CAP forms on $U(2, 2)$ II. Cusp forms *

Takuya KONNO †, Kazuko KONNO ‡

Abstract

This is a report of our work on non-tempered automorphic representations of $U_{E/F}(2, 2)$. Few years ago, we obtained a complete description of the local components of such automorphic forms. This time, we construct all the expected automorphic forms with these components.

Contents

1 Introduction to CAP forms

2 $A$-parameters

3 Review of the local theory

4 Presentation of the problem

5 Endoscopy for $U_{E/F}(2)$

6 Automorphic forms

1 Introduction to CAP forms

The term CAP is a short hand for the phrase “Cuspidal but Associated to Parabolic subgroups”. This is the name given by Piatetski-Shapiro [PS83] to those cuspidal automorphic representations which apparently contradict the generalized Ramanujan conjecture. An up-to-date definition of CAP forms might be given as follows.

---

*Talk at the conference “Automorphic representations, $L$-functions and periods”, RIMS, Kyoto Univ., January, 2006

† Graduate School of Mathematics, Kyushu University, Hakozaki, Higashi-ku, Fukuoka, 812-8581, Japan

E-mail: takuya@math.kyushu-u.ac.jp

URL: http://knmac.math.kyushu-u.ac.jp/~tkonno/

‡ Dept of Edu., Fukuoka University of Education, 1-1 Bunkyomachi Akama, Munakata-city, Fukuoka, 811-4192, Japan

E-mail: kazukokonno@vesta.ocn.ne.jp

URL: http://www15.ocn.ne.jp/~tkonno/kkonno/Kazuko.html
Let $G$ be a connected reductive group defined over a number field $F$. We write $\mathbb{A} = \mathbb{A}_F$ for the adele ring of $F$. By an automorphic representation of $G(\mathbb{A})$, we mean an irreducible subquotient of the right regular representation

$$R(g)\phi(x) = \phi(xg), \quad g \in G(\mathbb{A})$$

of $G(\mathbb{A})$ on the Hilbert space

$$L^2(G(F)\mathbb{A}_G \backslash G(\mathbb{A})) := \left\{ \phi : G(\mathbb{A}) \rightarrow \mathbb{C} \text{ measurable} \mid \begin{array}{l}
(i) \phi(\gamma ag) = \phi(g), \\
\quad \gamma \in G(F), \; a \in \mathbb{A}_G, \; g \in G(\mathbb{A}) \\
(ii) \; \int_{G(F)\mathbb{A}_G \backslash G(\mathbb{A})} |\phi(g)|^2 \, dg < \infty
\end{array} \right\}.$$

Here, $\mathbb{A}_G$ is the maximal $\mathbb{R}$-vector subgroup in the center $Z(G)(\mathbb{A})$ of $G(\mathbb{A})$ and the measure is taken to be $G(\mathbb{A})$-invariant. The discrete spectrum $L^2_{\text{disc}}(G(F)\mathbb{A}_G \backslash G(\mathbb{A}))$ is the maximum subspace of $L^2(G(F)\mathbb{A}_G \backslash G(\mathbb{A}))$ which is a direct sum of irreducible subrepresentations. Further this decomposes as

$$L^2_{\text{disc}}(G(F)\mathbb{A}_G \backslash G(\mathbb{A})) = L^2_0(G(F)\mathbb{A}_G \backslash G(\mathbb{A})) \oplus L^2_{\text{res}}(G(F)\mathbb{A}_G \backslash G(\mathbb{A})).$$

Here $L^2_0(G(F)\mathbb{A}_G \backslash G(\mathbb{A}))$ is the completion of the space of cusp forms with respect to the Petersson (i.e., $L^2$-) norm and called the cuspidal spectrum. On the other hand, $L^2_{\text{res}}(G(F)\mathbb{A}_G \backslash G(\mathbb{A}))$ is spanned by certain iterated residues of Eisenstein series

$$\text{Res}_{\lambda = s} E^G_F(\phi), \quad \phi \in \text{Ind}^G_F(\tau, \kappa) \subset L^2_0(M(F)\mathbb{A}_M \backslash M(\mathbb{A})),
$$

where $P = MU \subset G$ is a proper parabolic subgroup. We observe that

- Let us write $t(\tau_v)$ for the Hecke (formerly called Satake) matrix of $\tau$ at any unramified place $v$ for $M$ and $\tau$. Then the Hecke matrix for the residue $\text{Res}_{\lambda = s} E^G_F(\tau, \kappa)$ is $q_v^{-s} t(\tau_v)$. Here $q_v$ is the cardinality of the residue field of $F_v$.

- According to Langlands’ criterion for square integrability, we must have $\Re \varpi^\vee(s) > 0$ for any “fundamental coweight” $\varpi$ for $P$.

In particular, even if $\tau$ satisfies the Ramanujan conjecture for $M$ (i.e., $t(\tau_v)^2$ is bounded), any residue $\text{Res}_{\lambda = s} E^G_F(\tau, \kappa)$ in the discrete spectrum cannot satisfy the same conjecture for $G$.

Now let $G^*$ be the quasisplit inner form of $G$. At almost all places $v$ of $F$, $G_v := G \otimes_F F_v$ is isomorphic to $G_v^*$.

**Definition 1.1.** An irreducible cuspidal representation $\pi = \otimes_v \pi_v \subset L^2_0(G(F)\mathbb{A}_G \backslash G(\mathbb{A}))$ of $G(\mathbb{A})$ is a CAP form if there exists an irreducible residual automorphic representation $\pi^* = \otimes_v \pi_v^* \subset L^2_{\text{res}}(G^*(F)\mathbb{A}_G \backslash G^*(\mathbb{A}))$ of $G^*(\mathbb{A})$ such that the absolute values of the eigenvalues of the Hecke matrices $t(\pi_v)$ and $t(\pi_v^*)$ coincide at almost all $v$.

