Infiniteness of A_∞-types of gauge groups

Mitsunobu Tsutaya (Kyoto Univ.)

joint work with Daisuke Kishimoto (Kyoto Univ.)

East Asian Conference on Algebraic Topology 2015, Daejeon

December 3, 2015

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1. Overview of the classification problems of gauge groups

1-1. Setting of the problem

1-2. Results on homotopy types

1-3. Results on A_n-types

1-4. Main result

1. Overview of the classification problems of gauge groups

1-1. Setting of the problem

Problem

Fix a topological space B, a topological group G and an equivalence relation between topological groups.

Take arbitrary principal G-bundles P and P' over B.

Then, when the gauge groups

$$\mathcal{G}(P) \quad \text{and} \quad \mathcal{G}(P')$$

are “equivalent” ?

All the equivalence relations we choose are homotopy theoretical.

1-2. Results on homotopy types

General result

Theorem (Crabb-Sutherland, 2000)

Fix a finite CW complex B and

a compact connected Lie group G.

Then, the following finiteness holds:

$$\#\left(\frac{\{\mathcal{G}(P) | P\text{: prin. } G\text{-bdl. over } B\}}{\text{homotopy equivalence}}\right) < \infty.$$

There are many classification results when G is a classical Lie group.

(Kono 1991, Sutherland 1992, Kono-Tsukuda 1996, Hamanaka-Kono 2006, 2007, Kamiyama-Kishimoto-Kono-Tsukuda 2007, Choi-Hirato-Mimura 2008, Claudio-Spreafico 2009, Theriault 2010, Kishimoto-Kono-T 2013, etc.)

1. Overview of the classification problems of gauge groups

1-3. Results on A_n-types

What are A_n-types?

Definition (Sugawara 1960, Stasheff 1963)

Let G, G' be topological monoids.

A based map $f : G \to G'$ is said to be an A_n-map if it admits an A_n-form.

If an A_n-map is also a homotopy equivalence, f is said to be an A_n-equivalence.

A_n-type = A_n-equivalence type

1. Overview of the classification problems of gauge groups

1-3. Results on A_n-types

What is an A_n-form?

