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Abstracts

Quasi hypergeometric functions and a few problems related to them

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Let be given $r+s$ complex numbers $\alpha'_1,...,\alpha'_r;\alpha_1,...,\alpha_s$ and two real matrices $(\beta'_{i,j})_{1\le i\le r,1\le j\le n}$ and $(\beta_{i,j})_{1\le i\le s,1\le j\le n}$. We assume the following condition of regular singularity. For each j, we have

$$\sum_{i=1}^{r}\beta'_{ij}=\sum_{i=1}^{s}\beta_{ij}+1 \tag{1}$$

Quasi hypergeometric functions are defined to be the following power series in $z_1,...,z_n$,

$$F(z)=\sum_{\nu_1\ge 0,...,\nu_n\ge 0}\frac{\prod_{i=1}^{r}\Gamma(\alpha'_i+\sum_{j=1}^{n}\beta'_{i,j}\nu_j)}{\prod_{i=1}^{s}\Gamma(\alpha_i+\sum_{j=1}^{n}\beta_{i,j}\nu_j)}\frac{z_1^{\nu_1}\cdots z_1^{\nu_1}}{\nu_1!\cdots\nu_n!}. \tag{2}$$

Because of the condition (1) and (2) converge at the origin at least if $\beta'_{i,j}$ are all positive. Typical examples of quasi hypergeometric functions (we shall abbreviate by QHGF) are given as follows.
(1) Lambert series.

$$F(z)=\sum_{\nu=0}^{\infty}\binom{\alpha+\beta\nu}{\nu}z^{\nu} \tag{3}$$

for $\alpha\in\mathbf{C}$,a positive β and $z\in\mathbf{C}$ such that $|z|<\beta^{-\beta}(\beta-1)^{\beta-1}$ or $|z|<\beta^{-\beta}(1-\beta)^{\beta-1}$ according as $\beta>1$ or $\beta<1$.

(2) Single variable QHGF.

$$F(z)=\sum_{\nu=0}^{\infty}\frac{\Gamma(\alpha'_1+\beta'_1\nu)\cdots\Gamma(\alpha'_r+\beta'_r\nu)}{\Gamma(\alpha_1+\beta_1\nu)\cdots\Gamma(\alpha_s+\beta_s\nu)}\frac{z^{\nu}}{\nu!} \tag{4}$$

where $\beta'_j,\beta_k\in\mathbf{R}$ such that $\beta'_j>0$ with the identity

$$\beta'_1+\cdots+\beta'_r=\beta_1+\cdots+\beta_s+1$$

(3) Quasi algebraic functions(see [2],[5],[7]).

Let $\lambda_0,\lambda_1,\cdots\lambda_n$, be positive real numbers. Consider the following quasi algebraic equation with respect to w

$$w^{\lambda_0}+z_1w^{\lambda_1}+\cdots+z_nw^{\lambda_n}-1=0. \tag{5}$$

This has the unique solution $w=F(z_1,...,z_n)$ which has the power series expansion in $z_1,...,z_n$

$$w=\frac{1}{\lambda_0}\sum_{\nu_1\ge 0,...,\nu_n\ge 0}(-1)^{|\nu|}\frac{\Gamma(A_\nu)}{\Gamma(A_\nu-|\nu|+1)}\frac{z_1^{\nu_1}\cdots z_n^{\nu_n}}{\nu_1!\cdots\nu_n!}$$

where A_ν and $|\nu|$ denote

$$A_\nu = \frac{1}{\lambda_0}(1 + \lambda_1\nu_1 + \cdots + \lambda_n\nu_n)$$
$$|\nu| = \nu_1 + \cdots + \nu_n$$

respectively.

Similarly we have for an arbitrary real ρ

$$w^\rho = \frac{\rho}{\lambda_0} \sum_{\nu_1\geq 0,\dots,\nu_n\geq 0} (-1)^{|\nu|} \frac{\Gamma(A_\nu + \frac{\rho-1}{\lambda_0})}{\Gamma(A_\nu + \frac{\rho-1}{\lambda_0} - |\nu| + 1)} \frac{z_1^{\nu_1}\cdots z_n^{\nu_n}}{\nu_1!\cdots\nu_n!} \tag{6}$$

(4) Solutions to the Wu's equations (see [13], [19]).

Let an n th order square real matrix $(\beta'_{i,j})_{i,j=1}^n$ be given. Wu's equations can be represented by the use of affine coordinates $w = (w_1, ..., w_n)$ as

$$w_i - 1 = z_i w_1^{\beta'_{1,i}} \cdots w_n^{\beta'_{n,i}} \quad \text{for} \quad 1 \leq i \leq n \tag{7}$$

There exists the unique solution to (7) $w_i = F_i(z_1, ..., z_n)$, which has the power series expansions in $z_1, ..., z_n$ at the origin. These can be obtained as follows.

We denote by $f_i(w) = f_i(w_1, ..., w_n)$ the functions

$$f_i(w) = w_i - 1 - z_i w_1^{\beta'_{1,i}} \cdots w_n^{\beta'_{n,i}}$$

Then the Jacobian $J = \frac{\partial(f_1,...,f_n)}{\partial(w_1,...,w_n)}$ at a point w where the equalities hold

$$f_1(w) = \cdots = f_n(w) = 0$$

is given by

$$J = \frac{\varphi(w_1, ..., w_n)}{w_1 \cdots w_n}$$

where $\varphi(w) = \varphi(w_1, ..., w_n)$ denotes the determinant of the n th order matrix $(\delta_{i,j}w_j - \beta'_{i,j}(w_j - 1))_{i,j=1}^n$. $\varphi(w)$ is a polynomial of n th degree which can be expressed as

$$\varphi(w) = B + \sum_{r=1}^n \sum_{1\leq i_1<\ldots<i_r\leq n} C_{i_1,\ldots,i_r} w_{i_1} \cdots w_{i_r}$$

where B equals the determinant of the matrix $(\beta'_{i,j})_{i,j=1}^n$.

We denote by $F(\alpha_1, ..., \alpha_n; z)$ the power series in $z_1, ..., z_n$

$$F(\alpha_1, ..., \alpha_n; z) = \sum_{\nu_1\geq 0,\dots,\nu_n\geq 0} \frac{\prod_{i=1}^n \Gamma(\alpha_i + \sum_{j=1}^n \beta'_{i,j}\nu_j)}{\prod_{i=1}^n \Gamma(\alpha_i + \sum_{j=1}^n \beta_{i,j}\nu_j)} \cdot \frac{z_1^{\nu_1}\cdots z_n^{\nu_n}}{\nu_1!\cdots\nu_n!} \tag{8}$$

where $\beta_{i,j}$ denotes $\beta_{i,i} = \beta'_{i,i} - 1$ and $\beta_{i,j} = \beta'_{i,j}$ for $i \neq j$. Then the following identity holds.

$$F(\alpha_1, ..., \alpha_n; z) = \left[\frac{w_1^{\alpha_1}\cdots w_n^{\alpha_n}}{\varphi(w)}\right]_{f_1=\cdots=f_n=0} \tag{9}$$

the function $w_1^{\alpha_1}\cdots w_n^{\alpha_n}$ itself has the power series

$$w_1^{\alpha_1}\cdots w_n^{\alpha_n} = BF(\alpha_1,...,\alpha_n;z)+\sum_{r=1}^{n}\sum_{1\le i_1<\cdots<i_r\le n} C_{i_1,...,i_r}F(\alpha_1,...,\alpha_{i_1}+1,...,\alpha_{i_r}+1,...,\alpha_n;z)$$

QHGF have recently appeared both in mathematics and physics litteratures (see [2],[8],[13]). They are analytic functions of $z=(z_1,...,z_n)$ satisfying linear fractional differetial equations in z and also satisfying difference-differetial equations in z and $\alpha'_1,...,\alpha'_r,\alpha_1,...,\alpha_s$.

In this talk, we want to discuss the following problems (some of these problems have been discussed in [3].)

(1) *Analytic integral representations of QHGF.*

(2) *Singularities of QHGF when they are prolonged as an analytic continuation.*

(3) *The structure of monodromy of QHGF.*

(4) *The symmetry related to QHGF, especially in the case of* 2 *dimensional Wu's equations.*

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Comparisons between Dirichlet and Neumann eigenvalues

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For bounded domains in Euclidean space, various inequalities between the Dirichlet and Neumann eigenvalues of the Laplacian are known. The main contributions are due to Payne (1955), Aviles (1986), and Levine and Weinberger (1986), culminating in the 1991 proof by Leonid Friedlander that the $(k+1)$th Neumann eigenvalue is always less than or equal to the kth Dirichlet eigenvalue. We will report on our recent progress toward extending some of these prior results to bounded domains in homogeneous spaces (such as the sphere S^n).

Another look at linear first order q-difference equations

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The simple linear first order q-difference equation $(1-ax)f(qx) = (1-bx)f(x)$ seems so simple that one would not expect surprises in its solutions. Yet, if one has not seen this before, Ramanujan's solution is a surprise, and contains a lot of deep information. I was surprised when reading a paper of Hahn about a new class of solutions he found. There are others, which will be described.

Combinatorics of symplectic Schur functions

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The Schur function $s_\lambda(x_1, \dots, x_n)$ in n variables can be defined as the character of the irreducible representation of the Lie algebra $gl(n)$ labeled by its highest weight λ. This gives rise to the combinatorial definition $s_\lambda(x) = \sum_T x^T$ where the summation is over all semi-standard tableaux of shape λ. The characters of the irreducible representations of the other classical Lie algebras have similar interpretations in terms of generalized tableaux. Here, we focus on the symplectic case and discuss the combinatorics of the $sp(2n)$ tableaux defined by King and those defined by Kashiwara and Nakashima, the latter tableaux being intimately connected with the theory of crystal bases.

For the usual semi-standard $(gl(n))$ tableaux, important combinatorial operations include insertion and jeu de taquin, which involve bumping and sliding (respectively) boxes around the tableau, while maintaining the tableau properties. For the $sp(2n)$ tableaux of King, Berele (1986) defined an insertion scheme which involves bumping *and* sliding. Here we define an insertion scheme for the $sp(2n)$ tableaux of Kashiwara and Nakashima. Although it is more complicated than Berele's insertion scheme, it turns out to be a crystal isomorphism.

Inequalities and special functions

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We describe several new applications of special function theory.

Families of discrete and absolutely continuous solutions to some indeterminate moment problems

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If $(s_n)_{n\geq 0}$ is an arbitrary indeterminate moment sequence, it is well-known that the set V of solutions to the corresponding moment problem contains measures of a very different nature: There are measures $\mu \in V$ with a C^∞-density, discrete measures and measures which are continuous singular. By a result of Christensen and the author from 1981 each of these classes of solutions is dense in V in the weak topology. Since V is a compact convex set, we get:

Let $t \mapsto \mu_t$ be a continuous mapping of an interval I into V. For any probability measure τ on I the vector integral $\kappa = \int \mu_t d\tau(t)$ belongs to V.

The first part of the talk will be based on a recent paper [1], and we shall show how the statement above can be used in concrete examples. It will be like a climbing tour from relatively easy ground to more difficult territory.

The starting point will be the well-known log-normal distribution with density on $]0,\infty[$

$$d_\sigma(x) = (2\pi\sigma^2)^{-\frac{1}{2}} x^{-1} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right). \tag{1}$$

The moment sequence is given as

$$s_n(d_\sigma) = q^{-\frac{1}{2}n^2}, \quad n \geq 0, \tag{2}$$

where $0 < q < 1$ is defined by $q = e^{-\sigma^2}$.

For $a > 0$ we follow Chihara in defining the discrete probability

$$\lambda_a = \frac{1}{L(a)} \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} \varepsilon_{aq^k}, \tag{3}$$

where

$$L(a) = \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2}. \tag{4}$$

(Here ε_a denotes the unit mass at the point a.)

It is easy to see that all measures in this family have the log-normal moments (2).

We shall see how this family in a simple way can be used to generate absolutely continuous solutions with the same moments. These solutions were found earlier by Askey and Askey-Roy.

We also consider families of discrete solutions to other indeterminate moment problems and show how they can be used in a similar way to generate absolutely continuous solutions. The following cases are considered: generalized Stieltjes-Wigert, q-Laguerre and discrete q-Hermite II.

In a second part we consider convolution semigroups $(\mu_t)_{t>0}$ of probability measures on the real line, i.e. $\mu_s * \mu_t = \mu_{s+t}$ and $\lim_{t\to 0} \mu_t = \varepsilon_0$ in the weak topology.

Assuming that all the measures μ_t have moments of any order, we shall show that the n'th moment $s_n(\mu_t)$ is a polynomial in t of degree n. We shall further discuss questions of indeterminacy for these measures.

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One-sided Tauberian theorems for Dirichlet series methods of summability

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Suppose that $\{\lambda_n\}$ is an unbounded and strictly increasing sequence of positive numbers, that $\{a_n\}$ is a sequence of non-negative numbers, and that the Dirichlet series

$$a(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}$$

has abscissa of convergence $\sigma < \infty$. Let $\{s_n\}$ be a sequence of *real* numbers. The Dirichlet series summability method $D_{\lambda,a}$ is defined as follows:

$$s_n \to s\,(D_{\lambda,a}) \quad \text{if} \ \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \text{ is convergent for } x > \sigma, \text{ and}$$
$$\sigma(x) := \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \to s \text{ as } x \to \sigma +$$

through real values. It is well known that if $a(x) \to \infty$ as $x \to \sigma+$ when $\sigma > -\infty$, or if $a_n \neq 0$ for infinitely many n when $\sigma = -\infty$, then $D_{\lambda,a}$ is regular, i.e., $s_n \to s$ implies $s_n \to s\,(D_{\lambda,a})$.

The primary purpose of the talk is to discuss theorems (established in work done together with W. Kratz and U. Stadtmüller) concerning *one-sided* Tauberian conditions on $\{s_n\}$ under which $s_n \to s\,(D_{\lambda,a})$ implies $s_n \to s$. These theorems generalize known two-sided or O-Tauberian results. Suppose in addition that

$$a_n \sim e^{-g(n)} \text{ as } n \to \infty, \text{ and } \lambda_n = \lambda(n) \text{ for } n \geq x_0 \in \mathbb{N}$$

where g and λ are real functions in $C_2[x_0, \infty)$ satisfying certain monotonicity conditions, and that

$$L(x) := \lambda'(x) \left(\frac{g'(x)}{\lambda'(x)} \right)', \quad \ell(x) := \frac{1}{\sqrt{L(x)}}.$$

One of the one-sided Tauberian conditions considered is:

$$\lim_{\epsilon \to 0+} \liminf_{n \to \infty} \min_{n \leq m \leq n + \epsilon \ell(n)} (s_{m+1} - s_n) \geq 0.$$

The q-Wronskian and Rubel's n-nomial theorem

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We develop the basic properties of q-Wronskians and as an application give a q-analogue of the theorem of Lee Rubel which constructs a differential equation of order $2n$ that annihilates all polynomials with exactly n terms. Here we obtain analogous results with q-difference equations.