**Example 1.2.** (i) Combining the results of Jacquet-Shalika [JSSTB], [JSSTA] and Moeglin-Waldspurger [MW89], one finds that there are no CAP forms on $G = GL(n)$.

(ii) If $G = D^\times$, the unit group of a central division algebra over $F$, the trivial representation $1_{G(\mathbb{A})}$ is a CAP form.
(iii) The CAP forms on $U_{E/F}(3)$ (any unitary group in 3 variables) are the $\theta$-liftings of automorphic characters on $U_{E/F}(1, A)$ [GR90], [GR91].

(iv) The CAP forms on $Sp(2)$ are either the Saito-Kurokawa liftings ($\theta$-liftings of automorphic representations of the metaplectic cover $SL(2, A)$) or the $\theta_{10}$-type representations constructed by Howe-Piatetski-Shapiro [PS83] ($\theta$-liftings of automorphic representations of various orthogonal groups in 2-variables). It is expected but I do not know if these two families are disjoint.

(v) Some CAP forms on the split exceptional group of type $G_2$ are studied by Gan-Gurevich-Jiang [GGJ02].

(vi) The Ikeda lift on $Sp(2n)$ and the Miyawaki lift on $Sp(3)$ [Ike01] are CAP forms.

Besides its importance as counter examples to the Ramanujan conjecture, we propose the following three motivation of studying CAP forms.

- Construct and explicitly describe certain mixed motives associated to Shimura varieties. This point of view is discussed in detail in [Har93].
- Capture some periods of automorphic forms. This is related to the Ikeda-Ichino conjecture.
- Construct unipotent and other singular supercuspidal representations of $p$-adic groups.

In 2003, we have described the expected local components of the CAP forms of the quasisplit unitary group $U_{E/F}(2, 2)$ in 4-variables [KKa]. In this talk, we construct the cusp forms with those local components.

2 $A$-parameters

In order to put non-tempered automorphic forms into the framework of Langlands’ conjecture, J. Arthur proposed a series of conjectures [Art89]. The conjectural description is given through the $A$-parameters. On the other hand, these parameters are not well related to the definition of CAP forms, because the Ramanujan conjecture is not yet established for any non-abelian reductive group $G$. In order to obtain a nice framework to study CAP forms, it is best to introduce the following ad hoc notion of $A$-parameters for unitary groups.

Let $E/F$ be a quadratic extension of number fields, and write $\sigma$ for the generator of $Gal(E/F)$. We fix an algebraic closure $\bar{F}$ of $E$ (or $F$) and write $W_F$ (resp. $W_E$) for the Weil group of $\bar{F}/F$ (resp. $\bar{F}/E$). Recall the (non-split) extension $1 \rightarrow W_E \rightarrow W_F \rightarrow Gal(E/F) \rightarrow 1$. We fix an inverse image $w_\sigma \in W_F$ of $\sigma$.

First we consider the group $H_n := \text{Res}_{E/F}GL(n)$. Its $L$-group is given by $LH_n = \bar{H}_n \rtimes \rho_{H_n}$ $W_F$ with $\bar{H}_n = GL(n, \mathbb{C})^2$ and

$$\rho_{H_n}(w)(h_1, h_2) = \begin{cases} (h_1, h_2) & \text{if } w \in W_E, \\ (h_2, h_1) & \text{otherwise.} \end{cases}$$

We write $\Phi_0(H_n)$ for the set of (isomorphism classes of) irreducible unitary cuspidal representations of $H_n(A)$. Conjecturally, this should be in 1-1 correspondence with the set of isomorphism classes of irreducible $n$-dimensional representations with bounded image of the hypothetical
Langlands group $\mathcal{L}_E$ of $E$. We adopt this latter point of view, since it is convenient for some observations. There should be a natural morphism $p_{W_F} : \mathcal{L}_F \rightarrow W_F$. As in the Weil group case, $\mathcal{L}_F$ should be an extension $1 \rightarrow \mathcal{L}_E \rightarrow \mathcal{L}_F \rightarrow \text{Gal}(E/F) \rightarrow 1$. Again we take an inverse image $w_\sigma \in \mathcal{L}_F$ of the above fixed $w_\sigma \in W_F$. By [Bor79, Prop. 8.4], each $\varphi_E \in \Phi_0(H_n)$ is identified with the homomorphism $\varphi : \mathcal{L}_F \rightarrow \mathbf{H}_n$ given by

$$\varphi(w) := \begin{cases} (\varphi_E(w), \varphi_E(w w_\sigma^{-1})) \times p_{W_F}(w) & \text{if } w \in \mathcal{L}_E, \\ (\varphi_E(w w_\sigma^{-1}), \varphi_E(w w_\sigma)) \times p_{W_F}(w) & \text{otherwise.} \end{cases} \quad (2.1)$$

**Definition 2.1.** An $A$-parameter for $H_n$ is a homomorphism $\phi : \mathcal{L}_F \times SL(2, \mathbb{C}) \rightarrow \mathbf{H}_n$ such that

(i) $\phi|_{SL(2, \mathbb{C})} : SL(2, \mathbb{C}) \rightarrow \widehat{H}_n$ is analytic.

(ii) $\mathcal{L}_F \phi \rightarrow L H_n \rightarrow W_F$ coincides with $p_{W_F} : \mathcal{L}_F \rightarrow W_F$. Thus $\phi$ is determined by the representation $\phi_E : \mathcal{L}_E \times SL(2, \mathbb{C}) \rightarrow L H_n \rightarrow GL(n, \mathbb{C})$ (under (2.1)).

