Mapping spaces from projective spaces

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1. Introduction

- A_n-map
- Main theorem

A_n-map

G, G', G'': topological monoids

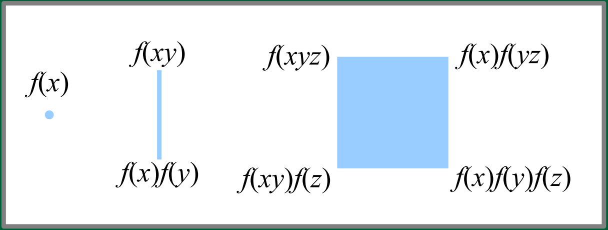
A_n-map ($n = 1, 2, \ldots, \infty$)

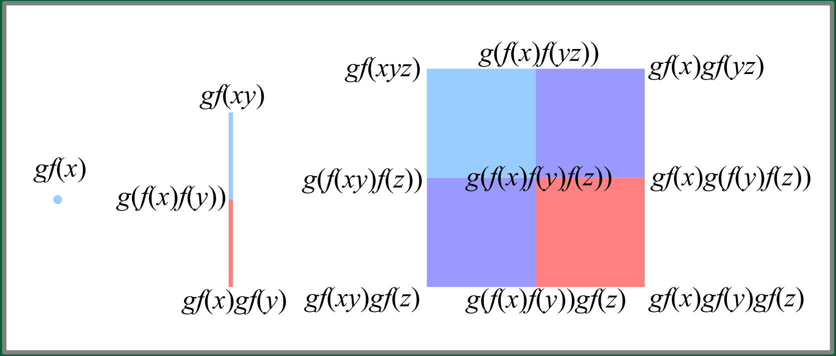
A family of maps $\{f_i : [0,1]^{i-1} \times G^i \to G'\}_{i=1}^n$ is called an A_n-form of f if

1. $f_1 = f$,
2. $f_i(t_1, \ldots, t_{i-1}; g_1, \ldots, g_i)$
$$= \begin{cases} f_{i-1}(t_1, \ldots, \hat{t}_k, \ldots, t_{i-1}; g_1, \ldots, g_k g_{k+1}, \ldots, g_i) & \text{for } t_k = 0 \\ f_k(t_1, \ldots, t_{k-1}; g_1, \ldots, g_k) f_{i-k}(t_{k+1}, \ldots, t_{i-1}; g_{k+1}, \ldots, g_i) & \text{for } t_k = 1 \end{cases}$$
3. $f_i(t_1, \ldots, t_{i-1}; g_1, \ldots, \overset{k}{\check{*}}, \ldots, g_i) =$
$f_{i-1}(t_1, \ldots, \max\{t_{k-1}, t_k\}, \ldots, t_{i-1}; g_1, \ldots, \hat{g}_k, \ldots, g_i)$.

A triple $(f, \{f_i\}, \ell)$ is called an A_n-map. We denote the space of A_n-maps from G to G' by $\mathcal{A}_n(G, G')$.

There is a composition of A_n-maps
$$\mathcal{A}_n(G', G'') \times \mathcal{A}_n(G, G') \to \mathcal{A}_n(G, G'').$$


f(x)
f(xy)
f(x)f(y)
f(xyz)
f(x)f(yz)
f(xy)f(z)
f(x)f(y)f(z)


gf(x)
gf(xy)
g(f(x)f(y))
gf(x)gf(y)
gf(xyz)
g(f(x)f(yz))
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g(f(x)f(y))gf(z)
gf(x)gf(y)gf(z)

Main theorem

G: topological monoid, G': grouplike topological monoid.
Both of them are CW complex.

Main Theorem (recognition theorem for A_n-maps) (T)

The composition

$$\mathcal{A}_n(G, G') \xrightarrow{B_n} \mathrm{Map}_0(B_nG, B_nG') \xrightarrow{(i_n)_\#} \mathrm{Map}_0(B_nG, BG')$$

is a natural weak equivalence, where $i_n \colon B_nG' \to BG'$ is the natural inclusion.

2. Recognitions of A_n-map

- Projective spaces
- Recognitions of A_n-map

Projective spaces

G: topological monoid

The n-th projective space B_nG is defined by

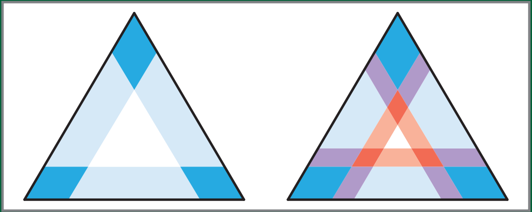
$$B_nG := \left(\coprod_{0 \le i \le n} \Delta^i \times G^i \right) \Big/ \sim,$$

where $\sim$ is the usual simplicial identification.

$$* = B_0G \subset \Sigma G = B_1G \subset B_2G \subset \cdots \subset B_\infty G = BG.$$

An A_n-map induces a based map between the n-th projective spaces:

$$B_n \colon \mathcal{A}_n(G, G') \to \mathrm{Map}_0(B_nG, B_nG').$$



Recognitions of A_n-map

G: topological monoid, G': grouplike topological monoid.
Both of them are CW complex.