Harmonic analysis on the quantum Lorentz group, hierarchy of continuations of $su_q(2)$'s $6j-$symbols and Askey-Wilson scheme

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We will describe the recent progress in the study of Harmonic analysis on the quantum Lorentz group. The representation theory, Plancherel theorems, and Fusion theory are explicitely given in terms of a hierarchy of complex continuations of usual quantum $su(2)$'s $6j-$symbols and identities verified by these new objects are deduced from representation theoretic arguments. The askey-Wilson scheme enters this construction as a particular case.

Extended incomplete gamma functions with applications

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An extension of the generalized inverse Gaussian density function is proposed. Analogous to the recent useful generalization of the incomplete gamma functions, extensions of the generalized gamma functions are presented for which the useful properties and representations are naturally and simply extended. Several classical functions including, Abromowitz,s functions, Dowson,s integral function, Goodwin and Stalon's function,and astrophysical thermonuclear functions are proved to be special cases of these extensions.

In addition, extended Meijer G- functions and Fox's H- functios are defined. The decomposition formulae of these functions are developed as well. The present work provides a uniform approach to the study of the special functions.

Hypertranscendental quantum group

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The hypertranscendental quantum group is calculated by "hypertrans or $\circledast$". The nonhopfian groups are in the subvariety of the hypertrans cendental fundations. The reformulation of harmonicity has been obtained by a hypertranscendental Laplacian analysis. The example is extended to hypertranscendental zeta function.

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Generating orthogonal polynomials for exponential weights on a finite interval

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Generating orthogonal polynomials given a positive weight is simple in theory but difficult in practice. The actual construction of the polynomials is fraught with numerical instabilities.

In theory, the process of constructing orthogonal polynomials is the Gram-Schmidt orthogonalization process applied to a subspace of polynomials, called the Stieltjes procedure. The sequence of orthogonal polynomials satisfies a three term recurrence whose coefficients are the weighted centroids and the ratios of the weighted norms of the successive polynomials. From another perspective, it is also a multiple scales problem. The scales are the inherent scale of the weight and the polynomials and the interval length.

In practice, the calculations involve integrating a $2n$th degree polynomial for each nth-order coefficient and marching forward via the three term recurrence relation. This process is extremely delicate and is susceptible to numerical perturbations. Numerical stability is a major concern.

In this presentation, we will discuss an alternative formulation that focuses directly on a set of nonlinear recurrence equations governing the coefficients of the three-term recurrence and the values of the orthogonal polynomials at the end points. This set of algebraic equations is redundent with a core of linearly independent equations. We will subject these equations to an asymptotic analysis to extract their local behaviors and to select the final set of four nonlinear recurrence equations. We will discuss the resulting asymptotics induced domain decomposition method to the orthogonal polynomial generation problem.

The spectral geometry of polynomial hypergroups

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A number of families of orthogonal polynomials can be thought of as the characters associated with discrete and continuous hypergroups or more general measure algebras. For discrete polynomial hypergroups, the support of the orthogonality measure is contained in the real spectrum which is contained in the maximal ideal space, and both of these inclusions may be proper. This and other geometric inequalities will be described, and their consequences for the harmonic analysis associated with the family of polynomials will be developed.

Powers of Euler's product and related identities

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Let $(q)_\infty^r = \sum_{n\geq 0} a_r(n)q^n$ where $(q)_\infty = \prod_{j=1}^{\infty}(1-q^j)$. A theorem of M. Newman (1955) says that if $r = 2, 4, 6, 8, 10, 14$ or 26 and $p > 3$ is a prime satisfying $r(p+1) \equiv 0 \pmod{24}$, then

$$a_r\left(np + \frac{r(p^2-1)}{24}\right) = (-p)^{r/2-1} a_r\left(\frac{n}{p}\right)$$

where $a_r(n/p)$ is defined to be zero if n is not a multiple of p. The cases $r = 4, 6$ and 10 of Newman's theorem imply the Ramanujan partition congruences $p(5n+4) \equiv 0 \pmod 5$, $p(7n+5) \equiv 0 \pmod 7$, and $p(11n+6) \equiv 0 \pmod{11}$, respectively.

We shall give explicit, elementary proofs of all cases of Newman's theorem except $r = 26$, several new results for products of the form $(q)_\infty^r (q^a)_\infty^s$ and some conjectures. The proofs utilise Macdonald identities of ranks 1 and 2.

This is joint work with M. Hirschhorn and R. Lewis.

Entropy-like functionals for orthogonal polynomials: asymptotic and applications

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The study of the internal disorder of the physical systems by means of quantum mechanics and information theory often reduces to the analysis of the functional

$$E_n(w(x)) = -\int_\Delta p_n(x)^2 \, \log[p_n(x)^2] \, w(x)\, dx \tag{1}$$

where $p_n(x) = p_n(w(x); x)$ is the nth-polynomial orthonormal with respect to a weight $w(x)$ on an interval $\Delta \subseteq \mathbf{R}$. It is called, for obvious reasons, information entropy of the (orthonormal) polynomial $p_n(x)$.

In this talk we will review the knowledge of this analytical notion (and some related functionals) from its birth [1,2,3] up to now. The asymptotic behavior of these entropic integrals, which is intimately related to that of the polynomial norms in the space $L_w^p(\Delta)$, will be considered for general weights with compact and non-compact supports, as well as for varying weights. Also, the difficulties to obtain closed formulas for these objects in the classical orthogonal polynomials will be discussed. Indeed, only in the two Chebychev cases an explicit, closed and simple formula has been found.

Finally, some specific quantum-mechanical problems (prototypic potentials, entropic uncertainty relations,...) are shown to require the determination of the information entropies of all classical orthogonal polynomials.

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Multi-variable orthogonal polynomials of types A and B

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In the analysis of orthogonal polynomials of several variables there is a much more detailed structure when there is an action by a group of symmetries. Two important examples are the group of permutations of coordinates (type A) and the group generated by permutations and sign-changes (type B). An important problem is the construction of orthogonal bases for the L^2-spaces of the measures

$$A: \prod_{1\le m<n\le N} |\exp(i\theta_m)-\exp(i\theta_n)|^{2k} d\theta_1\dots d\theta_N,\ (\theta_1,\dots,\theta_N)\in \mathbf{T}^N$$

$$B: \prod_{1\le m<n\le N} |x_m^2-x_n^2|^{2k} \prod_{1\le n\le N} |x_n|^{2k_1}\exp(-\sum_{n=1}^{N} x_n^2/2)d\mathbf{x},\ \mathbf{x}\in\mathbf{R}^N$$

on the torus $\mathbf{T}^N$, and on Euclidean space; where k and k_1 are arbitrary positive parameters. The commutative algebra of differential-difference ("Dunkl") operators, which is analogous to the algebra of partial derivatives, provides crucial tools for the analysis. The construction of these orthogonal polynomials and formulas for norms and special values will be discussed. There will be an overview of related inner-product structures, some algebraic while others are of L^2-type. The L^2-space of the measure of type B above can be analyzed in two different ways, by generalized Hermite polynomials and by products of harmonic polynomials with Laguerre polynomials in the argument $|x|^2$. Some special cases of harmonic polynomials are now explicitly known. These measures occur in exactly solvable quantum-mechanical Calogero-Sutherland models.

Uniform asymptotic expansions for the generalised Bessel polynomials

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This paper provides a comprehensive asymptotic picture of the Bessel polynomials, which are generalisations of the well-known Modified K Bessel functions. A general turning point theory for ordinary differential equations is applied to obtain asymptotic expansions as the degree n of the polynomials tends to infinity. The results, involving Airy functions, are uniformly valid for the argument z lying in unbounded subdomains of the complex plane, and are complete with explicit error bounds. Moreover, there is a parameter a involved, and the results are valid for real or complex a, the only restriction on this being that $a = O(n)$ as n tends to infinity, and that a must not be close to the value $-n$.

Orthogonal matrix polynomials

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During the last few years, orthogonal matrix polynomials on the real line are receiving an increasing amount of interest due to the applications of matrix orthogonality to solve some problems of scalar orthogonality, but also because of the extension of some important results in the classical theory of orthogonal polynomials to orthogonal matrix polynomials. We will show a survey of that results concerning matrix orthogonality (properties of zeros, quadrature formulae, asymptotic properties, etc.) with especial emphasis on examples and relevant applications to solve scalar problems.

Computations and open problems for the Scorer functions

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We have recently completed a Fortran 90 package which computes the Scorer functions for complex arguments. These functions are solutions of the inhomogeneous Airy differential equation

$$\frac{d^2 f}{dz^2} - zf = \frac{1}{\pi},$$

where z is a complex variable. The general solution $f(z)$ is a linear combination of the Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ together with one of the Scorer functions $\mathrm{Hi}(z)$ or $-\mathrm{Gi}(z)$ (the minus sign is a historical consequence of Scorer's original work). These functions are computed by either summing their respective asymptotic expansions or by integrating their defining differential equation given above in the framework of a boundary valued problem. We discuss the challenges in writing such a code, emphasize some recently discovered functional identities, and state some open problems which we encountered, e.g., the lack of error bounds in truncating the asymptotic expansions and the lack of knowledge about the zeros of the Scorer functions in the complex plane.

A modified Reiter P_1-condition for hypergroups

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For a commutative hypergroup K we denote by $\Upsilon_b(K)$ the set of all characters of K. The multiplicative functionals of the Banach algebra $L^1(K,\mu)$ are given by

$$h_\alpha(f) = \int_K f(x)\,\overline{\alpha(x)}\;d\mu(x)\,, \quad f \in L^1(K,\mu)\,, \quad \alpha \in \Upsilon_b(K)\,,$$

where μ is the Haar measure on K. The Fourier transform of a function $f \in L^1(K,\mu)$ is defined by $\mathcal{F}f(\alpha) = h_\alpha(f)$.

We will give a sufficient and necessary condition for the existence of a bounded approximate identity in the maximal ideal $I(\alpha) := \ker(h_\alpha)$ with help of a modified Reiter P_1-condition. This condition is defined as follows.

Definition. We say that the Reiter P_1-condition with bound $M > 0$ is satisfied in $\alpha \in \Upsilon_b(K)$, if for each $\varepsilon > 0$ and every compact set $C \subset K$ there is some $g \in L^1(K,\mu)$ with the following properties:

(i) $\mathcal{F}g(\alpha) = 1$

(ii) $\|g\|_1 < M$

(iii) $\|L_{\tilde{y}}g - \overline{\alpha(y)}\;g\|_1 < \varepsilon$ for all $y \in C$.

Finally we will apply the general result to polynomial hypergroups and discuss some problems.

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Phase space and path integral methods in classical elliptic wave propagation modeling

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The analysis and fast, accurate numerical computation of direct and inverse wave propagation problems are often quite difficult for rapidly varying, multidimensional environments extending over many wavelengths. The atmosphere/earth and complex, multiscale material environments provide just such examples. The application of classical, "macroscopic" methods has resulted in direct wave field approximations, derivations of approximate wave equations, and discrete numerical algorithms. In the last several decades, however, developments in Fourier analysis, partial differential equations, asymptotic analysis, and mathematical physics, among others, have been synthesized into what is now called harmonic analysis in phase space. This analysis has produced sharp, "microscopic" tools (pseudodifferential and Fourier integral operators, wave packets) appropriate for attacking wave propagation problems in extended inhomogeneous environments. In conjunction with global path integral techniques, the multidimensional, fundamental wave field solutions can be represented explicitly and computed directly. The phase space methods and path (functional) integral representations provide the appropriate framework to extend homogeneous Fourier methods to extended inhomogeneous environments, in addition to providing the basis for the formulation and solution of the corresponding multidimensional (nonlinear) inverse problems.

In the context of fixed-frequency (elliptic) wave propagation problems, modeled at the level of the scalar Helmholtz equation, for example, the application of the above modern methods results in the exact reformulation of the two-way equation in terms of one-way equations which are well-posed for marching. This reformulation has obvious application for (1) large-scale, deterministic, direct wave propagation modeling, (2) the construction of computational boundary conditions, (3) the development of layer-stripping inverse algorithms and their corresponding regularization methods, (4) the incorporation of the exact inverse methods into optimization/iteration algorithms to account for realistic, noisy and limited data, and (5) the reformulation of stochastic wave propagation modeling. The one-way wave equations are constructed in terms of appropriate square root Helmholtz and Dirichlet-to-Neumann (DtN) operators, which are explicitly represented in terms of their operator symbols in a pseudodifferential operator calculus. The fundamental wave equation solutions (propagators) are expressed as path integrals, directly in terms of the operator symbols, which immediately result in explicit, marching (one-way) computational algorithms. Essentially, the uniform characterization over phase space and subsequent exploitation of the (singularity and oscillatory) structure of the operator symbols provide the focus for the direct and inverse analysis. Since the relevant (frequency-domain) operators lie outside of the well-developed theory of elliptic pseudodifferential operators and the corresponding asymptotic analysis, new, uniform, high-frequency asymptotic operator symbol expansions are developed. This analysis uniformly incorporates the usual algebraic terms associated with the elliptic calculus and the terms of exponential order, corresponding to the contributions from the infinitely-smooth part of the kernel, lying outside of the standard theory.

Both low-frequency and multiscale, high-frequency asymptotic results are also obtained. These asymptotic results are compared with several exact operator symbol constructions incorporating canonical features and including (1) both the defocusing and focusing quadratic profiles, (2) a class of hyperbolic function profiles incorporating symmetric and asymmetric wells and large-gradient effects, (3) the delta profile, and (4) piecewise-constant profiles such as the discontinuity and rectangular well. Both the exact and asymptotic operator symbol constructions make extensive use of special function theory.

Combinatorics of Affine and Virasoro highest weight modules

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I would like to survey some of the recent developments in our understanding of the combinatorics of highest weight representations of affine and Virasoro algebras.

Jacobi, Laguerre, and Hankel multipliers

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A survey of the theory of multipliers for Jacobi and Laguerre expansions and for Hankel transforms is presented. The various methods employed to derive necessary conditions and sufficient conditions for multipliers are discussed, along with some important open problems.

Incomplete gamma functions: the last 50 years

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The theory of the incomplete gamma function was consolidated in the early 1950s by F.G. Tricomi as part of the theory of confluent hypergeometric functions. Here we review progress made since then. This includes work on uniform asymptotics, the generalized exponential integral, inverse functions and zeros, inequalities and monotonicity, as well as computational methods.

Asymptotics of eigenfunctions on type A affine buildings

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Hall polynomials appear as the structure constants for the algebras of spherical functions on the general linear groups over local fields. The spherical functions correspond to invariant kernels on the set of special points of the associated affine building. The asymptotic behavior of the eigenfunctions are described, and a Poisson formula is given for an unramified eigenvalue, for all type A affine buildings with rank ≥ 2.

This is a joint work with King Fai Lai, University of Sydney.

Electrostatics and the Darboux process

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More that 100 years ago Stieltjes found a nice and simple electrostatic interpretation for the zeros of the certain families of orthogonal polynomials, including those of Jacobi. I will show how a small modification of his method allows one to give in some cases a simple electrostatic interpretation of the polynomials obtained from those of Jacobi by repeated applications of the Darboux process.