(iii) $\phi_E$ is semisimple, so that we have an irreducible decomposition $\phi_E \simeq \bigoplus_{i=1}^r \varphi_i, E \otimes \rho_d$. Here, $\varphi_i, E$ is a semisimple irreducible representation of $\mathcal{L}_E$ and $\rho_d$ denotes the $d$-dimensional irreducible representation of $SL(2, \mathbb{C})$. Note $\sum_{i=1}^r d_i m_i = n$.

(iv) $\varphi_i, E \in \Phi_0(H_m)$. $A$-parameters $\phi, \phi'$ for $H_n$ are equivalent if they are $\widehat{H}_n$-conjugate, or equivalently, if $\phi_E$ and $\phi'_E$ are isomorphic. An $A$-parameter $\phi$ contributes to the discrete spectrum if and only if it is elliptic, i.e., $\phi_E$ is irreducible.

Now we turn to the quasisplit unitary group $G = G_n$ in $n$-variables for $E/F$. This can be realized in such a way that

$$G_n(R) := \{ g \in M_n(R \otimes_F E) \mid \theta_n(g) = \sigma(g) \},$$

for any abelian $F$-algebra $R$. Here $\theta_n(g) := \text{Ad}(I_n)^gy^{-1}$ with

$$I_n := \begin{pmatrix} \vdots & & 1 \\ & \ddots & \\ & & -1 \end{pmatrix}.$$ 

The $L$-group $L G_n = \widehat{G}_n \rtimes_{\rho_{G_n}} W_F$ is given by $\widehat{G}_n = GL(n, \mathbb{C})$ and

$$\rho_{G_n}(w) = \begin{cases} \text{id} & \text{if } w \in W_E, \\ \theta_n & \text{otherwise.} \end{cases}$$

**Definition 2.2.** An $A$-parameter for $G$ is a homomorphism $\phi : \mathcal{L}_F \times SL(2, \mathbb{C}) \rightarrow LG$ such that

$$\phi_E : \mathcal{L}_E \times SL(2, \mathbb{C}) \rightarrow LG \rightarrow GL(n, \mathbb{C})$$

coincides with $\phi^H_E$ for some $A$-parameter $\phi^H$ for $H_n$. 

---

Note: The above text contains mathematical content that is beyond the scope of a natural language model to fully interpret. The text includes advanced algebraic and number-theoretic concepts, and the reader is encouraged to consult relevant mathematical literature for a deeper understanding.
Two $A$-parameters are equivalent if they are $\hat{G}$-conjugate. Let $\Psi(G)$ be the set of equivalence classes of $A$-parameters for $G$. For an $A$-parameter $\phi$, we write $S_\phi(G)$ for the centralizer of $\phi(\mathcal{L}_F \times SL(2, \mathbb{C}))$ in $\hat{G}$, and $S_\phi(G)$ for the group of connected components of $S_\phi(G)/Z(\hat{G})_{Gal(F/F)}$. $\phi \in \Psi(G)$ is called elliptic if the identity component $S_\phi(G)^0$ of $S_\phi(G)$ is contained in $Z(\hat{G})_{Gal(F/F)}$. We write $\Psi_0(G)$ for the subset elliptic classes in $\Psi(G)$. An elliptic $\phi$ is of CAP-type if $\phi|_{SL(2, \mathbb{C})}$ is non-trivial. We write $\Psi_{CAP}(G)$ for the set of classes of CAP-type in $\Psi_0(G)$.

An elementary exercise in representation theory shows that each $\phi \in \Phi_0(G_n)$ can be written as

$$\phi_E \simeq \bigoplus_{i=1}^{r} \xi_i \cdot \varphi_{i,E} \otimes \rho_{d_i} \quad (2.2)$$

where,

- $\varphi_i \in \Psi(G_m)$ is such that $\varphi_{i,E}|_{\mathcal{L}_E}$ is irreducible;
- $\xi_i$ is an idele class character of $E$ such that $\xi_i|_{A^\times} = \omega_{E/F}^{n-d_i-m_i+1}$. Here $\omega_{E/F}$ is the quadratic character of $A^\times/F^\times$ associated to $E/F$ by the classfield theory.
- $\xi_i \cdot \varphi_{i,E} \nprec \xi_j \cdot \varphi_{j,E}$, ($1 \leq i \neq j \leq r$).

Thus it suffices to describe the set

$$\Phi_{st}(G_m) := \{ \varphi \in \Psi_0(G_m) \mid \varphi_E|_{\mathcal{L}_E} \text{ is irreducible} \}.$$

For $\varphi \in \Phi_{st}(G_m)$, $\varphi_E$ viewed as a parameter for $H_m$ corresponds to a cuspidal automorphic representation $\pi_E$ of $H_m(\mathbb{A})$. According to Langlands’ functoriality conjecture, the map $\varphi \mapsto \varphi_E$ corresponds to the standard base change lifting from $G_m(\mathbb{A})$ to $H_m(\mathbb{A})$ [Rog90]. Hence the description of $\Phi_0(G_m)$ amounts to that of the image of the standard base change. As for this question, the following expectation is well-known.

**Conjecture 2.3.** Let $\pi_E$ be an irreducible cuspidal representation of $H_m(\mathbb{A})$ and $\varphi^H : \mathcal{L}_F \rightarrow L_{H_m}$ be its Langlands parameter. Take an idele class character $\mu$ of $E$ such that $\mu|_{A^\times} = \omega_{E/F}$. Then $\varphi^H = \varphi_E$ for some $\varphi \in \Phi_{st}(G_m)$ (i.e., $\pi_E$ is the standard base change lift of some stable $L$-packet of $G_m(\mathbb{A})$) if and only if

(i) $\sigma(\pi_E) := \pi_E \circ \sigma \simeq \pi_E^\vee$ (the contragredient);

(ii) the twisted tensor $L$-function $L_{Asai}(s, \mu^{n+1}(\det)\pi_E)$ [Go94] has a pole at $s = 1$.

