$ htpy. eq.

$A_2$-eq. $\Leftrightarrow$ H-eq. (eq. as H-sp.s).

$A_\infty$-eq. $\Leftrightarrow$ the classifying sp.s are htpy. eq. $\square$

<u>Thm</u> (Crabb-Sutherland $(n=1,2)$, 2000; T $(n\geq 3)$, 2012)

G: cpt. conn. Lie gp., B: fin. cpx., $1\leq n<\infty$.

$\Rightarrow \#(\{\mathcal{G}(P) \mid P\to B\text{: prin. }G\text{-bdl.}\}/A_n\text{-eq.})<\infty$. $\square$

<u>Ex</u>

$G=SU(2)$, $B=S^4$

(Kono, 1991)

$$\mathcal{G}(P_k)\simeq_{A_1}\mathcal{G}(P_\ell)\Leftrightarrow (12,k)=(12,\ell) \quad 12=2^2 3$$

(Crabb-Sutherland, 2000)

$$\mathcal{G}(P_k)\simeq_{A_2}\mathcal{G}(P_\ell)\Leftrightarrow (180,k)=(180,\ell) \quad 180=2^2\cdot 3^2\cdot 5$$

(T, 2015)

$$\underbrace{\mathcal{G}(P_k)[\tfrac{1}{2}]}_{\text{localized away from 2}}\simeq_{A_n}\mathcal{G}(P_\ell)[\tfrac{1}{2}]\Leftrightarrow \Big(\prod_{p:\text{ odd prime}} p^{[\frac{2n}{p-1}]},k\Big)=\Big(\prod_{p:\text{ odd prime}} p^{[\frac{2n}{p-1}]},\ell\Big).$$

(Tsukuda, 2001)

$$\mathcal{G}(P_k)\simeq_{A_\infty}\mathcal{G}(P_\ell)\Leftrightarrow |k|=|\ell|$$

$$(\text{or } (k,0)=(\ell,0)). \quad \square$$

In [Tsukuda, 2001] and [T, 2015], the extension problem

$$\begin{array}{ccc} S^4\vee \mathbb{H}P^n & \xrightarrow{k\vee id} & S^4\vee\mathbb{H}P^\infty \\ \downarrow & & \downarrow (incl, id) \\ S^4\times\mathbb{H}P^n & \overset{\exists ?}{\dashrightarrow} & \mathbb{H}P^\infty_{(p)} \end{array}$$

is studied. If $\dashrightarrow$ exists, $\mathcal{G}(P_k)_{(p)}\simeq_{A_n}\mathcal{G}(P_0)_{(p)}$.

<u>Ex</u>

(Kishimoto-T, 2015)

G: simple cpt. conn., $\#\pi_d(BG)=\infty$

$\Rightarrow \#(\{\mathcal{G}(P)\mid P\to S^d\text{: prin. }G\text{-bdl.}\}/A_\infty\text{ eq.})=\infty$. $\square$

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Open problems (remaining rank 2 case over $S^4$).

- $G = G_2$, $B = S^4$

  $(84, k) = (84, l) \overset{?}{\Rightarrow} \mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_l)_{(p)}$ for any prime $p$ and $p=0$.

  ("$(168, k) = (168, l) \Rightarrow \cdots$" OK! cf. Kishimoto-Theriault-T, 2017)

- (genus problem)

  Is there any case when $\mathcal{G}(P_k)_{(p)} \simeq \mathcal{G}(P_l)_{(p)}$ for any prime $p$ and $p=0$ but $\mathcal{G}(P_k) \not\simeq \mathcal{G}(P_l)$.

- ($A_n$-types)

  Classification of $A_n$-types of $\mathcal{G}(P_k)$ for $G \neq SU(2)$.

- (non-connected case)

  Classification of htpy types of $\mathcal{G}(P_k)$ for $G$ : non-conn.

  (e.g. $G = O(n)$, $Pin(n)$)