Contractions, De Leeuw's theorem and asymmetry of Fourier multipliers

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Two physicists, Inönu and Wigner pointed out that Mehler-Heine type of formulas relating special functions, arise from a notion of a continuous deformation (known as contraction) of one Lie group into another. Using contractions, it has been shown in several articles in the literature that:

(i) the matrix coefficients of the starting Lie group can be recovered from that of the deformed group

(ii) De Leeuw kind of Fourier multiplier transference between the two groups.

Recently, in joint work with Dooley, I have shown that:

(a) matrix coefficients of the Heisenberg group H^{p-1} can be approximated by the matrix coefficients of the unitary group $U(p)$ using a contraction between the groups H^{p-1} and $U(p)$

(b) De Leeuw kind of Fourier multiplier transference from $U(p)$ to H^{p-1}.

Our results (a) and (b) generalise the earlier results of Ricci-Rubin. Our methods for the proof of (a) and (b) also yield a strong converse of De Leeuw's tranference result for the pair $(H^{p-1}, U(p))$, which, so far has not been known in the noncommutative set up. Our results resolve a conjecture of Herz (1977) regarding the asymmetry of Fourier multipliers on compact nonabelian Lie groups. In other joint work with Dooley and Ricci, I have proved a very general result using contractions regarding the transference of Fourier multipliers which again resolve the Herz conjecture.

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q-Statistics and perfect matchings with forbidden edges

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We apply methods from the theory of q-rook polynomials to develop a q-analog of the number of k-edge matchings in a threshold graph. This leads to a q-analog of a result of Reiner and White, as well as closed form expressions for these q-matching numbers. We then introduce a version of the q-hit polynomial from rook theory involving a family of statistics on perfect matchings which satisfy an analog of the Mahonian property.

Partitions into squares

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Jacobi's four-square theorem says that the number of representations of the number n as a sum of four squares is eight times the sum of those divisors of n not dvisible by four. Thus, for example, 30 has 576 representations. However, 30 has only two partitions into four squares! We find the generating function for $p_{4\Box}(n)$, the number of partitions of n into four squares, and show that $p_{4\Box}(n)$ satisfies certain arithmetic relations.

Asymptotics of $Gl(3,R)$ Whittaker functions and applications

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This work is joint with Nam Jong Moh and is an extension of previous work witth Dan Bump.

In the paper "Unramified Whittaker functions for $Gl(3,R)$" (Journal D'Analyse Mathematicque, 65), Bump and Huntley study the properties of the decaying Whittaker function for the group $Gl(3,R)$. This function is a generalization of the K-Bessel function (and in fact may be written as an integral of such functions) and is important in the study of the symmetric space associated to $Gl(3,R)$,which we denote by H. It is a function of two positive real numbers and is an eiegenfunction a two differential operators that are derived from studying the ring of invariant differential operators on H. One operator is second order and the other is third order. The Whittaker function thus also depends on two parameters λ and μ, corresponding to the eigenvalues. For simplicity we will supress this and denote the Whittaker function by $W(y_1,y_2)$.

It is shown by Bump and Huntley that on any curves of the form $y_1 = y_2^\alpha$ (or $y_2 = y_1^\alpha$) with $\alpha > -1/2$ and y_2 (Resp. y_1) tending to infinity, the Whittaker function has an aymptoitc expansion. Given the irregular singularities of the differential equations at infinity this is the most that one would expect.

The first term of the expansion is $y_1^{\frac{1}{3}} y_2^{\frac{1}{3}} (y_1^{\frac{2}{3}} + y_2^{\frac{2}{3}})^{\frac{-1}{4}} \exp(-(y_1^{\frac{2}{3}} + y_2^{\frac{2}{3}})^{\frac{3}{2}})$. The second term in the expansion depends on λ but not μ. The later tems depend on both parameters.

In this talk, recent work of Huntley and Moh is discussed. Error estimates for the expansion will be considered. We will also describe work on extending the range of validity of the expansion and applications to harmonic analysis on H. In particular we will discuss problems in the theory of automorphic forms.

Symmetry classification for Jackson integrals associated with irreducible reduced root systems

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We use notation $(\xi;q)_\infty := \prod_{i=0}^\infty (1-\xi q^i)$ and $\vartheta(\xi;q) := (\xi;q)_\infty (q/\xi;q)_\infty (q;q)_\infty$.

Let R be an irreducible reduced root system, spanning a real vector space E of dimension n, and let $\langle \cdot,\cdot \rangle$ be a positive definit scalar product on E under the Weyl group W of R. We denote by R^+ the set of positive roots relative to a fixed basis $\{\alpha_1,\cdots,\alpha_n\}$ of R. For each $\alpha \in R$, let $\alpha^\vee = 2\alpha/\langle \alpha,\alpha\rangle$. Let P be the *coweight lattice* $\{\chi \in E; \langle \alpha,\chi\rangle \in \mathbf{Z}$ for any $\alpha \in R\}$ and let Q be the *coroot lattice* of R defined by $Q = \mathbf{Z}\alpha_1^\vee + \cdots + \mathbf{Z}\alpha_n^\vee \subset P$. Let L be any sublattice of P of rank n. The scalar product $\langle \cdot,\cdot \rangle$ is uniquely extended linearly to $E_\mathbf{C} = E \otimes_\mathbf{R} \mathbf{C} \simeq \mathbf{C}^n$.

For each $\alpha \in R$, let q_α be a complex number such that $q_\alpha = q_\beta$ if $|\alpha| = |\beta|$. We assume $|q_\alpha| < 1$. For $x \in E_{\mathbf{C}}$, we define

$$\begin{aligned}
&\Phi_R(\beta_1, \cdots, \beta_s, \gamma_1, \cdots, \gamma_l, x) = \Phi_R(\{\beta_i\}, \{\gamma_j\}; x) \\
&= \prod_{i=1}^{s} \prod_{\substack{\alpha>0 \\ \alpha:short}} q_\alpha^{(\frac{1}{2}-\beta_i)\langle\alpha,x\rangle} \frac{(q_\alpha^{1-\beta_i+\langle\alpha,x\rangle}; q_\alpha)_\infty}{(q_\alpha^{\beta_i+\langle\alpha,x\rangle}; q_\alpha)_\infty} \cdot \prod_{j=1}^{l} \prod_{\alpha>0 \ \alpha:long} q_\alpha^{(\frac{1}{2}-\gamma_j)\langle\alpha,x\rangle} \frac{(q_\alpha^{1-\gamma_j+\langle\alpha,x\rangle}; q_\alpha)_\infty}{(q_\alpha^{\gamma_j+\langle\alpha,x\rangle}; q_\alpha)_\infty}
\end{aligned}$$

where $s, l \in \mathbf{Z}_{\geq 0}$, $\beta_i, \gamma_j \in \mathbf{C}$ and $\alpha > 0$ means $\alpha \in R^+$. If all roots $\alpha \in R$ have same length, we regard the roots as all short. We denote by $\Delta_R(x)$ the Weyl denominator as

$$\Delta_R(x) := \prod_{\alpha>0} (q_\alpha^{\frac{1}{2}\langle\alpha,x\rangle} - q_\alpha^{-1/2}\langle\alpha, x\rangle).$$

For $z \in E_{\mathbf{C}}$, we define the *Jackson integral associated with* R as

$$J_R(\{\beta_i\}, \{\gamma_j\}; L; z) := \sum_{\chi\in L} \Phi_R(\{\beta_i\}, \{\gamma_j\}; z + \chi)\ \Delta_R(z + \chi).$$

Theorem 1. *Assume that $q_\alpha = q$ for all $\alpha \in R$. For any sublattice $L \subset P$ of rank n, the sum $J_R(\{\beta_i\}, \{\gamma_j\}; L; z)$ is expressed as*

$$C_R(\{\beta_i\}, \{\gamma_j\}; L) \prod_{\substack{\alpha>0 \\ \alpha:short}} \frac{q^{((s-1)/2-\sum_{i=1}^{s}\beta_i)\langle\alpha,z\rangle}\vartheta(q^{\langle\alpha,z\rangle}; q)}{\prod_{i=1}^{s}\vartheta(q^{\beta_i+\langle\alpha,z\rangle}; q)} \cdot \prod_{\substack{\alpha>0 \\ \alpha:long}} \frac{q^{((l-1)/2-\sum_{j=1}^{l}\gamma_i)\langle\alpha,z\rangle}\vartheta(q^{\langle\alpha,z\rangle}; q)}{\prod_{j=1}^{l}\vartheta(q^{\gamma_j+\langle\alpha,z\rangle}; q)}$$

where $C_R(\{\beta_i\}, \{\gamma_j\}; L)$ is a constant not depending on $z \in E_{\mathbf{C}}$, if and only if
$s = 1$ for A_n, E_6, E_7 and E_8-type,
$(s, l) = (1, 1)$ or $(2n - 1, 0)$ for B_n-type,
$(s, l) = (1, 1)$ or $(\frac{n+1}{2}, 0)$ for C_n-type if n is odd,
$(s, l) = (1, 1)$ or $(4, 0)$ for G_2-type,
$(s, l) = (1, 1)$ or $(3, 0)$ for F_4-type.

In [1, 3, 5], the Jackson integrals with $(s, l) = (1, 1)$ are investigated and an explicit form of the constant $C_R(\beta_1, \gamma_1; L)$, which is written as a product of q-gamma functions, is obtained when $L = P$ and Q. The cases $(s, l) = (2n-1, 0)$ for B_n-type and $(s, l) = (4, 0)$ for G_2-type are investigated by Gustafson and explicit forms of the constants $C_R(\{\beta_i\}; P)$ of them are known (See [2]). For the case $(s, l) = (3, 0)$ for F_4-type and its constant $C_{F_4}(\{\beta_i\}; L)$, see [4].

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The little q-Jacobi functions of complex order

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We use Ismail's argument and an elementary combinatorial identity to prove the q-binomial theorem, the symmetry of the Rogers-Fine function, Ramanujan's ${}_1\psi_1$ sum, and Heine's q-Gauss sum and give many other proofs of these results. We prove a special case of Heine's ${}_2\phi_1$ transformation and write Ramanujan's ${}_1\psi_1$ sum as the nonterminating q-Chu-Vandermonde sum. We show that the q-Saalschütz and q-Chu-Vandermonde sums are equivalent to the evaluations of certain moments and to the orthogonality of the little q-Jacobi polynomials; hence the q-Chu-Vandermonde sum implies the q-Saalschütz sum. We extend the little q-Jacobi polynomials naturally to the little q-Jacobi functions of complex order. We show that the nonterminating q-Saalschütz and q-Chu-Vandermonde sums are equivalent to the evaluations of certain moments and, using the Liouville-Ismail argument, to two orthogonality relations. We show that the nonterminating q-Chu-Vandermonde sum implies the nonterminating q-Saalschütz sum.

Raising operators of row type for Macdonald polynomials

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This is a joint work with Prof. M. Noumi (Kobe University).

We consider the raising operators B_m of row type for the integral forms $J_\lambda(x;q,t)$ of Macdonald polynomials of type A with partition $\lambda = (\lambda_1, \dots, \lambda_n)$. These are a dual version of the raising operators of column type by A.N. Kirillov and M. Noumi. The oprators B_m have the following property,

$$B_m J_\lambda(x;q,t) = \begin{cases} J_{(m,\lambda)}(x;q,t) & \text{if} \quad \ell(\lambda) < n, \\ 0 & \text{if} \quad \ell(\lambda) = n. \end{cases} \tag{1}$$

Macdonald symmetric polynomials are characterized as the joint eigenfunction of a certain family of commuting q-difference oerators. The uniqueness and existence of the q-difference operators B_m can be proven by analysing the action of the commuting family of q-differnce operators on the generating function of dual type for Macdonald polynomials.

Theorem 1. *For each $m = 0, 1, 2, \dots,$ there exists a unique q-difference operator*

$$B_m = \sum_{|\gamma| \le m} b_\gamma^{(m)}(x) T_{q,x}^\gamma \tag{2}$$

of order $\le m$ satisfying the condition (1), where $b_\gamma^{(m)}(x)$ are rational functions in x with coefficients in $\mathbb{Q}(q,t)$. Furthermore, the operator B_m is invariant under the action of the symmetric group $\mathfrak{S}_n$ of degree n.

We can also determine the explicit form of B_m. In caluculation of explicit forms, there arises a special case of ${}_6\psi_6$ Bailey summation formula in $SU(n)$ due to R.A. Gustafson. Lowering operators can be constructed similarly.

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Algebraic aspects of quantum Calogero models

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Among quantum integrable models in one dimension, Calogero-Sutherland type models catch renewed interests.

A typical example of such models is the Sutherland model:

$$\mathcal{H}_s = -\sum_{j=1}^{N} \frac{\partial^2}{\partial \theta_j^2} + \frac{1}{2}\sum_{j<k} \frac{\beta(\beta-1)}{\sin^2[(\theta_j-\theta_k)/2]}. \tag{1}$$

From algebraic viewpoint, this model is related to a degenerate version of the double affine Hecke algebra introduced by Cherednik.

The Calogero models of A-type and B-type,

$$\mathcal{H}_a = \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2} + x_j^2\right) + \sum_{j<k}\frac{\beta(\beta-1)}{(x_j-x_k)^2}, \tag{2}$$

$$\mathcal{H}_b = \frac{1}{2}\sum_{j=3D1}^{N}\left\{-\frac{\partial^2}{\partial x_j^2} + x_j^2 + \frac{\gamma(\gamma-1)}{x_j^2}\right\} + \sum_{j<k}\left\{\frac{\beta(\beta-1)}{(x_j-x_k)^2} + \frac{\beta(\beta-1)}{(x_j+x_k)^2}\right\}, \tag{3}$$

share the same algebraic structure; that is, these three Hamiltonians are connected to individual representation of the degenerate double affine Hecke algebra. To construct the representations, the Dunkl operators play a crucial role.

Wavefunctions of these Hamiltonians are written in terms of multivariable orthogonal polynomials. The Jack polynomials appear in the case of the Sutherland model. In the cases of the Calogero models (2) and (3), the polynomial part of the wavefunctions can be regarded as a multivariable generalization of the Hermite and Laguerre polynomials and has been studied by several authors [1–4].

We will discuss several applications of the algebraic structure to the properties of these multivariable orthogonal polynomials, such as the shift operators, the norm formula. We also give a new differential formula for the Jack polynomials J_λ:

$$e_n(D)J_\lambda(x) = \sum_{|\mu|=|\lambda|-n} f_{n\lambda}^{\mu} J_\mu(x), \tag{4}$$

where $e_n(D)$ denotes the n-th elementary symmetric function whose arguments are the Dunkl operators. We will give the coefficients $f^{\mu}_{n\lambda}$ explicitly.

