2017.02.13. Mon.

Higher homotopy commutativity in localized Lie groups and gauge groups. (joint work with Sho Hasui and Daisuke Kishimoto)

No. 1

Malaga Meeting

§1 Intro.

G: htpy assoc. H-sp. with htpy inversion: $G \to G$.

G is <u>htpy comm</u>

$\overset{def}{\Leftrightarrow}$ $G \wedge G \longrightarrow G$: null-htpic.

$(x, y) \longmapsto xyx^{-1}y^{-1}$

<u>Thm</u> (Hubbuck, 1969)

G: fin. conn H-sp. $\Rightarrow$ $G \simeq T^n$ $\square$

<u>p-local case</u>

G: cpt. conn. simple Lie gp.

$H^*(G;\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_1, \dots, x_\ell)$, $|x_i| = 2n_i - 1$, $n_1 \leq \cdots \leq n_\ell$.

<u>Thm</u> (McGibbon, 1984, Saumell, 1995, HKT).

(1). $p > kn_\ell \Rightarrow G_{(p)}$: Williams C_k-sp. (W-C_k-sp.)

(2) $p < kn_\ell \Rightarrow G_{(p)}$: NOT W-$C_k$-sp.

or $(G, p, k) = (Sp(2), 3, 2), (G_2, 5, 2)$.

(3) $Sp(2)_{(3)}$: W-C_2-sp. but NOT W-C_3-sp.

(4) $G_{2\,(5)}$: W-C_2-sp. but <u>NOT W-C_3-sp.</u>

HKT. $\square$

<u>Main results</u>

<u>Thm A</u> Higher htpy. commutativity of $G_{(p)}$ in the sense of Sugawara.

<u>Thm B</u> ——〃—— of gaugegroups ——〃——.

§2. Higher htpy. commutativity. (HHC)

G: top. gp.

<u>Def-Prop.</u> (Williams, 1969)

G: <u>Williams C_k-sp.</u> (W-C_k-sp)

$\overset{def}{\Leftrightarrow}$ $\exists \{Q_i : \underset{\sim}{P_i} \times G^{\times i} \to G\}_{i=1}^{k}$: "C_k-form"

i-th permutohedron.

$$\Leftrightarrow \quad \begin{array}{ccc} \Sigma G^{\vee k} & \longrightarrow & BG \\ \cap & \nearrow_{\exists} & \\ (\Sigma G)^{\times k} & & \end{array}$$

wedge sum of inclusions. $\square$

generalized

$\Leftrightarrow$ Any higher Whitehead prod. in BG contains 0 up to the order k.

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<u>Def.-Prop.</u> (Sugawara 1960, McGibbon 1989).

G : <u>Sugawara C_k-sp.</u> (S-C_k-sp.)

$\overset{\text{def}}{\Longleftrightarrow}$ the multiplication $G\times G\to G$ is an "A_k-map."

$\Longleftrightarrow$

$$\begin{tikzcd} \Sigma G\vee\Sigma G \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & BG \\ \bigcup\limits_{i+j=k} \underbrace{B_iG}_{i\text{-th proj.sp.}}\times B_jG \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$$\left(\begin{array}{l} B_0G\subset B_1G\subset B_2G\subset\cdots\subset BG \\ \;\;\| \qquad\;\; \| \\ \;\;* \qquad\; \Sigma G \end{array}\right)$$

$\square$

<u>Def.</u> (Kishimoto-Kono 2010, HKT).

G : <u>$C(k_1,\dots,k_r)$-sp.</u> $(k_1,\dots,k_r\geq 1)$.

$\overset{\text{def}}{\Longleftrightarrow}$

$$\begin{tikzcd} (\Sigma G)^{\vee r} \arrow[r] \arrow[d, hook] & BG \\ B_{k_1}G\times\cdots\times B_{k_r}G \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$\square$

<u>Prop.</u>

G: S-C_k-sp. $\Rightarrow$ G: $C(k_1,\dots,k_r)$-sp. $\Rightarrow$ G: W-C_k-sp.

G: S-C_2-sp $\Leftrightarrow$ G : W-C_2-sp. $\Leftrightarrow$ G: htpy. comm. ($k=k_1+\cdots+k_r$). $\square$

§3 H.H.C. of localized Lie gp.s

<u>Thm A</u> (HKT)

$p>kn_\ell \Rightarrow G_{(p)}$: S-C_k-sp. $\square$

<u>Outline of proof</u>

(i) $G_{(p)}\simeq S^{2n_1-1}_{(p)}\times\cdots\times S^{2n_\ell-1}_{(p)}$ as A_k-spaces.

$\rightsquigarrow$ We can replace B_iG by $\bigcup\limits_{i_1+\cdots+i_\ell=i} B_{i_1}S^{2n_1-1}\times\cdots\times B_{i_\ell}S^{2n_\ell-1}$.