$$\{f_i : I^{\times i-1} \times G^{\times i} \to G'\}_{i=1}^{n}$$

$n = 1$

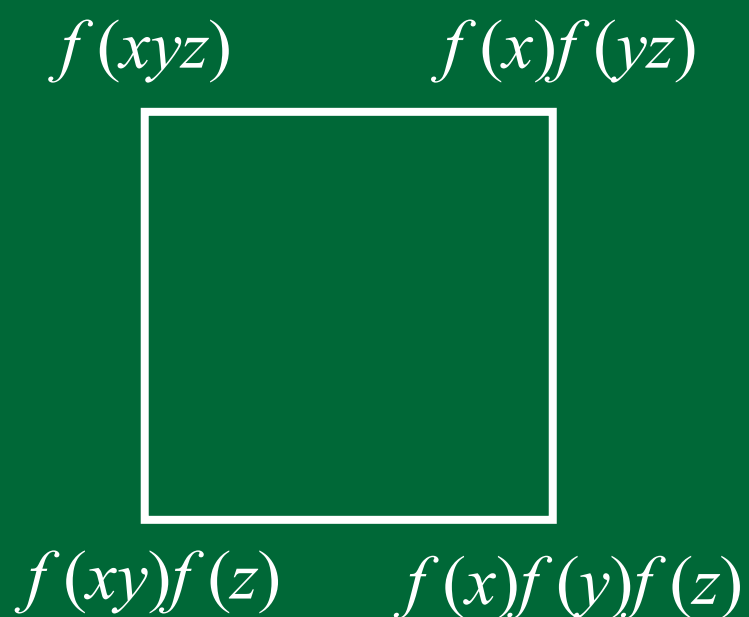
$f(x)$

$n = 2$

$f(xy)$

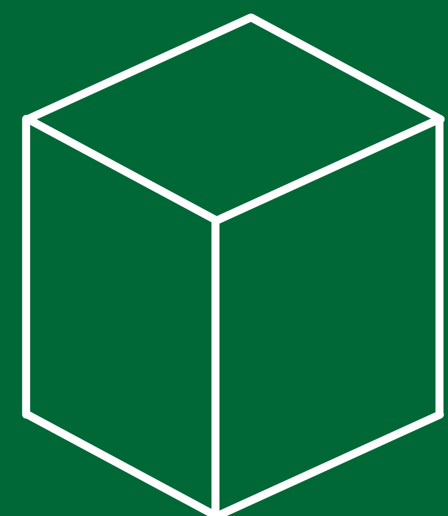
$f(x)f(y)$

$n = 3$



$n = 4$

parametrized by a cube



and so on.

1-3. Results on A_n-types

Why A_n-types?

Proposition (Stasheff)

A based map $f: G \to G'$ is an A_n-map if and only if

$$\begin{array}{ccc} B_1G & \xrightarrow{B_1 f} & B_1G' \\ \downarrow & & \downarrow \\ B_nG & \overset{\exists}{\dashrightarrow} & BG' \end{array}$$

- B_nG is the n-th projective space of G.
- B_1G is the reduced suspension of G.
- G and G' are A_∞-equiv. $\Leftrightarrow$ BG and BG' are homotopy equiv.

1-3. Results on A_n-types

Why A_n-types of gauge groups?

$B\mathcal{G}(P)$ is homotopy equivalent to the following spaces:

- a path component of $\mathrm{Map}(B,BG)$,
- the moduli space of connections on P.

Further interest:

- the relation of multiplicative properties between G and the gauge groups.

(cf. Kishimoto-Kono-Theriault 2013)

1-3. Results on A_n-types

General result

Theorem (Crabb-Sutherland 2000, T. 2012)

Fix a finite CW complex B,

a compact connected Lie group G and $n < \infty$.

Then, the following finiteness holds:

$$\#\left(\frac{\{\mathcal{G}(P) | P\text{: prin. } G\text{-bdl. over } B\}}{A_n\text{-equivalence}}\right) < \infty.$$

T. (2015) also gave a complete classification of the A_n-types of the gauge groups of prin. SU(2)-bdl.s over S^4 localized away from 2.

1-4. Main result

What happens when $n = \infty$?

Masbaum (1992) proved the previous finiteness breaks down when $n = \infty$. In fact, his result is rather typical.

Main theorem (Kishimoto-T 2016)

Fix an integer m and

a compact connected simple Lie group G.

Then, if there are infinitely many distinct isomorphism

classes of the principal G-bundles over S^m, then:

$$\#\left(\frac{\{\mathcal{G}(P) \mid P\text{: prin. } G\text{-bdl. over } S^m\}}{A_\infty\text{-equivalence}}\right) = \infty.$$

2. Outline of the proof

2-1. Adjoint bundles and their triviality

2-2. Strategy

2-3. Two key propositions

2. Outline of the proof

2-1. Adjoint bundles and their triviality

P: a principal G-bundle over B.

The left action of G on G itself given by the conjugation induces the adjoint bundle

$$\mathrm{ad}\,P = (P \times G)/G.$$

$\mathrm{ad}\,P$ is a bundle of topological groups.

Proposition

$$\mathcal{G}(P) \cong \Gamma(\mathrm{ad}\,P)$$

as topological groups.

2. Outline of the proof

2-1. Adjoint bundles and their triviality

· Fiberwise A_n-equivalences are similarly defined.

Proposition (Kishimoto-Kono 2010)

Let P be a principal bundle classified by $\alpha\colon B \to BG$.

ad P is fiberwise A_n-equivalent to the trivial bundle $B\times G$

if and only if

$$\begin{array}{ccc} B \vee B_nG & \xrightarrow{(\alpha, i_n)} & BG \\ \downarrow & \nearrow_{\exists} & \\ B \times B_nG & & \end{array}$$

2. Outline of the proof

2-2. Strategy

Fix m and G as in the main theorem.

For simplicity, we assume $\epsilon \in \pi_m(BG)$ generates $\pi_m(BG) \otimes \mathbb{Z}_{(p)}$ except for a finite number of primes p.