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Special functions and black holes

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The occurrence of spheroidal type special functions arises naturally when the linearised equations for metric perturbations of a Black Hole space time are considered. It is somewhat of a surprise that these perturbations are explicitly solvable using Spheroidal type functions as a corresponding Debye potential. This is a consequence of the fact that the corresponding Debye potential equations can be solved by the method of separation of variables. This in turn is related to the fact that the Spacetime of a Black hole is known to admit a so called Killing Yano symmetry tensor. From this tensor one can construct an angular momentum type constant of the geodesic motion as well as the corresponding symmetry operator of the corresponding Klein Gordan equation. The separability mechanism can be extended to the case of metric perturbations (spin2). Black Hole space time as well as electromagnetic perturbations (spin1). We discuss the state of completeness of these problems an give some indication of how to deal with the case of perturbations of a charged rotating black hole. Associated with the discussion will be some ideas on the mathematical significance of the existence of a Killing Yano tensor.

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On a nonlinear recurrence related to Nevai polynomials

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Orthogonal polynomials on the real line corresponding to the weight function $w(x) = \exp(-x^4)$ have been studied by Freud and, most notably, Nevai. These "Nevai" polynomials $\{p_n(x)\}$ have been shown to satisfy the recurrence relation

$$x \cdot p_n(x) = a_{n+1}p_{n+1}(x) + a_n p_{n-1}(x),$$

where the coefficients $\{a_n\}$ in the recursion in turn can be generated from the nonlinear recurrence

$$n = 4a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2) \tag{1}$$

for $n = 1, 2, 3, \ldots$, with starting values for the sequence of $a_0 = 0$ and $a_1^2 = \Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})$. Efficient iterative schemes have been developed for computing the sequence $\{a_n\}$ numerically, but a closed form representation of the $\{a_n\}$ suitable for use in developing a generating function has proven elusive.

In this talk, I will describe work underway towards the construction of a generating function for the $\{a_n\}$ and related nonlinear recurrences. In particular, I will show how a simple transformation of the three-term recurrence (1) can yield a nonlinear recurrence involving just two terms, which is solvable in closed form.

Forrester's contant term identity II

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The Forrester's conjectural constant term identity, in the form of integral evaluation, is

$$\int_{[0,1]^N} \prod_{i=1}^{N} t_i^{x-1}(1-t_i)^{y-1} \prod_{\substack{i,j=N_0+1 \\ i\neq j}}^{N} \left(1-\frac{t_i}{t_j}\right) \prod_{N_0+1\le i<j\le N} (t_i-t_j)^2 dt_1\cdots dt_N$$

$$= \prod_{l=0}^{N_0-1} \frac{\Gamma(y+kl)\Gamma(x+k(N-1-l))\Gamma(1+k(l+1))}{\Gamma(x+y+k(N-1+l))\Gamma(1+k)}$$

$$\times \prod_{j=0}^{N_1-1} \frac{(j+1)\Gamma(y+(k+1)j+kN_0)\Gamma(x+k(N-1-N_0-j)-j)\Gamma((k+1)(j+1)+kN_0)}{\Gamma(x+y+k(N-1+N_0+j)+j)\Gamma(1+k)}$$

where $N = N_0 + N_1$.

The special cases of $k = 1$ (general x, y, N_0, N_1) and $N_1 = 2, 3$ (general x, y, k, N_0) have been verified so far. In this talk we shall report progress including the proof of $N_1 = 4$ case (general x, y, k, N_0).

On the non-commutative neutrix product $B^{(s)}(x_+, n) \circ x_+^r \ln x_+$

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In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

Let f, g be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$, $g_n = g * \delta_n$, where $\{\delta_n\}$ is a certain sequence converging to the Dirac delta-function, see Fisher [1]. The non-commutative neutrix product $f \circ g$ is defined as the neutrix limit of the sequence $\{fg_n\}$, providing the limit h exists in the sense that

$$N\langle fg_n, \phi\rangle = \langle h, \phi\rangle,$$

for all ϕ in $\mathcal{D}$, where N is the neutrix, see van der Corput [2], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, ...)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In this work, the Beta function $B(x, n)$ and the associated Beta functions $B(x_\pm, n)$ are defined as distibutions and neutrix product $B^{(s)}(x_+, n) \circ x_+^r \ln x_+$ is evaluated, from which other neutrix products are deduced.

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A decoding scheme for the lexicodes $S_{9,3}$ with base 4

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Let $S_{n,d}$ be the lexicographic codes with length n, minimum distance d and $q = 4$. For the q-ary $S_{9,3}$, 6 basis vectors are obtained. These vectors are used in decoding scheme of $S_{9,3}$. So we propose a simple decoding algorithm for this code.

Inversion formulas involving orthogonal polynomials and some of their applications

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We derive inversion formulas involving orthogonal polynomials which can be used to solve different kind of problems involving these orthogonal polynomials. We show how they can be used to find connection coefficients in some very special cases. Further we show how they can be used to find coefficients of differential equations satisfied by generalizations of the classical orthogonal polynomials. As an example we consider special symmetric generalizations of the Hermite and the ultraspherical polynomials.

Harmonic analysis on the quantum $SU(1,1)$ group

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The spherical Fourier transform on $SU(1,1)$ gives the Legendre function transform, and consideration of other K-types gives rise to the Jacobi function transform. For the quantum group case we can parametrise the subgroup K by a parameter, and this gives rise to a two-parameter analogue of the Legendre function transform. For this purpose we have to describe the analogue of the Haar functional as a weight on a C*-algebra. We consider two-cases: (i) the Haar functional as a Jackson type integral over $[-1,\infty)$, (ii) the Haar functional as an Askey-Wilson type integral with unbounded support. The asssociated spherical Fourier transforms give rise to the definitions of the big q-Jacobi function transform and the Askey-Wilson function transform. See also Jasper Stokman's lecture at this conference for the discussion of the analytic aspects of the Askey-Wilson function transform.

This is joint work with Jasper Stokman, and in the course of this work we have received essential help from Mizan Rahman.

A convolution product associated with a pair of q-Fourier transforms

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The speaker introduced in [1, §8] a q-Fourier transform pair which has a quantum group interpretation on the braided line. This has later been extended by Giovanna Carnovale [2] to the case of the braided A_n quantum space. In the lecture definition and properties of a q-convolution product on the braided line will be given which is mapped by the q-Fourier transform to the pointwise product. This is joint work with Giovanna Carnovale.

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On certain Gegenbauer polynomial sums

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We prove that for all $\lambda \geq 0$,

$$\sum_{k=0}^{n} a_k \frac{C_k^\lambda(\cos\theta)}{C_k^\lambda(1)} \cos(k+\frac{1}{2})\phi > 0, \quad n = 1, 2, \ldots, 0 \leq \phi \leq \theta < \pi,$$

if and only if $\sum_{k=0}^{n} a_k \cos kx > 0$, for all $n \geq 1$ and $0 < x < \pi$, where $C_k^\lambda(x)$ is the Gegenbauer polynomial of degree k and order λ. This result enables us to show that, for $\lambda \geq 0$, the polynomials $P_n^\lambda(z;x) := \sum_{k=0}^{n} a_k \frac{C_k^\lambda(x)}{C_k^\lambda(1)} z^k$ have no zeros for z in the unit disk $\mathbf{D}$, if and only if $\sum_{k=0}^{n} a_k \cos kx > 0$, for all $n \geq 1$ and $0 < x < \pi$.

The method of the proof of these results uses several integral representations for Gegenbauer polynomials. In a similar way, we are able to prove that for all positive integers n, we have

$$\sum_{k=0}^{n} C_k^\lambda(x)\, z^k \neq 0, \qquad |z| \leq 1,$$

when $-\frac{1}{2} < \lambda < 0$ or $0 < \lambda \leq \frac{1}{2}$.

This result is best possible and extends in the complex plane some old results of Fejér and Szegő.

All the above results can be extended to sums of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which include the $C_n^\lambda(x)$ as the special case where $\alpha = \beta = \lambda - \frac{1}{2}$.

Bochner-Krall orthogonal polynomials

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Let $\tau = \sigma + \nu$ be a point masses perturbation of a classical moment functional σ, which is quasi-definite. We find conditions for orthogonal polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ to be Bochner-Krall orthogonal polynomials, that is, $\{Q_n(x)\}_{n=0}^{\infty}$ are eigenfunctions of a finite order linear differential operator of the spectral type with polynomial coefficients. For example, we show that σ cannot be the moment functional for Bessel or Hermite polynomials and $supp(\nu)$ contains at most two points. These solve partially Magnus' conjecture and explain completely the phenomena for infinite order differential equations (found by J. Koekoek and R. Koekoek), which have generalized Jacobi polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$ and generalized Laguerre polynomials $\{L_n^{\alpha,M}(x)\}_{n=0}^{\infty}$ as eigenfunctions.

Backlund transformations and product formulas for special functions

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We review the theory of Backlund transformations (BT's) for finite-dimensional integrable systems and construct new BT's for several families of such systems, like Toda lattice, sl(2) Gaudin and Heisenberg magnet. Quantum Backlund transformations are studied and applied to special functions theory. They are linked to (integral) product formulas and to their multi-variable generalisation. As a result, we find several new product formulas for classical special functions and orthogonal polynomials. Their generalisation to the multi-variable case are also studied.

Proof of a conjecture on the gamma function

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Anderson and Qiu [1] proved that the function $h(x) = \log\Gamma(x+1)/(x\log x)$ strictly increases from $1-\gamma$ to 1 as x increases from 1 to ∞, where $\gamma = 0.577\cdots$ denotes the Eulero Mascheroni constant. Moreover they conjectured that $h(x)$ is concave for $x > 1$.

A. Elbert and A. Laforgia proved this conjecture. The proof is based on the infinite product formula for the gamma function and on some bounds for $\log\Gamma(x)$, $\Psi(x)$ and $\Psi'(x)$, where $\Psi(x) = \Gamma'(x)/Gamma(x)$ and $' = d/dx$.

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The fine stucture of Jack polynomials

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Fifty years ago the great chinese mathematician Hua Loo-Keng set the problem of obtaining an explicit expression for the “zonal polynomials”, familiar to statisticians. This problem kept open in spite of many attempts.

Zonal polynomials are now understood as a special case of Jack polynomials. We present results showing that an explicit expression for Jack polynomials might be within reach, thus giving some (yet very partial) answer to the Hua’s problem.

Jack polynomials J_λ^α form a basis of the algebra of symmetric polynomials in N independent variables, with coefficients rational in a parameter α. Another (explicit) basis of this algebra is given by the “power sums” p_λ. Both basis are indexed by partitions $\lambda = 3D(\lambda_1, \dots, \lambda_N)$.

We investigate the structure of θ_μ^λ, the transition matrix between these two basis defined by $J_\lambda^\alpha = 3D \sum \theta_\mu^\lambda p_\mu$.

First we (explicitly) define new combinatorial quantities $d_\rho(\lambda)$. They form a basis of the algebra of “shifted symmetric” polynomials in λ, that is polynomials with coefficients rational in α and symmetric in the “shifted” variables $\lambda_i - i/\alpha$. Then we expand θ_μ^λ upon these quantities and obtain

$$\theta_\mu^\lambda = 3D \sum_{i\geq 0} \sum_{j=3D0}^{r-2i} \alpha^{i+j}(1-\alpha)^{r-2i-j} \sum_{|\rho|=3Dj} c_i^\rho(\mu) d_\rho(\lambda).$$

Here n is the common weight of λ and μ, $r = 3Dn - l(\mu)$. The coefficients $c_i^\rho(\mu)$ are polynomials in the variable n, with degree $2i$, dependent on μ but independent of α and λ: $c_i^\rho(\mu) \in \mathbf{Q}[n]$.

This formula gives the explicit dependence of the Jack polynomial J_λ^α with respect to the parameter α and to the partition λ. It reduces the Hua's problem to the knowledge of a collection of polynomials in one variable. Being independent of α and λ, these polynomials $c_i^\rho(\mu)$ can be considered as the "structure constants" of Jack polynomials.

It is remarkable that the most elementary of these coefficients can be expressed in terms of Stirling numbers of the first kind.

Our results are new even in the particular case $\alpha = 1$, corresponding to Schur functions. In that case they lead to new combinatorial problems linked with the Littlewood-Richardson rule.

Kernel functions for orthogonal polynomial approximation $L^p(\mu)$- and C-spaces

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Pollard, Muckenhoupt, Bavinck and many more have studied orthogonal polynomial approximation in $L^p(\mu)$– and C–spaces especially in the case of Jacobi polynomials. After a short overview of their results we introduce two classes of kernel functions for general orthogonal polynomials. One class has nonnegativity properties (hence we call it Fejér-type kernel) and the other class has properties as the de la Vallée Poussin kernel. We can derive sufficient conditions such that the corresponding Lebesgue constants are bounded. Thus we can apply approximation processes not only for Jacobi polynomials but also for generalized Chebyshev polynomials, Bernstein-Szegö polynomials and some further families of orthogonal polynomial sequences.

Boundedness of Fourier operators for exponential weights

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Let $W = e^{-Q}$ be a Freud weight and $\{P_n(x)\}_{n=0}^\infty$ be the corresponding orthonormal polynomials with respect to W^2. For a given function f, $S_n[f](x)$ denotes the n-th Fourier expansion of orthonormal polynomials for f. Let $1 < p < \infty$. Then we find a sufficient condition that for $fW_B \in L^p(\mathbb{R})$,

$$\|S_n[f]Wu_b\|_p \le C\|fWu_B\|_p, \tag{1}$$

and for $f^{(r-1)} \in AC_{loc}$ and $f^{(r)}Wu_B \in L^p(\mathbb{R})$,

$$\|S_n[f]^{(r)}Wu_b\|_p \le C\|f^{(r)}Wu_B\|_p, \quad k = 0, 1, 2, \dots, r, \tag{2}$$

where $\|\cdot\|_p$ is the L^p-norm and $u_b(x)=(1+|x|)^b$. We also give a sufficicient condition for the inequality

$$\|S_n[f]W(1+Q)^{-b}\leq C\|fW(1+Q)^B\|_p \tag{3}$$

when W is an Erdös weight.

Left-definite spectral theory with applications to differential equations and orthogonal polynomials

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If A is a a self-adjoint operator in a Hilbert space H that is bounded below by a positive constant, we show that there is a continuum of Hilbert spaces $H(r)$ (for $r>0$) and a continuum of self-adjoint operators $A(r)$ in $H(r)$, generated by A. These spaces and operators are called left-definite Hilbert spaces and left-definite operators associated with (H,A); the origins of left-definite theory can be traced to H. Weyl's 1910 treatise on second-order differential equations. We apply our general results to some of the classical second-order equations of Laguerre and Hermite. This is joint work with R. Wellman.

Mean convergence of rational functions interpolating on the unit circle

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Let $\mathbf{A}$ denote the disc algebra. Assume α_j, $j=0,1,2,...$, are (not necessarily distinct) points in the open unit disc. Then there are $n+1$ distinct points $z_{n,k}$, $k=0,1,...,n$, on the unit circle such that $w_n(z_{n,k})=w_n^*(z_{n,k})$, where $w_n(z)=\prod_{j=0}^{n}(z-\alpha_j)$ and $w_n^*(z)=\prod_{j=0}^{n}(1-\overline{\alpha}_j z)$. We study the mean convergence of the rational functions of the form $p_n(z)/w_n^*(z)$, with p_n being a polynomial of degree at most n, that interpolates a function from $\mathbf{A}$. In particular, we establish that the mean convergence is guaranteed by the Blaschke divergence condition: $\sum_{j=0}^{\infty}(1-|\alpha_j|)=\infty$.