Thoerem (Stasheff, 1963)

A based map $f\colon G \to G'$ admits an A_n-form if and only if the composite

$$\Sigma G \xrightarrow{\Sigma f} \Sigma G' \xrightarrow{i_1} BG'$$

extends to a map $B_n G \to BG'$.

Thoerem (Fuchs, 1965)

There is a one-to-one correspondence between $\pi_0(\mathcal{A}_\infty(G, G'))$ and the homotopy set of based maps $[BG, BG']$.

Thoerem

The model categories of simplicial groups and reduced simplicial sets are Quillen equivalent by the Kan's loop group construction.

3. Main theorem

- Main theorem
- Adjointness of B_n and Ω

Main theorem

G: topological monoid, G': grouplike topological monoid.
Both of them are CW complex.

Main Theorem (recognition theorem for A_n-maps) (T)

The composition

$$\mathcal{A}_n(G, G') \xrightarrow{B_n} \mathrm{Map}_0(B_nG, B_nG') \xrightarrow{(i_n)_\#} \mathrm{Map}_0(B_nG, BG')$$

is a natural weak equivalence, where $i_n \colon B_nG' \to BG'$ is the natural inclusion.

Proof of Theorem

When $n = 1$, this is the well-known adjunction of Σ and Ω.
Suppose this is true for A_{n-1}-maps. Consider the following commutative diagram of homotopy fiber sequences:

$$\begin{array}{ccccc} F & \longrightarrow & \mathcal{A}_n(G, G') & \longrightarrow & \mathcal{A}_{n-1}(G, G') \\ \downarrow & & \downarrow & & \downarrow \simeq \\ F' & \longrightarrow & \mathrm{Map}_0(B_n G, BG') & \longrightarrow & \mathrm{Map}_0(B_{n-1} G, BG'). \end{array}$$

In fact, the map $F \to F'$ coincides with the composite $F \simeq \mathrm{Map}_0(S^{n-1} \wedge G^{\wedge n}, G') \simeq \mathrm{Map}_0(S^n \wedge G^{\wedge n}, BG') \simeq F'$. Then by the five lemma, we have the desired conclusion.

Adjointness of B_n and Ω

G: topological monoid which is a CW complex, X: a based space.

Corollary (adjointness of B_n and Ω) (T)

There is a natural weak equivalence

$$\mathcal{A}_n(G, \Omega X) \xrightarrow{\simeq} \mathrm{Map}_0(B_n G, X).$$

G': grouplike topological monoid which is a CW complex.

Corollary

The following map is a weak equivalence.

$$\mathcal{A}_\infty(G, G') \xrightarrow{\simeq} \mathrm{Map}_0(BG, BG').$$

4. Related topics

- Evaluation fiber sequence
- Higher homotopy commutativity
- Gottlieb-type filtration on $\pi_*(X)$

Evaluation fiber sequence

X, Y: based CW complexes.
The homotopy fiber sequence

$$\cdots \to \Omega Y \to \mathrm{Map}_0(X,Y) \to \mathrm{Map}(X,Y) \to Y$$

is called the evaluation fiber sequence. In general, this fiber sequence does not extend to the right.
If $Y = X$, there is a homotopy fiber sequence

$$\begin{aligned}\cdots \to \Omega X \to \mathrm{Map}_0(X,X)_{\mathrm{id}} \to \mathrm{Map}(X,X)_{\mathrm{id}} \to X \\ \to B\,\mathrm{Map}_0(X,X)_{\mathrm{id}} \to B\,\mathrm{Map}(X,X)_{\mathrm{id}}\end{aligned}$$

where the subspaces $\mathrm{Map}_0(X,Y)_f \subset \mathrm{Map}_0(X,Y)$ and $\mathrm{Map}(X,Y)_f \subset \mathrm{Map}(X,Y)$ consist of maps freely homotopic to a based map $f\colon X \to Y$.