Using the base change for $GU_{E/F}(2)$, we deduced the case $m = 2$ of the conjecture from [HLR86, Th.3.12] ([KKa, Cor.3.3]). This avails us to deduce the following description of $\Psi_{CAP}(G_4)$ from (2.2). Note that this does not involve the hypothetical Langlands group $\mathcal{L}_F$ anymore.

**Proposition 2.4.** The set $\Psi_{CAP}(G_4)$ consists of the following classes. We write $\eta$, $\mu$ for typical idele class characters of $E$ such that $\eta|_{A^\times} = 1$, $\mu|_{A^\times} = \omega_{E/F}$, respectively.
our problem can be stated as follows.

It is conjectured that any CAP-form on $G$

(ii) Describe the multiplicity of each $\pi$

Problem 3.1.

Review of the local theory

Here, in (1.b), (2.b), $\pi_E$ runs over the set of irreducible cuspidal automorphic representation of $H_2(\mathbb{A})$ such that $\sigma(\pi_E) \simeq \pi_E^\vee$ and $L_{\text{Asai}}(s, \pi_E)$ is holomorphic at $s = 1$. In (2.a) $\mu = (\mu, \mu')$ where $\mu'$ can be $\mu$. In (2.c) $\eta = (\eta, \eta')$ modulo permutation, with $\eta \neq \eta'$. Finally, in (2.d) $\mu = (\mu, \mu')$ modulo permutation and $\mu \neq \mu'$.

3 Review of the local theory

Let $\phi$ be an $A$-parameter for $G = G_4$. By restriction, we obtain the local component

$$\phi_v : \mathcal{L}_{F_v} \times SL(2, \mathbb{C}) \to L^1_{\text{disc}}G_v$$

of $\phi$ at each place $v$ of $F$. Here the local Langlands group $\mathcal{L}_{F_v}$ is given by $W_{F_v} \times SU(2, \mathbb{R})$ if $v$ is non-archimedean and $W_{F_v}$ otherwise [Kot84, §12]. $L^1_{\text{disc}}G_v$ is the $L$-group of the scalar extension $G_v = G \otimes_F F_v$. Arthur’s local conjecture, among other things, associates to each $\phi_v$ a finite set $\Pi_{\phi_v}(G_v)$ of isomorphism classes of irreducible unitarizable representations of $G(F_v)$, called an $A$-packet. At all but finite number of $v$, $\Pi_{\phi_v}(G_v)$ is expected to contain a unique unramified element $\pi_v^1$. Using such elements, we can form the global $A$-packet associated to $\phi$:

$$\Pi_{\phi}(G) := \left\{ \bigotimes_v \pi_v \mid (i) \quad \pi_v \in \Pi_{\phi_v}(G_v), \quad \forall v; \quad (ii) \quad \pi_v = \pi_v^1, \quad \forall v \right\}.$$ 

It is conjectured that any CAP-form on $G$ is contained in $\Pi_{\phi}(G)$ for some $\phi \in \Psi_{\text{CAP}}(G)$. Thus our problem can be stated as follows.

Problem 3.1. (i) Describe $\Pi_{\phi}(G)$ (or equivalently, its local components $\Pi_{\phi_v}(G_v)$.

(ii) Describe the multiplicity of each $\pi \in \Pi_{\phi}(G)$ in $L^2_{\text{disc}}(G(F) \backslash G(\mathbb{A}))$. (Note $\mathfrak{A}_G = \{1\}$ for the unitary group $G$.)

Example 3.2. The $A$-packets associated to some of the parameters listed in Prop. 2.4 can be easily described.

(1.a) For $\phi_\eta$, we have $\Pi_{\phi}(G) = \{ \eta_G := \eta_u(\det) \}$, where $\eta_u : U_{E/F}(1, A) \ni z/\sigma(z) \mapsto \eta(z) \in \mathbb{C}^\times$.

(1.b) For $\phi_{\pi E, \mu}$, $\Pi_{\phi}(G)$ consists of the unique irreducible quotient $J^2_E(\mu(\det) \pi_E \vert \det \vert_{\mathbb{A}})^{1/2}$, of the global parabolically induced representation from the Siegel parabolic subgroup $P = MU$.

(2.a) For $\phi_\mu$, $\Pi_{\phi}(G)$ consists of the $\theta$-lifting $\theta_u((\mu/\mu')_u, W)$ of the automorphic character $(\mu/\mu')_u$ of $U_{E/F}(1, A)$.
In particular, no CAP forms occur in these cases. All of these representations are known to occur in the residual discrete spectrum [Kon98]. Hence from now on, we concentrate on the rest cases (2.b–d).

**Local A-packets** Now let $E/F$ be a quadratic extension of non-archimedean local fields of characteristic zero. We also have corresponding results in the archimedean case, but we need some extra notation to state them. Let $\phi$ be (local analogue of) an $A$-parameter of type (2.b–2.d). In [KKa], we have constructed $\Pi_\phi(G)$ by the local $\theta$-correspondence. Let us briefly recall the construction. First note that $\phi$ can be written in the form

$$\phi_E = \varphi_{\eta_E} \otimes (\eta \otimes \rho_2).$$  

(3.1)