Preserving properties of (Jacobi) conjugate functions and multipliers of Zygmund's type for Jacobi expansions

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One of the preserving properties of classical conjugate functions is that if f belongs to Lipα for $\alpha \in (0,1)$, so does the conjugate function. This is called Privalov's theorem. The multiplier theorem of Zygmund, which can be viewed as a generalization of Privalov's theorem, states that for $\alpha \in (0,1)$, a necessary and sufficient condition for that $\sum \lambda_n c_n e^{inx}$ belongs to Lipα whenever $\sum c_n e^{inx}$ does, is that $\sum (in)^{-1} \lambda_n e^{inx}$ is a Fourier series of a function in Λ_*^1 (Zygmund's class in L^1-norm).

In the present paper, we study the preserving properties of (Jacobi) conjugate functions and also the multiplier theorems of Zygmund's type for Jacobi expansions and conjugate Jacobi expansions. For this purpose, a new translation operator in terms of the conjugate Jacobi system will be introduced and the characterizations of the corresponding Lipschitz classes by the (Jacobi) Poisson and conjugate Poisson integrals will be used.

Five asymptotic expansions for the three symmetric standard elliptic integrals

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Symmetric standard elliptic integrals are considered when one of their parameters is larger than the others. Distributional approach is then used for deriving five convergent expansions of these integrals in inverse powers of the respective five possible asymptotic parameters. Coefficients of these expansions are obtained by recurrence and, most of times, they involve nothing but elementary functions. Error test is used for providing the approximation supplied by any of these expansions of an error bound at any order. Numerical experiments show that this bound is considerably accurate. Convergence speed of any of these expansions increases for increasing difference between the asymptotic variable and the remaining ones.

The DLMF project: A new initiative in classical special functions

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The National Institute of Standards and Technology has started an ambitious project that aims to produce a successor to Abramowitz and Stegun's *Handbook of Mathematical Functions* [1].

This *Handbook* was published by the National Bureau of Standards in 1964, and was reprinted by Dover in 1965. Errata were corrected through the 10th government printing in 1972. However, having never been revised, the basic information is current only up to about 1960.

The *Handbook* has been a great success. The government and Dover editions have been in print continuously since their original appearance, and they continue to sell briskly in technical bookstores and the technical sections of general bookstores. Total government sales approach 150,000. Because of superior marketing and lower price, Dover's total sales are estimated to be two or three times this number. The amazon.com sales rank of the Dover edition falls consistently in the top 10,000 in its stock of over 4.5 million titles. For comparison, on a recent day the sales rank of Gradshteyn and Ryzhik's *Tables of Integrals, Series, and Products* was 62,400. Another measure of success is the record of citations. According to a study of Science Citation Index data, the *Handbook* was cited more than 7000 times in 1992–96, and the trend of citations relative to the index as a whole is increasing.

In view of the many advances in the theory, computation and application of special functions in recent decades, this trend cannot continue. The current project is intended to meet the need for a new standard reference. With the help of a distinguished board of associate editors, the project leaders at NIST have constructed an outline for the new work. The plan of work calls for engaging leading researchers to write and validate the new reference material.

The project will result in a *Digital Library of Mathematical Functions*, published on the Web and in a book with CD-ROM. The *DLMF* will provide many capabilities that are impossible in print media alone. These include advanced search for formulas, downloading of formulas into programs and documents, links to software descriptions and source code, user-controllable visualization of functional surfaces (with rotation, zoom and animation), and support of auxiliary Web sites that link to the DLMF site for basic information on special functions. For a prototype, see http://www.nist.gov/DigitalMathLib. Project background is provided in [2].

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Symmetry constraints and separation of variables of soliton equations

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Symmetry constraints are imposed on soliton equations, which transforms Lax pairs into integrable finite-dimensional Hamiltonian systems. The constrained Lax pairs provide a kind of separated variables for soliton equations and Jacobi inversion problems of the constrained Lax pairs exhibit integrability by quadratures for soliton equations. The idea is illustrated by the KdV and AKNS hierarchies.

On a semiclassical sequence of Electrostatic interpretation of their zeros

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Semiclassical linear functionals $\tilde{\mathcal{U}}$ are introduced as the solutions of distributional differential equations

$$D\left(\sigma\tilde{\mathcal{U}}\right) = \tau\tilde{\mathcal{U}}$$

where σ, τ are polynomials and degree $\tau \geq 1$.

In this paper we consider the linear functional

$$\delta(-1) + M_1\delta'(1) + N_1\delta'(-1)$$

where $\mathcal{J}_{\alpha,\beta}$ is the Jacobi one, and M_i, N_i, $i = 0, 1$ are real numbers.

First, we give necessary and sufficient conditions in order $\tilde{\mathcal{U}}$ be quasi-definite, i.e. there exists a sequence of monic polynomials Q_n orthogonal with respect to $\tilde{\mathcal{U}}$.

Next, we prove that such polynomials are hypergeometric functions ${}_6F_5$ and we deduce in an explicit way a second order differential equation

$$A(x;n)Q_n''(x) + B(x;n)Q_n'(x) + C(x;n)Q_n(x) = 0$$

that such polynomials verify. Here $A(x;n)$, $B(x;n)$ and $C(x;n)$ are polynomials of degree independent of n.

Finally, we use the above equation in order to establish an electrostatic interpretation of the zeros of Q_n.

Zeros of classical polynomials with varying non-classical parameters

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There are several tools suitable for the study of the zero distribution of the classical polynomials, like Jacobi, Laguerre or Bessel. Nevertheless, much less is known when the parameters defining these families vary together with the degree of the polynomial, and may take negative and even complex values.

We study the (weak) zero asymptotics of some classical families in this non-classical setting, establishing that the limit distribution of zeros is related with trajectories of quadratic differentials on a Riemann surface. Along with the positive results, some open problems are presented.

Geometric phase factor of Stark-Wannier resonance

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In a uniform electric field, a crystal has a localization of energy called Stark-Wannier resonance. The resonance points are located periodically in the complex plane along the real axis with a very small imaginary part. Though this phenomenon was found after the mathematical prediction, a precise explanation was first given by Buslaev-Dmitrieva [1] using the multiple scale method.

The equation we treat is

$$(-\frac{d^2}{dx^2} + V(x) + \varepsilon x)\Phi(x) = E\Phi(x),$$

where V is a periodic potential, ε is a small parameter which tends to 0, and E is an energy parameter. Tajima [2] developed a geometric phase factor in the first term of its asymptotic formal solution, as well as the Berry phase. The solution itself is obtained using the Bloch solution at a certain energy level. In Maruyama [3], the factor is calculated explicitely in the case where V is a Lamé potential, which is a finite–gap potential. The factor is devided into two parts; one depends only on the gap structure, and the other is a normalization part of the Bloch solution.

We calculate the phase factor for more general finite–gap potentials using the theta function after the Akhiezer and Its–Matveev formulae to verify its structure.

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A celebration of the Erdélyi-Kober operators

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The Erdélyi-Kober operators of fractional calculus have many uses. The lecture will survey some of these, including connectons with

- hypergeometric and Bessel functions
- dual integral equations of Titchmarsh
- fractional powers of Bessel-type differential operators
- Riesz potentials, via spherical harmonics.

Nonterminating q-Whipple transformations for basic hypergeometric series in $U(n)$

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The purpose of this talk is to discuss our multivariable generalizations of the classical nonterminating q-Whipple and q-Saalschütz transformations. We work in the setting of multiple basic hypergeometric series very well-poised on unitary groups $U(n+1)$. We derive these transformations from our previous $U(n+1)$ generalizations of the classical very well-poised ${}_{10}\phi_9$ transformation. This is joint work with my former PhD student John W. Newcomb.

A duality of the Macdonald-Koornwinder polynomials and its application to the integral representations

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A formula representing a duality of the Macdonald-Koornwinder polynomials will be presented. The formula induces several integral representations of the Macdonald-Koornwinder polynomials, of Heckman-Opdam's Jacobi polynomials of type BC_m, etc.

Zeros of derivatives of cylinder functions of negative order

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The zeros of Bessel and other cylinder functions of positive order and their derivatives have been well studied [7]. There are fewer results on functions of negative order, when some of the zeros are complex, though some of them date back more than a century [1] and some recent extensive numerical and analytic work has been done [2].

L. Lorch and the speaker [5] have studied the real x–zeros of the derivative $C'_\nu(x)$ of the cylinder function $C_\nu(x) = C_\nu(x,\alpha) = \cos\alpha J_\nu(x) - \sin\alpha Y_\nu(x)$ for $\nu < 0$. For fixed α, $0 \le \alpha < \pi$ and $n = 1, 2, \ldots$, we have shown the existence of a number ν_n in the interval $-n+\alpha/\pi-1 < \nu < -n+\alpha/\pi$ such that the first positive zero of $C'_\nu(x)$ occurs after the first positive zero of $C_\nu(x)$ when $-n-1+\alpha/\pi < \nu < \nu_n$ and before the first positive zero of $C_\nu(x)$ when $\nu_n \le \nu < -n+\alpha/\pi$. In case $\nu_n < \nu < -n-\alpha/\pi$, $C'_\nu(x)$ has two zeros which precede the first positive zero of $C_\nu(x,\alpha)$. The first is decreasing and the second is increasing as ν increases. Moreover, we show that in the case $\alpha = 0$, for $\nu_1 = -1.117\cdots \le \nu < -1$, $J'_\nu(x)$ has exclusively real zeros. The present results and earlier ones ([3,4,6] and unpublished work with Á. Elbert) on the zeros of $J''_\nu(x)$, $J'''_\nu(x)$ lead to some conjectures on the zeros of $J^{(n)}_\nu(x)$.

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The WKB method for 3rd order O.D.E.

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We consider a third order linear ordinary differential equation containing a small parameter:

$$\varepsilon^3 y''' + 3(ax+b)\varepsilon^2 y'' + 3(cx+d)\varepsilon y' + (ex+f)y = 0, \quad 0 < \varepsilon \le \varepsilon_0, \quad |x| \le x_0, \tag{1}$$

which is classified by using the discriminant $\mathbf{D}$ of (1)'s characteristic equation:

$$\lambda^3 + 3(ax+b)\lambda^2 + 3(cx+d)\lambda + (ex+f) = 0. \tag{2}$$

$$\begin{aligned}
\mathbf{D} = & \left(-3b^2d^2 + 4d^3 + 4b^3 f - 6bdf + f^2\right) \\
& + \left(-6b^2cd - 6abd^2 + 12cd^2 + 4b^3e - 6bde + 12ab^2 f - 6bcf - 6adf + 2ef\right) x \\
& + \left(-3b^2c^2 - 12abcd + 12c^2d - 3a^2d^2 + 12ab^2e - 6bce - 6ade + e^2 + 12a^2bf - 6acf\right) x^2 \\
& + \left(-6abc^2 + 4c^3 - 6a^2cd + 12a^2be - 6ace + 4a^3 f\right) x^3 + a^2\left(-3c^2 + 4ae\right) x^4.
\end{aligned}$$

By defining constants appropriately, we get the following typical examples. The symbol '∘' represents a number of n-ple zeros of $\mathbf{D}$. There exists no case where $\mathbf{D}$ has only one simple zero or a 4-ple zero.

type	1ple	2ble	3ple	characteristic equation	discriminant **D**	turning points
Ia	∘∘			$\lambda^3 - 3\lambda - 2x = 0$	$4(x^2-1)$	$-1, 1$
Ib	∘ ∘ ∘			$\lambda^3 - 4x\lambda - 2 = 0$	$4\{1-(4x/3)^3\}$	$3/4, 3\omega/4, 3\omega^2/4$
Ib′	∘ ∘ ∘			$\lambda^3 - 3x\lambda + 2 = 0$	$4(1-x^3)$	$1, \omega, \omega^2$
Ib″	∘ ∘ ∘			$\lambda^3 - 3x\lambda^2 + 4 = 0$	$16(1-x^3)$	$1, \omega, \omega^2$
Ic	∘ ∘ ∘∘			$\lambda^3 + 3ax\lambda^2 - 3a\lambda + x = 0$	$4a^3(x^4-1),\ a := \sqrt{1+2/\sqrt{3}}$	$-1, 1, -i, i$
IIa		∘		$\lambda^3 - x = 0$	x^2	0,0
IIb	∘	∘		$\lambda^3 - \lambda^2 - x\lambda + x = 0$	$-4x(x-1)^2/27$	0,1,1
IIc	∘∘	∘		$\lambda^3 + 3x\lambda^2 - 4x = 0$	$16x^2(1-x^2)$	$-1, 1, 0, 0$
IId		∘∘		$\lambda^3 - x\lambda^2 - \lambda + x = 0$	$-4(x^2-1)^2/27$	$-1, -1, 1, 1$
IIIa			∘	$\lambda^3 - x\lambda = 3D0$	$-4x^3/27$	0,0,0
IIIb	∘		∘	$\lambda^3 + x\lambda^2 + x\lambda + 1 = 0$	$-(x-3)^3(x+1)/27$	$-1, 3, 3, 3$

The equation coming from type Ib with $\varepsilon = 1$:

$$y''' - 4\,x\,y' - 2y = 0 \tag{3}$$

has the solutions $\mathrm{Ai}(x)^2$, $\mathrm{Ai}(x)\cdot\mathrm{Bi}(x)$ and $\mathrm{Bi}(x)^2$, where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the Airy functions, the solutions of the Airy equation $Y'' - xY = 3D0$. Three independent solutions of (3) are given by the integrals:

$$\mathrm{Ap}(x) := \int_0^{+\infty} \frac{1}{\sqrt{s}} e^{xs - s^3/12}\, ds, \qquad \mathrm{Bp}(x) := \int_0^{\infty\omega} \frac{1}{\sqrt{s}} e^{xs-s^3/12}\, ds, \tag{4}$$
$$\mathrm{Cp}(x) := 3D \int_0^{\infty\omega^2} \frac{1}{\sqrt{s}} e^{xs-s^3/12}\, ds.$$

References [1], [6] and [8] give no name to them, and so we propose here to name the equation (3) the **Pairy equation** and three functions Ap, Bp and Cp **Pairy functions**. The name "Pairy" is originated from **P**airy=(**P**roducts+**Airy**)/2.

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Knot invariants and new quantum field theory

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Recently Witten derived knot invariants such as the Jones polynomial from quantum field theory based on the Chern-Simon Lagrangian. Inspired by Witten's work in this paper we derive knot invariants from a new quantum field theory. Our approach will be simpler and more rigorous and we can derive new knot invariants which extend the Jones polynomial.