P_k: the principal bundle classified by $k\epsilon$.

Then we find positive integers $N(G,p)$ and $r(G,p)$ such that

- if k is not divisible by p, then $\mathcal{G}(P_k)$ and $\mathcal{G}(P_0)$ are NOT $A_{N(G,p)}$-equivalent p-locally,
- if k is divisible by $p^{r(G,p)}$, then $\mathcal{G}(P_k)$ and $\mathcal{G}(P_0)$ are $A_{N(G,p)}$-equivalent p-locally (relatively easy).

This can be done for all except finitely many p!

2. Outline of the proof

2-3. Two key propositions

Proposition 1

Localizing at sufficiently large prime p,

the following conditions are equivalent for large $N(G,p)$:

· the gauge groups $\mathcal{G}(P_k)$ and $\mathcal{G}(P_0)$ are $A_{N(G,p)}$-equivalent,

· the adjoint bundles $\mathrm{ad}\,P_k$ and $S^m \times G$ are fiberwise $A_{N(G,p)}$-equivalent.

Proof.

It is easy to see the second condition implies the first.

The essential part is the converse.

2. Outline of the proof

2-3. Two key propositions

Proof (continued).

Step 1. An A_n-map $\mathcal{G}(P_0) \to \mathcal{G}(P_k)$ induces a f.w. A_n-map ad $P_0 \longrightarrow$ ad P_k such that the restriction on the fiber is

$$G \to \mathcal{G}(P_0) \to \mathcal{G}(P_k) \to G$$

Step 2. The above composite is nontrivial on mod p cohomology.

(This follows from the evaluation fibration

$$\Omega^m G \to \mathcal{G}(P_k) \to G$$

and the fact that G is p-locally a product of spheres.)

2. Outline of the proof

2-3. Two key propositions

Proof (continued).

Step 3. For sufficiently large $N(G,p)$, the induced map

$$H^*(BG;\mathbb{Z}/p) \to H^*(B_{N(G,p)}G;\mathbb{Z}/p)$$

lifts to a map of unstable algebras

$$H^*(BG;\mathbb{Z}/p) \to H^*(BG;\mathbb{Z}/p)$$

Step 4. Such a nontrivial map must be isomorphism.

(This follows from Galois theory of unstable algebras and the irreducibility of the natural representation of the Weyl group.)

Therefore, ad $P_0 \to$ ad P_k is a f.w. $A_{N(G,m,p)}$-equiv. ■

2. Outline of the proof

2-3. Two key propositions

If ad P_k is f.w. A_n-equiv. to the trivial bundle, there is a map

$$H^*(BG) \to H^*(S^m) \otimes H^*(B_nG)$$

of unstable algebras such that the image of the generator

$x \in H^m(BG)$ is $ku \in H^m(S^m)$.

Now $N(G,p)$ is sufficiently large. Then there is a lift

$$H^*(BG) \to H^*(S^m) \otimes H^*(BG)$$

Proposition 2

In this situation, k is divisible by p.

2. Outline of the proof

2-3. Two key propositions

Proof.

Consider the composite

$$H^*(BG) \to H^*(S^m) \otimes H^*(BG) \to H^*(S^m) \otimes H^*(BT)$$
$$\to H^*(S^m) \otimes H^*(BV)$$

where T is the maximal torus of G and

V the corresponding elementary abelian group.

Using the Lannes' T-functor T_V, this map corresponds to

$$T_V H^*(BG) \to H^*(S^m)$$

The proposition follows from the computation of $T_V H^*(BG)$

by Aguadé (1989). ■

3. Further problem

Our method detects only the “primary obstruction” for the triviality of adjoint bundles. We can say only that there are at least two A_∞-types of the gauge groups at each large prime p.

For $G = \mathrm{SU}(2)$ and $B = S^4$, Tsukuda (2001) proved that there are infinitely many distinct p-local A_∞-types of the gauge groups. The following might be a reasonable generalization.

Problem

For a simple p-compact group G and an integer m such that $\pi_m(BG)$ is infinite. Then are there infinitely many distinct A_∞-types of the gauge groups of principal G-bundles over S^m?