Here $\varphi_{\eta_E} : \mathcal{L}_E \to GL(2, \mathbb{C})$ corresponds to an irreducible admissible representation $\pi_E$ of $H_2(F) = GL(2, E)$ under the local Langlands correspondence [HT01], [Kut80]. Also notice that $\mathcal{S}_\phi(G) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For a 2-dimensional hermitian space $(V, (\cdot, \cdot))$, we write $G_V$ for its unitary group. $(W_n, (\cdot, \cdot)_n)$ denotes the hyperbolic skew-hermitian space of dimension $2n$, so that $G = G_4$ is the unitary group $G_{W_2}$ of $W_2$. Fix a character pair $\xi = (1, \eta)$ of $E^\times$ such that $\eta_{|E^\times} = 1$, and a non-trivial character $\psi_F : F \to \mathbb{C}^\times$. These specify the Weil representation $\omega_{\psi, W_n} = \omega_{W_1} \times \omega_{\psi_\eta}$ of $G_V(F) \times G_W(F)$. As usual, this determines the local $\theta$-correspondence

$$\mathcal{R}(G_V, \omega_{W_1}) \ni \theta_\xi(\pi_V, W) \mapsto \xi(\pi_V, W) \in \mathcal{R}(G_W, \omega_{\psi_\eta})$$

between certain subsets $\mathcal{R}(G_V, \omega_{W_1}) \subset \Pi(G_V(F)), \mathcal{R}(G_W, \omega_{\psi_\eta}) \subset \Pi(G_W(F))$. Here $\Pi(G_V(F))$ denotes the set of isomorphism classes of irreducible admissible representations of $G_V(F)$.

**Definition 3.3.** In the notation of (3.1) let $\Pi_{\psi, W} \in \mathcal{R}(G_V, \omega_{W_1})$ be the $L$-packet of $G_V(F)$ whose standard base change to $H_2(F)$ is $\eta(\det)\pi_{W_1}$. (Empty if $V$ is anisotropic and $\pi_E$ is in the principal series.) We define

$$\Pi_\phi(G) := \bigoplus_v \theta_{\xi}(\Pi_{\psi, W} \in \mathcal{R}(G_V, \omega_{W_1}), W),$$

where $V$ runs over the set of isometry classes of 2-dimensional hermitian space over $E$.

**4 Presentation of the problem**

We now go back to the global setting. Let $\phi$ be an $A$-packet of type (2.b–d) in Prop2.4. Having defined the local $A$-packets, we have the global packet $\Pi_\phi(G) = \otimes_v \Pi_\phi (G_v)$ in the present case, the multiplicity formula in Arthur’s conjecture is stated as follows.

**Conjecture 4.1.** There exists a pairing $\langle \cdot, \cdot \rangle : \mathcal{S}_\phi(G) \times \Pi_\phi(G) \to \{ \pm 1 \}$ such that the multiplicity of $\pi = \otimes_v \pi_v \in \Pi_\phi(G)$ in $L^2_{\text{disc}}(G'(F) \backslash G(A))$ is given by

$$m(\pi) := \frac{1}{|\mathcal{S}_\phi(G)|} \sum_{s \in \mathcal{S}_\phi(G)} \epsilon(\pi) \prod_v \langle s, \pi_v \rangle.$$

Here, $\epsilon$ is the sign character of the first $\mathbb{Z}/2\mathbb{Z}$ of $\mathcal{S}_\phi(G)$ if $\varepsilon(1/2, \pi_E \times \eta^{-1}) = -1$, and is the trivial character otherwise.
Our main result states that $m(\pi)$ is equal to or larger than the right side hand of the conjectural formula. But this makes sense only after the pairing $\langle \cdot, \cdot \rangle : \mathcal{S}_{\phi_\pi}(G_{\epsilon}) \times \Pi_{\phi_\pi}(G_{\epsilon}) \to \{ \pm 1 \}$ is described.

**Pairing in the stable case**  The pairing $\langle \cdot, \cdot \rangle : \mathcal{S}_{\phi_\pi}(G_{\epsilon}) \times \Pi_{\phi_\pi}(G_{\epsilon}) \to \{ \pm 1 \}$ is given locally as the notation indicates. Thus we may go back to the local non-archimedean situation of $\mathfrak{L}_{E/F}$.

First we recall some basic requirements on $\Pi_{\phi_\pi}(G_{\epsilon})$ from [Art89].

(i) For $\phi \in \Psi(G)$, we have a Langlands’ parameter

$$\varphi_\phi : \mathcal{L}_F \ni w \mapsto \phi\left( w, \left( |w|_F^{1/2}, |w|_F^{-1/2} \right) \right) \times pw_F(w) \in L_G,$$

where $| \cdot |_F$ is the transport of the module of $F$ by the reciprocity isomorphism $F^\times \simeq W_{F,\text{ab}}$ (or its composite with $\mathcal{L}_F \xrightarrow{pw_F} W_F \xrightarrow{} W_{F,\text{ab}}$). Then the associated $L$-packet $\Pi_{\phi_\pi}(G_{\epsilon})$ should be contained in $\Pi(\phi(G_{\epsilon}))$.

(ii) More precisely, there exists a parabolic subgroup $P_\phi = M_\phi U_\phi$ such that $\phi(\mathcal{L}_F) \subset L M_\phi$ and

$$\mu : \mathcal{L}_F \ni w \mapsto \phi(1, \left( |w|_F^{1/2}, |w|_F^{-1/2} \right)) \in L G$$

is a $P_\phi$-dominant element of $\mathfrak{a}_{M_\phi}^0 = (\mathfrak{Lie \mathfrak{g}}_{M_\phi})^*$. Then $\Pi_{\phi_\pi}(G) = \{ J_{P_\phi}^G(\pi \otimes e^{\mu_\phi}) | \pi \in \Pi_{\phi_\pi}(M_\phi) \}$, where $J_{P_\phi}^G(\pi \otimes e^{\mu_\phi})$ is the “Langlands’ quotient”$^\text{M}$ of the standard parabolically induced representation $I_{P_\phi}^G(\pi \otimes e^{\mu_\phi})$. Now let us fix a Borel subgroup $B = TU$ and a non-degenerate character $\psi_U$ of $U(F)$. According to the **generic packet conjecture**, $\Pi_{\phi_\pi}(M_\phi)$ contains a unique generic representation $\pi_1$ with respect to $\psi(U_{\mathfrak{Lie \mathfrak{g}}_{M_\phi}}(F))$. Then, the pairing between $\Pi_{\phi_\pi}(G_{\epsilon})$ and $\Pi(\mathcal{S}_{\phi_\pi}(G_{\epsilon}))$ should be chosen in such a way that $\langle J_{P_\phi}^G(\pi_1 \otimes e^{\mu_\phi}), \cdot \rangle$ is the trivial character of $\mathcal{S}_{\phi_\pi}(G_{\epsilon})$.