Similar to the Wiener measure based on an integral $\int_{t_0}^{t_1}\left(\frac{dx}{dt}\right)^2 dt$ we propose the following integral for constructing a measure of quantum electrodynamics (QED):

$$-\frac{1}{2}\int_{s_0}^{s_1}[\frac{1}{2}\left(\frac{dA_1}{ds}-\frac{dA_2}{ds}\right)^2+\sum_{i=1}^{2}\left(\frac{dz}{ds}-ieA_i z\right)^*\left(\frac{dz}{ds}-ieA_i z\right)]ds \tag{1}$$

where s denotes the proper time in relativity; e denotes the electric charge and the complex variable z, real variables A_1, A_2 represent one electron and two photons respectively. By extending ds to $(ih+\beta)ds$ $(h,\beta>0)$ we get a quantum model of QED. We remark that (1) is not formulated with the space-time. In this new QED model the space-time is clarified as a statistical concept which is not as fundamental as the proper time. For statistics of many electrons and photons we introduce space-time to write ds^2 in the form $ds^2=dt^2-d\mathbf{x}^2$ which is the Lorentz metric. By this space-time statistics we can derive the results of the conventional QED theory and this new theory is free of difficulties of the conventional theory such as ultraviolet divergences. The integral (1) has a $U(1)$-gauge symmetry. We may extend it (as a nonabelian example) to a $SU(2)$-gauge symmetry with the following integral:

$$L:=-\frac{1}{2}\int_{s_0}^{s_1}[\frac{1}{2}\mathrm{Tr}(D_1A_2-D_2A_1)^*(D_1A_2-D_2A_1)+(D_1Z)^*(D_1Z)+(D_2Z)^*(D_2Z)]ds \tag{2}$$

where $Z=(z_1,z_2)^T$ is a two dimensional complex vector; $A_j=\sum_{k=1}^{3}A_j^k t_k$ $(j=1,2)$ where t_k denote generators of $SU(2)$ and $D_j=\frac{d}{ds}-igA_j$ $(j=1,2)$ where g is the charge. (2) has a $SU(2)$-gauge symmetry. From it we can develop a QCD type quantum theory. Since (2) is not formulated with the space-time, the following partition function will be a topological invariant for knots:

$$\langle \mathrm{Tr}W_R\rangle := \int DA_1DA_2DZe^L\mathrm{Tr}W_R(C) \tag{3}$$

where $W_R(C):=W(r_0,r_1):=Pe^{i\int_C A_j dx^j}$ which may be called a Wilson loop as analogous to the usual Wilson loop where C denotes a closed curve of the form: $C(s)=(x^1(r(s)),x^2(r(s)))$, $s_0\le s\le s_1$ where r is a real valued function such that $r_0:=r(s_0)=r(s_1)=r_1$. This closed curve C is in a two dimensional phase plane (x^1,x^2) dual to (A_1,A_2). P denotes a path-ordered product and R denotes a representation of $SU(2)$. Then under a gauge transformation we have a quantum chiral symmetry $W(z_0,z_1)\mapsto U(z_0)W(z_0,z_1)U^{-1}(z_1)$ where $U(z_i)$ denotes an element of the $SU(2)$ gauge group and we extend r to a complex variable z. Then the quantum current $J(z)$ of this symmetry satisfies an affine Kac-Moody algebra with a central extension $k>0$.

If we cut a knot we get two pieces of curves crossing (or not crossing) each other once. This gives two primary fields $W(z_1, z_2)$ and $W(z_3, z_4)$. Let us write $W(z_i, z_j) = W(z_i, w')W^{-1}(z_j, w')$. Then from the Kac-Moody algebra we can derive the Knizhnik-Zamolodchikov equation for the correlation $\langle W(z_1, w_1')W^{-1}(z_2, w_1')W(z_3, w_2')W^{-1}(z_4, w_2')\rangle$. This KZ equation has two independent conformal block solutions forming a two dimensional vector space. From this we can derive the skein relation for the Jones polynomial. On the other hand the correlation $\langle W(z_1, z_2)W(z_3, z_4)\rangle$ satisfies the KZ equation for z_1, z_3 and satisfies a dual KZ equation for z_2, z_4 from which we can derive a quantum group structure for $W(z_i, z_j)$ by the monodromy of the KZ equation. From this quantum group we can derive new knot invariants for which (3) is the knot invariant for the trivial knot. These new knot invariants extend the Jones polynomial and give a classification of knots.

Algebraic construction of the nonsymmetric Macdonald polynomials

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The Macdonald operators form a commuting family of difference operators and have the simultaneous eigenfunctions which are called the symmetric Macdonald polynomials. Cherednik reconstructed the Macdonald operators in terms of the representation of the affine Hecke algebra, that is, the symmetrization of the commuting elements in the affine Hecke algebra provide the Macdonald operators through the restriction of their operands to the Weyl-group-invariant ring of polynomials. These commuting elements are called the Cherednik operators which have the simultaneous eigenfunctions, i.e., the nonsymmetric Macdonald polynomials. Introducing the raising and lowering operators, we algebraically construct the nonsymmetric Macdonald polynomials. The raising and lowering operators are provided from the representation of the double affine Hecke algebra. By use of the algebraic expressions of the nonsymmetric Macdonald polynomials, Cherednik's scalar products are calculated.

Generating functions of m-simple branched covers and quasimodular forms

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We show that the generating functions of m-simple branched covers of the elliptic curve are quasimodular forms in the sense of Kaneko-Zagier. For $m = 2$, this has been proved by Douglas, Dijkgraaf.

Resurgence in difference equations, with an application to Legendre functions

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Resurgence is the phenomenon in which the "late" coefficients and remainder terms in a Poincare asymptotic expansion can be re-expanded as generalized asymptotic expansions whose coefficients are the "early" coefficients in the original expansion. An extensive theory, with applications, has been developed for linear differential equations during the past decade. This lecture describes analogous results that have been obtained recently for linear difference equations of the second order.

New re-expansions of the remainder terms in asymptotic expansions of Legendre functions of large order are derived by way of illustration.

Using methods of classical analysis to solve new problems in singular perturbations

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Singular perturbation problems have traditionally been solved by applied mathematicians and engineers using asymptotic matching and other relatively ad hoc techniques. In this time of exponential (and other more precise) asymptotic analyses, complex variable and other classical approaches can effectively provide the more careful results needed to successfully solve the more subtle problems of current interest and importance. This will be demonstrated using relatively simple boundary value problems.

Harmonic analysis for Hecke algebras

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We discuss various aspects of the analytic theory of Hecke algebras as was developed systematically by Cherednik, based on pioneering work of Dunkl. We will concentrate on the impact of this theory on some problems in classical representation theory.

On kernel polynomials and their zeros

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Given an orthogonal polynomial system $\{Q_n(x)\}_{n=0}^{\infty}$, define another polynomial system by

$$P_n(x) = Q_n(x) - \alpha_n Q_{n-t}(x), \quad n \geq 0$$

where α_n are complex numbers and t is a positive integer. We find conditions for $\{P_n(x)\}_{n=0}^{\infty}$ to be an orthogonal polynomial system. When $t = 1$ and $\alpha_1 \neq 0$, it turns out that $\{Q_n(x)\}_{n=0}^{\infty}$ must be kernel polynomials for $\{P_n(x)\}_{n=0}^{\infty}$.

New asymptotic formulas for the Riemann zeta function on the critical line

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The classical means of computing the Riemann zeta function $\zeta(s)$ on the critical line $s = \frac{1}{2} + it$ $(t \geq 0)$ are the Gram and the Riemann-Siegel formulas. Both these formulas consist of a finite main sum $\sum_{n=1}^{N} n^{-s}$, corresponding to the first N terms of the Dirichlet series, together with asymptotic correction terms. The essential difference between these two expansions is that the Gram formula is characterised by $N = O(t/2\pi)$, whereas the more powerful Riemann-Siegel formula has $N \sim (t/2\pi)^{1/2}$ which represents a considerable saving in computation time for very large values of t.

During the last 5 years, other methods have been discovered which consist of the finite main sum replaced by a convergent sum with the sharp cut-off after $N \sim (t/2\pi)^{1/2}$ terms 'smoothed' by an error function or an incomplete gamma function. Term-for-term these new expansions are found to be more accurate than the Riemann-Siegel formula. In this lecture we shall present the details of one of these expansions: this relies on the connection between $\zeta(s)$ and the incomplete gamma function, together with use of the uniform asymptotics of $\Gamma(a, z)$ for complex values of the variables. It is shown how other properties of the uniform asymptotics of $\Gamma(a, z)$ (bounds on the remainder and the behaviour of the late terms) enable us to obtain some insight into the asymptotic character of this new expansion for $\zeta(s)$.

We shall also mention a formula corresponding to a Gram-type expansion where the smoothing of the main sum is via a Gaussian exponential rather than an error function.

Fourier analysis on the unit sphere and its Lipschitz perturbations

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The talk will concentrate on introducing a complex structure in Euclidean spaces using Clifford algebras. We show that, in this frame, usng Fueter's and Sce's methods we can establish a corresponding relationship between the unit sphere and the unit circle. By means of this relationship, we are able to treat Euclidean spaces just as we treat the space of complex numbers. As example, we develop a theory of singular integral operators that form an oprator algebra on the unit sphere and its Lipschitz perturbations. It includs the Hilbert transform in the context as a particular case.

Uniform asymptotic formula for orthogonal polynomials with exponential weight

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Let $\{p_n(x)\}_{n\geq 0}$ be the set of orthogonal polynomials with respect to the exponential weight $w(x) = e^{v(x)}$, where $v(x) = x^{2m} + \cdots$ is a monic polynomial of degree $2m$ with $m \geq 2$ and is even. An asymptotic approximation is obtained for $p_n(x)$, as $n \to \infty$, which holds uniformly for $0 \leq x \leq O(n^{1/2m})$. As a corollary, a three-term asymptotic expansion is also derived for the zeros of these polynomials.

A q-analogue of a product formula of Bailey and related results

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We find a q-analogue of Bailey's product formula

$$F\left(a,b;c;\frac{1-x}{2}\right)\ F\left(a,b;a+b-c+1;\frac{1-y}{2}\right) = F_4\left(a,b;c,a+b-c+1;\frac{(1-x)(1+y)}{4},\ \frac{(1-y)(1+x)}{4}\right)$$

where $F(a,b;c;t)$ is the Gaussian hypergeometric function and F_4 is an Appell function. The analogue of the F function that we consider here is the very-well-poised ${}_8\phi_7$ series that corresponds to the quadratic lattice. It turns out that the resulting analogue of the F_4 function is not just one double series but a sum of six infinite double series containing very-well-poised ${}_{10}\phi_9$ series. We also find a product formula in an integral form and discuss some special cases.

Elliptic integrals of the first and second kind — comparision of Bulirsch's and Carlson's algorithms for numerical calculation

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Numerical calculations of elliptic integrals of the first and second kind are usually done using algorithms of R. Bulirsch [1] or B.C. Carlson [4]. These algorithms are based on the descending Landen transformation and the duplication theorem [3,4] respectively.

The algorithms will be compared concerning the number of floating point operations and computing time. The comparision is done in single precision with a given tolerance and leads to a rating concerning the range of the arguments.

A modification of Bulirsch's algorithm will be presented for calculating the elliptic integral of the second kind $\int_0^x \frac{1+k_c^2\,\xi^2}{(1+\xi^2)\sqrt{(1+\xi^2)(1+k_c^2\,\xi^2)}}\,d\xi$ avoiding cancellation occuring for arguments $0 < x \ll 1$ and values of the complementary modulus $k_c \gg 1$.

The duplication theorem will be interpreted in the sense of the addition formula for the elliptic integral of the first kind or the Jacobian elliptic function *sinus amplitudinis* and in the sense of Weierstraß.

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Positiviy of Dunkl's intertwining operator

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Dunkl operators are parametrized differential-reflection operators on a Euclidean space which are associated with a finite reflection group and generalize the usual partial derivatives. They are, for example, very useful in the analysis of certain exactly solvable quantum many particle systems of Calogero-Moser-Sutherland type. For a fixed reflection group and multiplicity parameters, the associated Dunkl operators generate a commutative algebra which is – under weak assumptions – intertwined with the algebra of partial differential operators by a unique linear and homogeneous isomorphism on polynomials.

In this talk we report on results concerning the structure of Dunkl's intertwining operator in case of nonnegative multiplicity parameters, and discuss some of their implications. Our main result, which proves a conjecture of Dunkl, states that the intertwining operator is positivity-preserving on polynomials. The proof of this result uses semigroup techniques for linear operators on spaces of polynomials. Moreover, Dunkl's interwining operator allows a positive integral representation on certain algebras of analytic functions. This in particular implies that the generalized exponential kernel of the Dunkl transform is positive-definite. We sketch some applications of these results and finally discuss ray asymptotics of the generalized exponential kernel.

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r^{th}-Associated Orthogonal Polynomials, classical with respect to the operators Δ and D_q

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For the operator $D = \frac{d}{dx}$ Classical Orthogonal Polynomials $P_n(x)$ are defined from a weight $\rho(x)$ solution of the Pearson equation $D(\sigma\rho) = \tau\rho$ (σ of degree ≤ 2, τ of degree 1). Then the r^{th} associated orthogonal polynomial $P_n^{(r)}(x)$ satisfy the coupled differential relations (r integer)

$$L_{2,r}[P_n^{(r)}(x)] = K_r[P_{n-1}^{(r+1)}]' , \qquad L_{2,r}^*[P_{n-1}^{(r+1)}(x)] = K_r^*[P_n^{(r)}]' ,$$

with $L_{2,r}$ and $L_{2,r}^*$ second order differential operators, K_r and K_r^* constants.

When

$$D \equiv \Delta \qquad \Delta f(x) = f(x+1) - f(x)$$

or

$$D \equiv D_q \qquad D_q f(x)) = \frac{f(qx) - f(x)}{(q-1)x} ,$$

the corresponding Classical Orthogonal Polynomials defined from

$$\Delta(\sigma\rho) = \tau\rho \qquad \text{or} \qquad D_q(\sigma\rho) = \tau\rho ,$$

have r^{th}-associated polynomials which satisfy coupled difference or q-difference equations similar to the one obtained when $D = \frac{d}{dx}$. From these equations a 4^{th}-order difference and q-difference equations are given. Situations for which the first associated Polynomials $P_n^{(1)}$ belong to the $P_n(x)$ family are also examined.

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Coupling coefficients and multivariable orthogonal poltynomials

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Some of the earliest explicit examples of multivariable orthogonal (and biorthogonal) polynomials were given by Appell in the late 19th century. These include polynomials expressible in terms of Lauricella functions and polynomials that factor as products of one-variable orthogonal polynomials. Related polynomial systems have since been studied by many authors, and they have also appeared in applications such as statistics and quantum physics.

One should point out that these polynomial systems are close to their one-varia ble special cases in the sense that their weight functions factor as products of elementary parts. In recent years several more "genuinely" multivariable polynomial systems have been discovered, such as the Koornwinder-Macdonald polynomials.

In this talk we provide group-theoretic interpretations of Appell's polynomials and their relatives. We show that they arise as coupling coefficients (generalized Clebsch–Gordan coefficients) for the simplest Lie groups. For the Lie group $\mathrm{SU}(1,1)$ one obtains multivariable generalizations of the Jacobi, Hahn and continuous Hahn polynomials, for the Heisenberg group multivariable Krawtchouk and Hermite polynomials. For the $\mathrm{SU}(1,1)$ quantum group, one may obtain multivariable q-Racah and Askey–Wilson polynomials as coupling coefficients. The group-theoretic interpretation leads to simple and unified proofs of various formal properties of these polynomial systems.