G: topological group which is a CW complex.
The conjugation on G defines an action on BG and hence on $\mathrm{Map}_0(X, BG)$. On the other hand, the conjugation defines a "homomorphism"

$$\alpha\colon G \to \mathcal{A}_n(G, G).$$

Theorem (T)

There is a homotopy fiber sequence

$$G \to \mathrm{Map}_0(B_nG, BG)_{i_n} \to \mathrm{Map}(B_nG, BG)_{i_n} \to BG \xrightarrow{B\alpha} B\mathcal{A}_n(G, G)_\alpha$$

where $\mathcal{A}_n(G, G)_\alpha$ is the union of path-components containing the image of α.

This theorem follows from the fact that the weak equivalence

$$\mathcal{A}_n(G, G) \xrightarrow{\simeq} \mathrm{Map}_0(B_nG, BG)$$

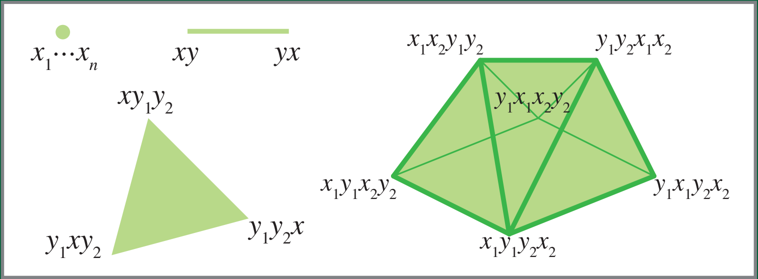
is G-equivariant.

Higher homotopy commutativity

G: topological monoid, $N_{r,s}$: resultohedron $(r, s \geq 0)$

Definition (Kishimoto–Kono, 2010)

If there is a family of maps $\{Q_{r,s} : N_{r,s} \times G^{r+s} \to G\}_{0\leq r\leq k, 0\leq s\leq \ell}$ satisfying appropriate compatibility, G is said to be a $C(k,\ell)$-space.



$C(1,1)$-space $\Leftrightarrow$ homotopy commutative

G: topological group which is a CW complex

Theorem (Kishimoto–Kono, 2010)

The following are equivalent:

- G is a $C(k,\ell)$-space,
- $(i_k, i_\ell)\colon B_kG \vee B_\ell G \to BG$ extends over $B_kG \times B_\ell G$,
- $i_k^* \operatorname{Map}(S^1, BG)$ is trivial as a fiberwise A_ℓ-space.

Theorem (T)

G is a $C(k,\ell)$-space if and only if the map $\alpha\colon G \to \mathcal{A}_\ell(G,G)$ is homotopic to the trivial map as an A_k-map.

Gottlieb-type filtration

X: based connected CW complex

Definition

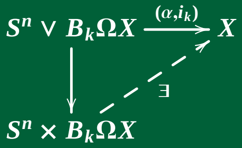
Define a subgroup $G_n^{(k)}(X) \subset \pi_n(X)$ by

$$G_n^{(k)}(X) = \mathrm{im}(ev_* : \pi_n(\mathrm{Map}(B_k\Omega X, X)_{i_k}) \to \pi_n(X)).$$

The group $G_n(X) := G_n^{(\infty)}(X)$ is called the n-th Gottlieb group.

$$G_n(X) = G_n^{(\infty)}(X) \subset \cdots \subset G_n^{(2)}(X) \subset G_n^{(1)}(X) \subset Z(\pi_n(X)) \subset \pi_n(X)$$

For $\alpha \in \pi_n(X)$, $\alpha \in G_n^{(k)}(X)$ if and only if



Example

If X is an H-space, then $G_n^{(\infty)}(X) = \pi_n(X)$ for any $n \geq 1$. More generally, if ΩX is a $C(1,k)$-space, then $G_n^{(k)}(X) = \pi_n(X)$ for any $n \geq 1$.

Example (T)

For an odd prime p and $\frac{r(p-1)}{2} \leq k < \frac{(r+1)(p-1)}{2}$, the subgroup

$$G_4^{(k)}(B\,SU(2))_{(p)} \subset \pi_4(B\,SU(2))_{(p)} \cong \mathbb{Z}_{(p)}$$

has index p^r and $G_4^{(\infty)}(B\,SU(2)) = 0$.

Example (Kishimoto–T)

G: compact connected simple Lie group
Suppose that $H^*(BG;\mathbb{Q})$ is a polynomial algebra on the generators of degree $2n_1, \ldots, 2n_\ell$. If $p > 2n_\ell$, then the subgroup

$$0 \neq G_{2n_i}^{(n_\ell+p-1)}(BG)_{(p)} \subset \pi_{2n_i}(BG)_{(p)} \cong \mathbb{Z}_{(p)}$$

has index $\geq p$.