(iii) The following diagram should commute.

\[
\begin{array}{ccc}
\Pi_{\phi_\pi}(G_{\epsilon}) & \ni & J_{P_\phi}^G(\tau \otimes e^{\mu_\phi}) \\
\text{inclusion} & & \langle \cdot, \tau \rangle \in \Pi(\mathcal{S}_{\phi_\pi}(M_\phi)) \\
\Pi_{\phi_\pi}(G) & \ni & \pi \\
\text{inclusion} & & \langle \cdot, \pi \rangle \in \Pi(\mathcal{S}_{\phi_\pi}(G_{\epsilon})).
\end{array}
\]

Going back to $\phi$ of type (2.b–d), the construction of the local packet $\Pi_{\phi_\pi}(G_{\epsilon})$ involved the following, so-called $\varepsilon$-**dichotomy** property of the local $\theta$-correspondence. Recall that there are only two isometry classes of 2-dimensional hermitian space $V$ over $E$. They are classified by the signature $\omega_{E/F}(-\det V)$.

**Theorem 4.2** ([KLa] Th.6.4). We adopt the notation of Def.3.3 The local $\theta$-correspondent $\theta_{\Sigma}(\Pi_{\phi_\pi}(G_{\epsilon})), V$ of the $L$-packet $\Pi_{\phi_\pi}(G_{\epsilon})$ to $G_V(F)$ is the $L$-packet $\Pi_{\phi_\pi}(G_V)$ if

$$\varepsilon(1/2, \pi_E \times \psi_E \mid \omega_{\Pi_{\phi_\pi}(G_{\epsilon})}(-1)) = \omega_{E/F}(-\det V),$$

\(^1\text{Again not precisely, because } \pi \text{ is not always tempered in our definition of } A\text{-parameters.}\)
and is zero otherwise. Here $\psi_E := \psi_F \circ \text{Tr}_{E/F}$ and $\varepsilon(s, \pi_E \times \eta^{-1}, \psi_E)$ is the Jacquet-Langlands local constant of $\pi_E \times \eta^{-1}$. Also $\omega_{\pi_E(G_2)}$ denotes the common central character of the members of $\Pi_{\pi_E}(G_2)$.

If we write $V$ for the (isometry class of the) 2-dimensional hermitian space over $E$ satisfying the condition of Th. 4.2 and $V'$ for the other one, the construction of $\Pi_{\phi}(G)$ is summarized in the following diagram.

Moreover, the induction principle of the local $\theta$-correspondence [Kud86], [MVW87, Ch.3] shows that $\Pi_{\phi}(G) = \theta_\xi(\Pi_{\eta \pi_E}(G_V), W_2)$. This together with the requirement (iii) above yield the following.

**Theorem 4.3.** Suppose $\Pi_{\pi_E}(G_2)$ is stable, i.e., consists of a single element, so that $S_{\phi}(G) \simeq \mathbb{Z}/2\mathbb{Z}$. Then we have

$$\langle \theta_\xi(\Pi_{\eta \pi_E}(G_V), W), \cdot \rangle = \text{sgn}, \quad \langle \theta_\xi(\Pi_{\eta \pi_E}(G_{V'}), W), \cdot \rangle = 1,$$

where $V$ and $V'$ are labeled as above.

## 5 Endoscopy for $U_{E/F}(2)$

It remains to consider the case where $\Pi_{\pi_E}(G_2)$ is endoscopic. This is the case (2.d) in Prop 2.4 (see [KKb 4.3]):

$$\varphi_E = \mu \oplus \mu', \quad \pi_E = I(\mu \otimes \mu').$$

We write $\Pi_{\mu}(G_V) := \Pi_{\pi_E}(G_V) = \{\pi_V(\mu)^\pm\}$ with $\mu = (\mu, \mu')$.

We briefly recall the endoscopic lifting for $G_V$ from [KKb]. The unique non-trivial elliptic endoscopic data for $G_V$ is $(H, H, s, \xi)$, where $H = U_{E/F}(1)^2$, $s = (\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ and $\xi : \text{L}H \hookrightarrow \text{L}G_2$ is the $L$-embedding given by

$$\hat{H} \ni (z_1, z_2) \mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \times 1$$

$$\xi : \quad W_E \ni w \mapsto \begin{pmatrix} \mu_0(w) & 0 \\ 0 & \mu'_0(w) \end{pmatrix} \times w \in \text{L}G_2,$$

$$w_\sigma \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times w_\sigma.$$

Here $\mu_0 = (\mu_0, \mu_0')$ are characters of $E^\times$ such that $\mu_0|_{F^\times} = \mu'_0|_{F^\times} = \omega_{E/F}$. The isomorphism class of the data is independent of $\mu_0$.\]
We fix a generator $\delta$ of $E$ over $F$ such that $\text{Tr}_{E/F}(\delta) = 0$, and take $\varepsilon \in F^\times \setminus N_{E/F}(E^\times)$.