Analytical study of 3 point Lotka-Volterra map

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A particular Lotka-Volterra system is known equivalent to the discrete time Toda lattice, hence it is completely integrable. We would like to know explicitly the behaviour of solutions of such systems. We are particularly interested in what is implied by the complete integrability of this system.

In this talk we consider a 3 point Lotka-Volterra map whose orbits we can follow explicitly. We will argue how the integrability is attained when it was viewed from nonintegrable regime.

Radon perfectness of conelike $*$-semigroups in $Q^{(\infty)}$

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Let $S = (S, +, *)$ be an abelian $*$-semigroup with the identity. The semigroup S is called *Radon perfect* if every positive definite function on S has a unique representing measure on the dual semigroup of S (cf.[1]). For example, the abelian $*$-semigroup

$$Q^{(\infty)} := \{(x_1, x_2, \cdots) \mid x_n \in Q, x_n = 0 \text{ eventually}\}$$

is Radon perfect for any involution $*$. In this talk, we characterize the structure of involutions on an abelian $*$-semigroup in $Q^{(\infty)}$, and show that any conelike $*$-semigroups in $Q^{(\infty)}$ are Radon perfect. Furthermore, we present some corollaries for positive definite and related functions (cf. [2],[3]).

Theorem 1. For every abelian $*$-semigroup S in $Q^{(\infty)}$, there exist an abelian $*$-semigroup T in $Q^{(\infty)}$ and an infinite diagonal matrix $E = diag\{\varepsilon_1, \varepsilon_2, \cdots\}$, where $\varepsilon_i = 1$ or -1, such that $(S, +, *)$ is $*$-isomorphic to $(T, +, E)$. Here the involution of T is given by $(x_1, x_2, \cdots)^* = (\varepsilon_1 x_1, \varepsilon_2 x_2, \cdots)$.

Definition. Let S be an abelian $*$-semigroup which is a subset of $Q^{(\infty)}$. S is called *conelike* if for every $s \in S$ there is a nonnegative rational number $\alpha(s)$ such that $\alpha s \in S$ for all rational numbers $\alpha \geq \alpha(s)$.

Theorem 2. Every conelike $*$-semigroup in $Q^{(\infty)}$ is Radon perfect.

Corollary 1. Let S be a conelike semigroup in $Q^{(\infty)}$ with the identical involution. Then every positive definite (resp. negative definite) function on S is completely positive definite (resp. completely negative definite).

Corollary 2. Let S be a conelike semigroup in $Q^{(\infty)}$ with the identical involution, ψ a function on S. Then the following conditions are equivalent:
(1) ψ is negative definite;
(2) ψ fulfils Hoeffding's inequality of order n for any $n \geq 2$;
(3) ψ is a Schur increasing function of order m for any $m \geq 2$.

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Remarks on the inner product formulas of Debnath and Grum

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In 1988, L. Debnath and W.J. Grum discussed an interesting property of Riemann-Liouville fractional integral operator $D^{-\alpha}$ and Weyl fractional integral operator $W^{-\alpha}$. Using the inner product notation, they obtain $(D^{-\alpha}f, g) = (f, W^{-\alpha}g)$. They stated that $D^{-\alpha}$ and $W^{-\alpha}$ behave like adjoint operators. We apply the formulas of Debnath and Grum to obtain the following results.

$$D^{-\alpha}t^{-1} = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(\log t)u(t) - \big(\gamma + \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\big)\frac{t^{\alpha-1}}{\Gamma(\alpha)}u(t), \quad \alpha > 0 \tag{1}$$

where $u(t)$ is a Heaviside unit-step function and γ is the Euler's constant.

$$\begin{aligned} D^{-\alpha}t^{-2} &= \big(\gamma(\alpha-1) - 1 + (\alpha-1)\Psi(\alpha) - 1\big)\mathrm{Pf}\,\frac{t^{\alpha-2}}{\Gamma(\alpha)}u(t) \\ &\quad - (\alpha-1)\Psi(\alpha)\mathrm{Pf}\ \log t\frac{t^{\alpha-2}}{\Gamma(\alpha)}u(t), \quad 0\ < \alpha\ <\ 1 \end{aligned} \tag{2}$$

where Pf is a pseudofunction and Ψ is the digamma function.

$$\begin{aligned} D^{\frac{1}{2}}t^{-\frac{1}{2}} &= \sqrt{\pi}\delta(t) \\ D^{\frac{2n+1}{2}}t^{-\frac{1}{2}-m} &= (-1)^{n+m}\Gamma\left(\frac{1}{2}-m\right)(m+n)!\delta(t)t^{-n-m}, \end{aligned} \tag{3}$$

where $\delta(t)$ is the Dirac delta function.

$$\begin{aligned} D^{\frac{3n+1}{3}}t^{-\frac{2}{3}} &= n!(-1)^n\Gamma\left(\frac{1}{3}\right)\delta(t)t^{-n} \\ D^{\frac{3n+2}{3}}t^{-\frac{1}{3}} &= n!(-1)^n\Gamma\left(\frac{2}{3}\right)\delta(t)t^{-n} \end{aligned}$$

Here the Riemann-Liouville fractional integral $D^{-\alpha}$ of the power function $f(t) = t^\nu$ is $D^{-\alpha}t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)}t^{\nu+\alpha}$, $(\alpha > 0, \nu > -1)$ and the Weyl fractional integral of $f(t)$ is defined by

$$W^{-\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}f(t)dt$$

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Solitons and Coulomb plasmas, similarity reductions and special functions

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Two applications of integrable nonlinear partial differential and difference equations, appearing from the compatibility conditions of two linear problems, are outlined.

First, tau-function of the N-soliton solution of the Korteweg-de Vries (KdV) equation is shown to describe fixed temperature partition functions of the specific Coulomb gases or Ising magnets on lines. A similar relation holds for Coulomb plasmas on the plane with some boundaries and soliton solutions of the Kadomtsev-Petviashvili (KP) equation. Partition functions are calculated exactly for the cases when the KdV and B-type KP soliton amplitudes form a number of geometric progressions (this is a joint work with I. Loutsenko).

Second, a (1+1)-dimensional discrete-time chain associated with the three-term recurrence relation of the R_{II}-type (Ismail and Masson, 1995) and some of its similarity reductions are described. The latter lead to large classes of biorthogonal rational functions including an eleven parameter family based upon the very-well poised, balanced ${}_{10}\varphi_9$ basic hypergeometric series. This is a joint work with A. Zhedanov.

Gaussian quadrature rules with simple node-weight relations

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Here we consider the Gaussian quadrature rules associated with the orthogonal polynomials $\{Q_n^{(p)}\}$ that satisfy the recurrence relations

$$Q_{n+1}^{(p)}(z) = zQ_n^{(p)}(z) - \alpha_{n+1}Q_{n-1}^{(p)}(z) \qquad n \geq 1,$$

with $Q_0^{(p)}(z) = 1$ and $Q_1^{(p)}(z) = z$, where

$$\alpha_2 = p\alpha_0, \qquad \alpha_{2n+1} = \alpha_1 \qquad \text{and} \quad \alpha_{2n+2} = \alpha_0, \qquad n \geq 1.$$

Here $0 < p$, α_0, $\alpha_1 < \infty$. The associated measure $d\phi_p$ is

$$\phi_p(x) = \frac{\Delta(\alpha_1 - \alpha_0)}{A(p)}\delta(x) + \frac{\Delta\left((p-1)^2\alpha_0 - \alpha_1\right)}{(p-1)A(p)}\left[\delta(x+\xi(p)) + \delta(x-\xi(p))\right] + \frac{1}{2\pi A(p)}\int_{-\infty}^{x} \frac{\sqrt{b^2-t^2}\sqrt{t^2-a^2}}{|t|(1+q(p)t^2)} I_B(t)dt,$$

where $a^2 = (\sqrt{\alpha_0} - \sqrt{\alpha_1})^2$, $b^2 = (\sqrt{\alpha_0} + \sqrt{\alpha_1})^2$, $A(p) = (p-1)\alpha_0 + \alpha_1$, $q(p) = -(p-1)/(pA(p))$, $(\xi(p))^2 = -1/q(p)$, $\Delta(x) = x\delta(x)$, $I_B(x) = \delta(x+b) - \delta(x+a) + \delta(x-a) - \delta(x-b)$ is the indicator function of the set $B = [-b,-a] \cup [a,b]$ and $\delta(x)$ is the Dirac delta function.

We show that the weights (Christoffel numbers) of the quadrature rules can be given as simple rational functions in terms of the corresponding nodes.

Applications of q-Hermite polynomials

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The q-Hermite polynomials are orthogonal polynomials with a parameter q which generalize the Hermite polynomials. As with the Hermite polynomials, their generating function is closely related to their measure. This allows for efficient integral evaluations, with applications to integer partitions. For example, the Rogers-Ramanujan identities follow immediately from an integral, as does a new quintic transformation. Many new multisum Rogers-Ramanujan identities are naturally motivated and follow from the q-Hermite polynomials, along with addition theorems for the quadratic q-exponential function. Some related combinatorial questions will be discussed.

On connections between Hankel, Laguerre, and Jacobi transplantation

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We prove two results showing connections between the Hankel transplantation and a transplantation for a certain kind of Laguerre and Jacobi expansions. An asymptotic formula of Hilb's type for Laguerre and Jacobi polynomials is used. As an application of this link we obtain an extension of Guy's transplantation theorem for the Hankel transform to the case α, $\gamma > -1$ also with more weights allowed. This is done by transferring a corresponding transplantation result for Jacobi expansions which was proved by Muckenhoupt. In the case when α, $\gamma \geq -1/2$ the same is obtained by using Schindler's explicit kernel formula for the transplantation operator $H_\alpha \circ H_\gamma$.

The Askey-Wilson function transform and its degenerate cases

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In this talk we discuss the spectral analysis of the Askey-Wilson second order q-difference operator as a formally symmetric operator on a one-parameter family of L^2-spaces. We define the associated Askey-Wilson function transform, which depends, besides q, on five parameters. The additional fifth parameter is related to the different choices of (non-compact) measures. We furthermore present the Plancherel formula and inversion formula for the Askey-Wilson function transform. In particular, it is shown that the Askey-Wilson function transform is self-dual, in the sense that its inverse is again a Askey-Wilson function transform, but with a different ("dual") choice of parameter values. In Erik Koelink's talk at this conference it is shown that the Askey-Wilson function transform is closely related to harmonic analysis on the $SU(1,1)$ quantum group.

The big and little q-Jacobi function transform, as well as several types of q-Hankel transforms, are presented as degenerate cases of the Askey-Wilson function transform. This leads to the interpretation of the self-dual Askey-Wilson function transform as a four parameter extension of the self-dual little q-Hankel transform. Furthermore, the Plancherel formula and inversion formula for the big q-Jacobi function transform give detailed information about orthogonality relations for the continuous dual q^{-1}-Hahn polynomials with respect to a one-parameter family of non-extremal orthogonality measures.

The degenerate cases which will be discussed in this talk are all related to harmonic analysis on the quantum $SU(1,1)$ group or to harmonic analysis on the quantum group of plane motions.

This talk is based on joint work with Erik Koelink.

Brahmagupta polynomials with two parameters and their application to crystallography

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Let $\mathbf{M}$ denote the set of matrices of the form

$$B = \begin{bmatrix} x & y \\ ty & x+sy \end{bmatrix}$$

where t and s are fixed reals number and x and y are real variables. Define B to be the Brahmagupta matrix. It is easy to check that $\mathbf{M}$ is a commutative algebra over the reals. Let

$\beta = \det B = x^2 + sxy - ty^2$. Define $\alpha_{pm} = \frac{-s\pm\sqrt{s^2+4t}}{2}$ and

$$B^n = \begin{bmatrix} x & y \\ ty & x+sy \end{bmatrix}^n = \begin{bmatrix} x_n & y_n \\ ty_n & x_n+sy_n \end{bmatrix} = B_n.$$

Using the eigenrelations of B, we obtain the following Binet forms for x_n and y_n:

$$x_n = \frac{1}{\alpha_+ - \alpha_-}[\alpha_+(x-\alpha_- y)^n - \alpha_-(x-\alpha_+ y)^n], \quad y_n = \frac{1}{\alpha_+ - \alpha_-}[(x-\alpha_- y)^n - (x-\alpha_+ y)^n]$$

and $x_n - \alpha_\pm y_n = (x-\alpha_\pm y)^n$. Notice that $x_0 = 1$ and $y_0 = 0$. x_n and y_n are homogeneous polynomials in x and y satisfying the following difference equations:

$$x_{n+1} = (2x+sy)x_n - \beta x_{n-1}, \;\; y_{n+1} = (2x+sy)y_n - \beta y_{n-1}.$$

If $s = t = y = 1$ and $x = 0$ then B reduces to the Fibonacci matrix $F = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ generating Fibonacci sequence and Penrose tilings with 5-fold symmetry; If $s = 0$, $x = y = 1$ and $t = 2$, then B becomes the Pell matrix $P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, which generate the Pell sequence and quasi-periodic tilings with 4-fold and 8-fold symmetries.

In the Brahmagupta matrix if we put $\sqrt{t}y = x^2 - 1$, we find that

$$x_n = \frac{1}{2}[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n], \quad ty_n = \frac{1}{2}[(x+\sqrt{x^2-1})^n - (x-\sqrt{x^2-1})^n].$$

Writing $x_n = T_n$ and $\sqrt{t}y_n = \sqrt{x^2-1}U_{n-1}$, we see that T_n and U_n are Chebychev polynomials of the first and second kind respectively.

The polynomials x_n and y_n satisfy a host of other relations; to name a couple, they both satisfy the wave equation:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{t}\frac{\partial^2}{\partial y^2}\right) U = 0 \text{ and } \sum_{k=1}^{\infty} \frac{1}{x_{k+1}}\left(\frac{2x+s}{x_{k-1}} - \frac{\beta+1}{x_k}\right) = \frac{1}{x}.$$

Basic Fourier series

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The study of Fourier series has a long and distinguished history in mathematics. They were introduced in order to solve the heat equation, and since then they have been frequently used in various applied problems. Much of modern real analysis including Lebesgue's fundamental theory of integration had its origin in some deep convergence questions in Fourier series. An interesting extension of Fourier series was discovered recently. In this talk we intend to lay a sound foundation for this study. We introduce basic Fourier series, investigate their main properties, and discuss some applications. Interesting open problems and further extensions of the theory will be given.

An operator theoretic characterization of the raising formula for Hermite polynomials

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We propose a characterization of the aforesaid formula, or rather the related formula which defines the creation operator of the quantum harmonic oscillator, in terms of some invariance property of weighted shifts. The same refers to Charlier polynomials and to still another class of special functions.