(ii) $B_n S^{2m-1}_{(p)}\simeq \underbrace{J_n S^{2m}_{(p)}}_{n\text{-th James construction}}$ for $n<p$.

(iii) $\pi_i(BG_{(p)})=0$ for odd $i<2p+1$. $\blacksquare$

§4. H.H.C. of localized gauge gp.s

P : prin. G-bdl. $/B$.

$\mathcal{G}(P) = \left\{ \underset{:G\text{-equivar.}}{f : P \to P} \;\middle|\; \begin{tikzcd} P \arrow[rr, "f"] \arrow[dr] & & P \arrow[dl] \\ & B & \end{tikzcd} \right\}$: gauge gp.

$$\begin{tikzcd} E_nG \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & EG \arrow[d] \\ \underset{n\text{-th proj. sp.}}{B_nG} \arrow[r, "i_n"] & \underset{\text{univ. bdl.}}{BG} \end{tikzcd}$$

$\mathrm{ad}\,P := P \times G/\sim \quad \left((ug, x) \sim (u, gxg^{-1})\right)$: adjoint bdl.

<u>Fact</u> $\Gamma(\mathrm{ad}\,P) \cong \mathcal{G}(P)$: naturally.

<u>Thm B</u> (HKT)

(1) $p > (n+k)n_\ell \Rightarrow \mathcal{G}(E_nG)_{(p)}$: S-C_k-sp.

(2) $(n+1)n_\ell < p < (n+k)n_\ell \Rightarrow \mathcal{G}(E_nG)_{(p)}$: NOT W-C_k-sp. $\square$

<u>Outline of Proof</u>

(1)
$$\begin{tikzcd} B_nG_{(p)} \vee B_kG_{(p)} \arrow[r] \arrow[d, hook] & BG_{(p)} \\ B_nG_{(p)} \times B_kG_{(p)} \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$\underset{\text{Kishimoto-Kono}}{\Longleftrightarrow}$ $(\mathrm{ad}\,E_nG)_{(p)}$ is trivial as a fiberwise A_k-sp.

$\Rightarrow \mathcal{G}(E_nG)_{(p)} \simeq \mathrm{Map}(B_nG_{(p)}, G_{(p)})$ as A_k-sp.

$\underset{G_{(p)}: S\text{-}C_k\text{-sp.}}{\Longrightarrow} \mathcal{G}(E_nG)_{(p)}$: S-C_k-sp.

(2) (i) $B\mathcal{G}(E_nG)_{(p)} \simeq \mathrm{Map}(B_nG_{(p)}, BG_{(p)})_{i_n}$: path comp. $\ni i_n$.
(Gottlieb, 1972)

(ii)
$$\begin{tikzcd} \Sigma G_{(p)} \vee B_nG_{(p)} \arrow[r] \arrow[d, hook] & BG_{(p)} \\ \Sigma G_{(p)} \times B_nG_{(p)} \arrow[ur, dashed] & \end{tikzcd}$$
$\exists f$ ($\because (n+1)n_\ell < p$)

$\underset{\text{adjoint}}{\rightleftarrows}$ $f' : \Sigma G_{(p)} \longrightarrow \mathrm{Map}(B_nG_{(p)}, BG_{(p)})_{i_n}$.

(iii) If $\mathcal{G}(E_nG)_{(p)}$: W-C_k-sp.,
$$\begin{tikzcd} (\Sigma G_{(p)})^{\vee k} \arrow[r, "\vee f'"] \arrow[d, hook] & \mathrm{Map}(B_nG_{(p)}, BG_{(p)})_{i_n} \\ (\Sigma G_{(p)})^{\times k} \arrow[ur, dashed, "\exists g"'] & \end{tikzcd}$$

$$\overrightarrow{\text{adjoint}} \quad \begin{tikzcd} (\Sigma G_{(p)})^{\vee k} \vee (B_nG)_{(p)} \arrow[r] \arrow[d, hook] & BG_{(p)} \\ (\Sigma G_{(p)})^{\times k} \times (B_nG)_{(p)} \arrow[ur, "g'"'] & \end{tikzcd} \quad g' : \text{adjoint of } g.$$

$\Longrightarrow G_{(p)} : C(\underbrace{1, \dots, 1}_{k}, n)\text{-sp.} \Longrightarrow G_{(p)} : W\text{-}C_{n+k}\text{-sp.}$

Contradiction. ■

The following theorems are proved similarly

Thm B′ (HKT)

B : fin. cpx. , P : prin. G-bdl. $/B$.

$p > (\mathrm{cat}\,B + k)n_\ell \Rightarrow \mathcal{G}(P)_{(p)} : S\text{-}C_k\text{-sp.}$ □

Thm B″ (Kishimoto-Kono-Theriault 2013 (for $k=2$), HKT)

P : prin G-bdl. $/ S^{2n_i}$

$p \geq n_i + kn_\ell \Rightarrow \mathcal{G}(P)_{(p)} : S\text{-}C_k\text{-sp.}$ □