We may realize $(V, (\cdot, \cdot))$ as $V = E^2$ and

$$(v, v') = \begin{cases} 
(\sigma(v) \begin{pmatrix} 0 & (2\delta)^{-1} \\ -(2\delta)^{-1} & 0 \end{pmatrix}) v' & \text{if } V \text{ is hyperbolic}, \\
(\sigma(v) \begin{pmatrix} -\varepsilon & 0 \\ 0 & 1 \end{pmatrix}) v' & \text{if } V \text{ is anisotropic}.
\end{cases}$$

Then we fix an embedding

$$\eta_V : H \ni \gamma_H = (zz', z\sigma(z')) \mapsto \begin{pmatrix} z \begin{pmatrix} x & -y\Delta \\ y & x \end{pmatrix} \end{pmatrix} \in G_V \quad \text{if } V \text{ is hyperbolic},$$

$$\begin{pmatrix} z' \begin{pmatrix} x & 0 \\ 0 & z\sigma(z') \end{pmatrix} \end{pmatrix} \in G_V \quad \text{if } V \text{ is anisotropic.}$$

Here, each element $\gamma_H \in H$ is written as $(zz', z\sigma(z'))$ for some $z, z' \in \text{Res}_{E/F}^G \mathbb{G}_m$ with $N_{E/F}(z) = N_{E/F}(z')^{-1}$ and $\Delta := -\delta^2$. These data together with the non-trivial character $\psi_F$ in \[3\] determines the Langlands-Shelstad transfer factor $\Delta_V : H(F)_{G\text{-reg}} \times G_V(F)_{\text{reg}} \to \mathbb{C}$.

This is characterized by the formula

$$\Delta_V(\gamma_H, \eta_V(\gamma_H)) = \lambda(E/F, \psi_F) \omega_{E/F}(\frac{z' - \sigma(z')}{-2\delta}) \mu_0(x_1) \mu'_0(x_2) \frac{|z' - \sigma(z')|^1_E}{|z'|^{1/2}_E}. \quad (5.1)$$

Here $\lambda(E/F, \psi_F)$ is Langlands’ $\lambda$-factor for $E/F$ with respect to $\psi_F$, and we have written $zz' = x_1/\sigma(x_1), z\sigma(z') = x_2/\sigma(x_2)$ for some $x_1, x_2 \in E^\times$.

**Fact 5.1** (Labesse-Langlands, [KKb] Ch.3). For any $f \in C_c^\infty(G_V(F))$,

$$f^H : H(F)_{G\text{-reg}} \ni \gamma_H \mapsto \sum_{\gamma \in \text{Ad}(G_V(F))\eta_V(\gamma_H) \cap G_V(F) \text{ mod. } G_V(F)_{\text{conj.}}} \Delta_V(\gamma_H, \gamma) O_\gamma(f) \in \mathbb{C}$$

extends to an element of $C_c^\infty(H(F))$. Here $O_\gamma(f)$ denotes the orbital integral of $f$ at $\gamma$.

The **endoscopic lifting** which we need is the adjoint map of $\gamma \mapsto f^H$ from the space of invariant distributions on $G(F)$ to that on $H(F)$. In particular, the $L$-packet $\Pi_{\gamma}(G_V) = \{ \pi_V(\mu)_{\pm} \}$ is labeled in such a way that

$$\text{tr} \pi_V(\mu)^+ (f) - \text{tr} \pi_V(\mu)^- (f) = ((\mu/\mu_0)_w \otimes (\mu'/\mu_0')_w)(f^H)$$

holds. If $V$ is hyperbolic in the realization $\math Metropolitan_\gamma(G_V)$, then $\pi_V(\mu)^{\pm}$ is the unique generic member in $\Pi_{\gamma}(G_V)$ with respect to the character [KKb] Prop.4.8]

$$\psi_U : U_2(F) \ni \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mapsto \psi_F(b) \in \mathbb{C}^\times.$$  

(This is a consequence of the Whittaker normalization of the transfer factor (5.1).) Combining these with the seesaw duality [Kud84]
we obtain a Saito-Tunnell type character formula for $\pi_V(\mu)^\pm$.

**Theorem 5.2.** For a character $\mu$ such that $\mu|_{F^\times} = \omega_{E/F}$, we introduce a sign $\varepsilon_{\psi_F}(\mu) := \varepsilon(1/2, \mu, \psi_E)\mu(-\delta)$.

(i) If $V$ is hyperbolic, the character (function) $\Theta_{\pi_V(\mu)^\pm}$ of $\pi_V(\mu)^\pm$ is given by (respecting signs)

$$
\Theta_{\pi_V(\mu)^\pm} \circ \eta_V = \sum_{\eta_{F^\times} = 1} 4 \left(1 \pm \varepsilon_{\psi_F}(\eta\mu^{-1})(1 \pm \varepsilon_{\psi_F}(\eta\mu^{-1}'))(\mu\mu'\eta)_{u} \otimes \eta_u.
\right)
$$

(ii) If $V$ is anisotropic, we have (respecting signs)

$$
\Theta_{\pi_V(\mu)^\pm} \circ \eta_V = \sum_{\eta_{F^\times} = 1} 4 \left(1 \mp \varepsilon_{\psi_F}(\eta\mu^{-1})(1 \pm \varepsilon_{\psi_F}(\eta\mu^{-1}'))(\mu\mu'\eta)_{u} \otimes \eta_u.
\right)
$$

Of course, these formulae indicates various interesting speculations. But this is not a place to discuss them. We only remark that the same formulae are also valid in the archimedean case. Now we combine the theorem with the seesaw duality

$$
G_V \quad U_{E/F}(1) \times U_{E/F}(1)
$$

$$
\eta_V \quad \times
$$

$$
U_{E/F}(1) \times U_{E/F}(1) \quad U_{E/F}(1)
$$

we obtain the following.