Integral equations, special functions and decomposition of some probability distributions

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The speaker will talk on an integral equation and analytical property of some probability distributions which are like the student t distributions and the F distributions. The speaker is interested in the analytical property of kernel function of an integral equation of the following type: For a given probability density function $g(x)$, it holds that

$$xg(x) = \int_0^x g(x-y)k(y)\,dy, \quad x > 0.$$

If we can transform the above integral equation to an integral equation of the second kind we can show that there exists a kernel function of the above integral equation and furthermore we can know the analytical property of $k(y)$ if the density function $g(x)$ is simple. In case of the t distribution with degree of freedom 5, there appears a function given by the following formula,

$$\frac{v^4}{\pi(3^2+3v^2+v^4)} = \frac{2}{\pi^2 v(J_{5/2}^2(v)+Y_{5/2}^2(v))}, \quad v > 0,$$

where $J_{5/2}(v), Y_{5/2}(v)$ are the Bessel functions of first kind and second kind with order $5/2$. From this concrete function, the convolutional decomposition of the t distribution can be obtained and we can find a clue to study the heigher dimensional case of similar probability density. The speaker will also report a formula of confluent hypergeometric functions and results obtained from it.

On Fourier-Laplace transforms of distributions with compact support

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Let T be a distribution with compact support on R. We prove that there is a positive function $g \in C_0^m(R)$ with m sufficiently large such that $T * g$ is positive if and only if $\hat{T}$, the Fourier-Laplace transform of T, never vanishes on the imaginary axis. Here, positive means nontrivial and nonnegative. If T is supported at one point, g can be chosen with arbitrarily small support. We discuss the implications of these statements, including those on the Beurling-Malliavin Multiplier Theorem.

The role of Hermite polynomials in asymptotic analysis

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Hermite polynomials show up in several problems of asymptotic analysis. We consider three different instances where these classical orthogonal polynomials can be used as main approximants:

1. As limits of other polynomials such as Laguerre and Jacobi orthogonal polynomials, but also of Bernoulli polynomials. We consider a rather elementary approach based on the generating functions of the polynomials.

2. In turning point problems for second order linear differential equations. In particular the Hermite polynomials can be used when two nearby turning points are present.

3. For functions defined as an integral in which the saddle points follow a certain pattern for certain values of the parameters

In all three cases, the results can be used for describing the asymptotic behaviour of polynomials. Several concepts of asymptotic analysis will be introduced, in particular those needed for uniform asymptotics. We briefly discuss that the Hermite polynomials are special cases of another set of special functions, the Weber parabolic cylinder functions, which can be used in similar and more general problems of asymptotic analysis.

Statistics of graph spectra for some finite matrix groups

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Physicists have long studied spectra of Schroedinger operators and random matrices, thanks to the implications for quantum mechanics. Analogously number theorists and geometers have investigated the statistics of spectra of Laplacians on Riemannian manifolds associated to arithmetic groups. This has been termed "arithmetic quantum chaos" by Sarnak. Equivalently one studies the zeros of Selberg zeta functions. Parallels with the statistics of the zeros of the Riemann zeta function have been evident for some time. Here we survey what may be called "finite quantum chaos" - seeking connections with the continous theory and with special functions. The results can also be formulated in terms of Ihara zeta functions of graphs.

Two linear transformations each tridiagonal with respect to an eigenbasis of the other

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Let $\mathcal{F}$ denote a field, and let V denote a finite dimensional vector space over $\mathcal{F}$. In this paper, we consider pairs of $\mathcal{F}$-linear maps $A : V \to V$ and $A^* : V \to V$ satisfying both conditions below:

1. There exists a basis for V with respect to which the matrix representing A is diagonal, and the matrix representing A^* is irreducible tri-diagonal.

2. There exists a basis for V with respect to which the matrix representing A^* is diagonal, and the matrix representing A is irreducible tri-diagonal.

We call such a pair a *Leonard pair* on V. We give a complete classification of Leonard pairs. In the process, we show that for any Leonard pair A, A^*, there exists a sequence of scalars β, γ, γ^*, ϱ, ϱ^* from $\mathcal{F}$ such that the commutators

$$\begin{aligned} 0 &= [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*], \\ 0 &= [A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A]. \end{aligned}$$

The sequence is uniquely determined by the Leonard pair if the dimension of V is at least 4. For $\beta = 2$, $\gamma = 0$, $\gamma^* = 0$ the above relations are known in statistical mechanics as the Dolan-Grady relations. We discuss how Leonard pairs are related to the q-Racah polynomials and other polynomials from the Askey scheme.

Convolution semi-groups and Calderon's formulas for compact k-variables continuous polynomial hypergroups

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We present results on convolution semi-groups on compact k-variables continuous polynomial hypergroups H, and we characterize some of them by Levy-Khintchine formulas.

Next we consider the Calderon's formula

$$f = \int_0^\infty f * \mu_a \frac{da}{a} \ ,$$

where $*$ is the convolution product on H and $\{\mu_a\}_{a>0}$ a family of complex-valued Borel measures supported in H, satisfying some conditions. We study the convergence of this formula and of its discretization, and we give links between the family $\{\mu_a\}_{a>0}$ and convolution semi-groups.

Conserved operators and their simultaneous eigenfunctions for the quantum Calogero models with distinguishable particles

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Due to the large degeneracy of their energy eigenvalues, the A_{N-1}- and B_N-Calogero models respectively have two inequivalent sets of conserved operators. One is given by the Dunkl-Cherednik operator formulation, and the other is given by a similarity transformation method. Two sets of conserved operators include, by definition, the Hamiltonians of the quantum Calogero models. However, the conserved operators of one set and those of the other do not commute. Thus the simultaneous eigenfunctions of each set of conserved operators are different. The simultaneous eigenfunctions of the Cherednik operators for the A_{N-1}- and B_N-Calogero models respectively are the nonsymmetric multivariable Hermite and Laguerre polynomials. We shall present the Rodrigues formulas for both polynomials. The simultaneous eigenfunctions of the conserved operators given by the similarity transformation method are algebraically constructed in completely parallel way to the construction of the Fock space of the quantum Harmonic oscillators.

Integrable systems, special functions and alagebraic combinatorics

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I shall discuss examples that illustrate aspects of the fruitful interplay that has developed recently between these three areas.

Harmonic analysis and Markov processes associated with Dunkl operators

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Dunkl operators are parametrized differential-difference operators on $\mathbf{R}^N$ which are related to finite reflection groups. They can be regarded as a generalization of partial derivatives and play a major role in the study of Calogero-Moser-Sutherland-type quantum many-body systems. Dunkl operators lead to generalizations of various analytic structures, like the Laplace operator, the Fourier transform, Hermite polynomials and the heat semigroup.

The purpose of this talk is to present some probabilistic aspects of this theory. In particular, stochastic processes on $\mathbf{R}^N$ associated with Dunkl's Laplace operators and heat semigroups will be introduced. These processes may be regarded as generalizations of classical Brownian motions; these processes may however admit jumps (coming from spin exchanges) and form, after symmetrization with respect to the underlying reflection groups, diffusions on the Weyl chambers. A major part of the talk will be devoted to connections of these processes with special functions like generalized Hermite polynomials. Moreover, related evolution semigroups and stochastic processes like generalizations of Cauchy and Ornstein-Uhlenbeck processes will be discussed. The results of this talk are joint work with Margit Rösler.

Hypergroups and special functions

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We will give first an overview of some of the connections between hypergroup theory and special functions, spherical hypergroups associated to symmetric spaces and wrapping maps. Then we will discuss work with N. Obata on generalized hypergroups and connections with orthogonal polynomials.

Uniform asymptotic expansions for orthogonal polynomials

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Asymptotic behaviors of the classical orthogonal polynomials such as Hermite $H_n(x)$, Laguerre $L_n^{(\alpha)}(x)$ and Jacobi $P_n^{(\alpha,\beta)}(x)$ are now well-known. The same can not be said about the more recent orthogonal polynomials such as Charlier $C_n^{(a)}(x)$, Meixner $m_n(x;\beta,c)$, Pollaczck $P_n(x;a,b)$, Meixner-Pollaczek $M_n(x;\delta,\eta)$ and Krawtchouk polynomials. In this lecture, I shall make a brief report on the development of uniform asymptotic expansions for these polynomials as well as those which are orthogonal with respect to an exponential weight function on the real line.

Classical and generalized classical orthogonal polynomials in several variables

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The classical orthogonal polynomials (OP) in several variables are polynomials orthogonal with respect to the Jacobi type weight functions on the cube $[-1,1]^d$, the unit ball B^d and the simplex T^d in $\mathbb{R}^d$, as well as multiple Hermite polynomials and multiple Laguerre polynomials.

We define the generalized classical OP on B^d as polynomials orthogonal with respect to weight functions that are invariant under reflection groups, and define the generalized classical OP on T^d by a relation between orthogonal structure on T^d and that on B^d. The classical OP on B^d and on T^d correspond to the rotation group and the abelian group $\mathbb{Z}_2^d$, respectively. A connection between orthogonal structure on B^d and that on S^d allows us to connect these polynomials to Dunkl's h-harmonics associated with reflection groups. In particular, the generalized classical OP on B^d or on T^d associated with the group G are eigenfunctions of a second order differential-difference operator which is derived from the h-Laplacian associated to the $G \times \mathbb{Z}_2$. Moreover, under proper limit of the $\mathbb{Z}_2$-parameter, the generalized OP on B^d are shown to converge to the generalized Hermite polynomials on $\mathbb{R}^d$ and the generalized OP on T^d converge to the generalized Laguerre polynomials on $\mathbb{R}_+^d$. In the case of Weyl group of type A and B, the latter two cases are associated to the Calogero-Sutherland models, which have been studied by many authors.

The connection between generalized OP and h-harmonics allows us to study the Cesàro summability of the Fourier orthogonal expansion on B^d and on T^d. In the case of classical OP, it yields sharp convergence results for the Cesàro summability and remarkable compact formulae for the reproducing kernels. We will also report results on the Cesàro summability in the case of $[-1,1]^d$, which requires separate treatment.

Schur's Q-functions and twisted affine Lie algebras

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This talk deals with the weight vectors of the basic representations of the twisted affine Lie algebras $A_{2l}^{(2)}$ and $D_{l+1}^{(2)}$. The weight vectors are expressed in terms of Schur's Q-functions. The up and down motion along a string of the fundamental imaginary root is described as a combinatorial game on an abacus. As a consequence one can determine the homogeneous polynomial solutions to the reduced BKP hierarchies as the maximal weight vectors. The results obtained will be applied to the case of $A_1^{(1)}$. The embedding of $A_1^{(1)}$ into $D_4^{(2)}$ gives rise to a modified realization of the basic representation of $A_1^{(1)}$. The isomorphism between the *standard* and this *non-standard* realizations of the basic representation induces a *funny* identity of Schur's S-functions and Q-functions indexed by certain partitions. In fact this identity reflects a nature of decomposition matrices of the modular spin representations of the symmetric groups with characteristic two.

Although the results presented in the talk might be known to experts, I could not find descriptions in the literature, at least in this form. Hence I decide to give a talk on this occasion. This is a joint work with Tatsuhiro Nakajima (Meikai University).

Finite element method for multi-dimensional hyperbolic systems of conservation laws

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The application of numerical simulation to fluid dynamics requires the scheme to be flexible and highly accurate. Unstructured grids are suitable to complicated geometry, so they are frequently used. We present here a new explicit finite element scheme, which is of first or second order and defined on unstructured grids, for the numerical computation to multi-dimensional hyperbolic systems of conservation laws. We obtain three basic estimates for the scalar cases: maximum norm estimate, discrete entropy inequality, and energy estimate, then based on these estimates we prove strong convergence in Lp of approximate solutions to entropy solutions. Moreover it is proved that if the mesh is uniform and the exact solution is smooth, then the order of convergence in L2 is exact two for the second order scheme. Numerical experiments show that the approximate solutions are in very good agreement with the exact ones.

Asymptotic zero distribution of solutions of oscillatory second order differential equations

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Let

$$P(x,n)y''(x) + Q(x,n)y'(x) + R(x,n)y(x) = 0 \tag{1}$$

be a sequence of second order differential equations, whose coefficients are polynomials in x of fixed degree and rational functions in n. For each n, let P_n be a solution of (1).

For several applications (orthogonal polynomials theory, special functions, mathematical physics) the problem of asymptotic distribution of zeros of $\{P_n\}$ (as $n \to \infty$) is of great interest (see e.g. [1, 2, 3]).

In this talk we present a general method which allows explicit computation of the absolutely continuous component of this distribution directly in terms of the coefficients of (1). Furthermore, this method also handles an eventually necessary scaling of the variable x, i.e. $\{P_n(a_n x)\}$, in order to obtain nontrivial densities. This method is based on recent results obtained in [4, 5], and is suitable to be implemented in any computer algebra system, such as Mathematica [6].

Our results are illustrated by considering the Lame differential equation, i.e. an equation as (1) with polynomial coefficients P, Q and R of degree $p+1$, p and $p-1$. A detailed discussion is done for the cases $p=0$ and $p=1$.

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About the Jackson's q-Gamma function

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Let $q \in]0,1[$; let us denote $[x] = (1-q^x)/(1-q)$ and $(x;q)_\infty = \prod_{n\geq 0}(1-xq^n)$ for $x \in \mathbb{C}$. Let $\mathbb{A} = \{x \in \mathbb{C} : \Re x > 0\}$. The Jackson's Gamma function, defined on $\mathbb{A}$ by $\Gamma_q(x) = (q;q)_\infty (1-q)^{1-x}/(q^x;q)_\infty$, satisfies the functional equation

$$y(x+1) = [x]y(x), \quad y(1) = 1. \tag{1}$$

Following a paper of R. Remmert for the Γ-function, we show how to obtain an integral representation of $1/\Gamma_q$ using the fact that Γ_q is the unic analytical solution of (1) on the half-plane $\mathbb{A}$, bounded on the vertical strip $\{x \in \mathbb{C} : 1 \leq \Re x \leq 2\}$. We introduce then the solution g_q :

$$g_q(x) = \int_0^{+\infty} \frac{(-t;q)_\infty(-qt^{-1};q)_\infty}{(-q^x t;q)_\infty(-q^{1-x}t^{-1};q)_\infty((q-1)t;q)_\infty}\frac{dt}{t},$$

which corresponds to a *divergent* formal solution for (1). By establishing a relation between g_q and Γ_q, we show that our function g_q converges to Γ when q tends to 1.

On a new special function in a wave-beach problem

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We present a new special function, which arises from a wave-beach problem in Geophysics. For some special cases, the new special function can be reduced to known functions, such as Bessel function, hypergeometric function and confluent hypergeometric function. For general case, the function, being solved with Frobenius' theory, can be written as a Taylor series, in which the coefficients can be computed with a recursive formula. The asymptotic expansion of the new special function is derived with WKB expansion technique.

Smoothing of Stokes discontinuity for the generalized Bessel function

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A modification of the steepest-descent method, as introduced recently by Berry and Howls, is used to re-investigate the asymptotic expansion of the generalized Bessel function

$$\phi(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(l+1)\Gamma(\rho l + \beta)}$$

where $-1 < \rho < \infty$ and β is any number, real or complex. Through this modified method, a uniform exponentially improved expansion is obtained, as well as the smooth transition across the Stokes lines is achieved, for the function $\phi(z)$.