**Theorem 5.3** (Howe duality for $\Pi_\mu(G_V)$). We write $\Pi_\mu(G_2) = \{\pi(\mu)^\pm\}$ as above. Suppose $(V, \langle \cdot, \cdot \rangle)$ satisfies the condition of Th.4.2. Then we have $\theta_{\xi}(\pi(\mu)^\pm, V) = \pi_V(\eta\mu^{-1})^{\pm\varepsilon_{\psi_F}(\mu)}$, where $\eta\mu^{-1} := (\eta\mu^{-1}, \eta\mu^{-1}')$.

**Pairing in the endoscopic case** We now define the pairing $\langle \cdot, \cdot \rangle : \Pi_\phi(G) \times \mathcal{S}_\phi(G) \to \{\pm 1\}$ for $\phi$ in Prop.2.4 (2.d). We retain the notation of the above discussion.

**Definition 5.4.** Recall that $\mathcal{S}_\phi(G)$ for $\phi_E \simeq (\eta \otimes \rho_2) \oplus \mu \oplus \mu'$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The pairing is defined by

$$
\langle \cdot, \theta_{\xi}(\pi_V(\eta\mu^{-1})^\pm, W_2) \rangle := \text{sgn}^{1-\varepsilon_{V,\eta}(\mu)/2} \otimes \text{sgn}^{1+\varepsilon_{\psi_F}(\mu)/2},
$$

where $\varepsilon_{V,\eta}(\mu) := \varepsilon_{\psi_F}(\eta\mu^{-1})\varepsilon_{\psi_F}(\eta\mu^{-1})\omega_{E/F}(-\det V)$. 


6 Automorphic forms

We now go back to the global situation, and consider the $A$-parameters $\phi$ of type (2.b)–(2.d) in Prop[2.4] As is announced in §4, our principal result is the following.

Theorem 6.1. Each $\pi = \bigotimes_v \pi_v \in \Pi_\phi(G)$ occurs in the discrete spectrum $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}))$ with the multiplicity at least:

$$\frac{1}{|S_\phi(G)|} \sum_{s \in S_\phi(G)} \epsilon_\phi(s) \prod_v \langle \pi_v, s \rangle.$$  \hspace{1cm} (6.1)

Here, $\epsilon_\phi$ is the sign character of the first $\mathbb{Z}/2\mathbb{Z}$ of $S_\phi(G)$ if $\epsilon(1/2, \pi E \times \eta^{-1}) = -1$, and is the trivial character otherwise.

The proof involves the global $\theta$-correspondence between $G_V(\mathbb{A})$ and $G(\mathbb{A})$ and the description of the discrete spectrum of $G_V(\mathbb{A})$ [KK5].

Remark 6.2. Those $\pi \in \Pi_\phi(G)$ such that $\epsilon(1/2, \pi E \times \eta^{-1}) = 1$ and $\langle \pi_v, \cdot \rangle$ are trivial on the first $\mathbb{Z}/2\mathbb{Z} \subset S_{\phi_v}(G_v)$ at all $v$ are the residual discrete automorphic representations of $G(\mathbb{A})$ [Kon98]. All the other $\pi$ with non-zero (6.1) are CAP automorphic forms.

References


Index

\((f, f^H)\), 10
\((H, L, H, s, \xi)\), 9
\((V, (\cdot, \cdot))\), 7
\((W_n, \langle \cdot, \cdot \rangle_n)\), 7
\(\delta\), 10
\(\Delta V(\gamma_H, \gamma)\), 10
\(\eta_V : H \rightarrow G_V\), 10
\(\mathcal{L}_F\) Langlands group
  global, 4
  local, 6
\(\mathfrak{A}_G\), 2
\(\Phi_{st}(G_n)\), 5
\(\Phi_0(H_n)\), 3
\(\phi_E\) for an \(A\)-parameter, 4
\(\phi_v\) \(v\)-component of \(\phi\), 6
\(\Pi(G(F))\), 7
\(\Pi_{\phi}(G)\), 6
\(\Pi_{\tau_E}(G_V)\), 7
\(\Pi_{\mu}(G_V)\), 9
\(\pi_V(\mu)^\pm\), 10
\(\Psi(G)\), 4
\(\Psi_{CAP}(G)\), 5
\(\Psi_{0}(G)\), 5
\(\psi_F\), 7
\(\sigma\), 3
\(\Theta_{\pi_V(\mu)^\pm}\), 11
\(\theta_{\xi}(\tau_V, W)\), \(\theta_{\xi}(\tau_W, V)\), 7
\(\theta_{n}\), 4
\(\mu\), 9
\(\mu_{0} = (\mu_0, \mu'_0)\), 9
\(\xi\), 7
\(\varphi_{\phi}\), 8
\(\varepsilon\)-dichotomy, 8
\(\varepsilon \in F^\times \setminus N_{E/F}(E^\times)\), 10
\(\varepsilon_{\psi_F}(\mu)\), 11
\(\varepsilon_{\psi_F}(\mu)\), 11
\(A\)-packet, 6
\(A\)-parameter
  elliptic, 5
  equivalent, 4
  for \(G_n\), 4
  for \(H_n\), 4
\(G_n\), 4
\(G_V\), 7
\(G_W\), 7
\(H_n\), 3
\(L_0^\varphi(G(F)\mathfrak{A}_G \setminus G(A))\), 2
\(L_0^\varphi(G(F)\mathfrak{A}_G \setminus G(A))\), 2
\(L_2^\varphi(G(F)\mathfrak{A}_G \setminus G(A))\), 2
\(S_{\phi}(G)\), \(S_{\phi}(G)\), 4
\(t(\tau_v)\) Hecke matrix, 2
\(w_\sigma\), 3
\(W_\sigma\) Weil group, 3
\(A = A_F\), 2
automorphic representation, 2
Ramanujan conjecture, 2
spectrum
  cuspidal, 2
discrete, 2