On doubling construction for real unitary dual pairs

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Abstract

In the theory of Howe duality correspondence over \( \mathbb{R} \), we usually consider certain (determinant) double cover of a unitary dual pair \[ \text{Pau98} \]. For the purposes of automorphic theory, one has to use the Weil representation of the unitary dual pair instead, which is constructed by certain doubling argument \[ \text{HKS96} \]. We calculate explicit formulas for the Fock model of this latter representation. Then we determine the Howe correspondence for \( K \)-types \[ \text{How89} \] under this Weil representation. These illustrate how to translate the results of \[ \text{Pau98} \] (and \[ \text{Pau00} \]) into the doubling method setting.

Contents

1 Introduction ................................................................. 2

2 Weil representations of metaplectic groups ......................... 3
   2.1 Schrödinger model .................................................. 3
   2.2 Fock model .......................................................... 5

3 Weil representation of unitary dual pairs.......................... 7
   3.1 Unitary dual pairs .................................................... 7
   3.2 Splitting on doubled dual pairs ................................... 8
   3.3 Splitting on unitary dual pairs and the Weil representation . 11

4 Doubling construction of the Fock model.......................... 12
   4.1 Derived representation of the Schrödinger model .............. 12
   4.2 Compatible Cartan decompositions .............................. 14
   4.3 Explicit formula for the Fock model ............................ 19

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1 Introduction

In this note, we present the translation rule for the local $\theta$-correspondence (Howe correspondence) of real unitary dual pairs between the two distinct settings. Let $V$ and $W$ be hermitian and skew-hermitian spaces over $\mathbb{C}$, $\mathbb{W} := V \otimes \mathbb{C} W$ becomes a symplectic space and the unitary groups $G_V, G_W$ of $V, W$ form a reductive dual pair in the symplectic group $Sp(\mathbb{W})$. For each non-trivial character $\psi$ of $\mathbb{R}$, we have the metaplectic extension $Mp(\mathbb{W})$ of $Sp(\mathbb{W})$ and its Weil representation $\omega_{\mathbb{W}}$. Let $\tilde{G}_V$ be the inverse image of $G_V$ in $Mp(\mathbb{W})$. We write $\mathcal{R}(\tilde{G}_V, \omega_{\mathbb{W}})$ for the set of isomorphism classes of irreducible Harish-Chandra modules of $\tilde{G}_V$ which appear as quotients of $\omega_{\mathbb{W}}|_{G_V}$. Then the Howe duality correspondence asserts that, for $\pi_V \in \mathcal{R}(\tilde{G}_V, \omega_{\mathbb{W}})$, there exists a unique $\pi_W \in \mathcal{R}(\tilde{G}_W, \omega_{\mathbb{W}})$ such that $\text{Hom}_{G_V \times \tilde{G}_W}(\omega_{\mathbb{W}}, \pi_V \otimes \pi_W) \neq 0$ [How89].

This correspondence plays an important role in some explicit construction of automorphic representations. In particular, one would like to have an explicit description of the correspondence. In this unitary dual pair case, such description is already obtained by A. Paul [Pau98], [Pau00]. Moreover her result is described in terms of Vogan’s Langlands classification, which is well adapted to applications. On the other hand, one usually needs a correspondence between irreducible Harish-Chandra modules of $G_V$ and $G_W$, not of $\tilde{G}_V$, $\tilde{G}_W$. The purpose of this note is to illustrate how to translate the results of [Pau98] into the local $\theta$-correspondence between $G_V$ and $G_W$ [HKS96]. This passage might be an easy exercise for experts. But we still believe that a concise description is of some use.

The Weil representation $\omega_{\mathbb{W}}$ is usually realized on its Schrödinger model. But to understand its Harish-Chandra module, it is often convenient to adopt another realization called Fock model. We briefly review these constructions in §2. As claimed above, we need a Weil representation of $G_V \times G_W$. The construction is due to Harris-Kudla-Sweet and uses the doubling technique. In §3 we review this from [HKS96], while prepare some notation for later calculation. In particular, the Weil representation $\omega_{V,W,\xi}$ obtained depends on certain pair of characters $\xi = (\xi, \xi')$ of $\mathbb{C}^\times$. The explicit formulas for the Schrödinger model are available only for the doubled representation $\omega_{V,W,\xi}$ (Lem. 3.1). In the next section §4 we calculate the derived representation of $\omega_{V,W,\xi}$. After fixing Cartan decompositions of the Lie algebras $g_V, g_W$, we specify the Fock model associated to that decomposition in §4.3. After a suitable change of variables, we obtain a decomposition of the doubled Fock model (for $\omega_{V,W,\xi}$) into the tensor product of those for $\omega_{V,W,\xi}$ and $\omega_{V,W,\xi}^{\mathbb{R}}$, together with some explicit formulas (Prop. 4.3).

The explicit description of Howe correspondence is a combination of two techniques, K-type correspondence in the space of joint harmonics [How89] and the refined induction principle due to Adams-Barbasch [AB95]. The latter is of qualitative nature and the presentation of the result in [Pau98] easily applies to our setting. Thus the rest task is to describe the K-type correspondence. We review the definition of joint harmonic polynomials and the Howe
correspondence between $K_V$ and $K_W$-types in [5.1]. Then using an explicit construction of highest weight vectors, the $K$-type correspondence is given as Th.5.4. The calculation of this part essentially follows [KV78]. Finally, as an example of usage of Th.5.4, we describe the local $\theta$-correspondence for limit of discrete series in the case $\dim V = \dim W$. Comparing the result (Prop.6.1) with [Pau98 Lem.5.2.5], one finds how to translate more general correspondence [Pau98 Th.6.1] into the setting of [HKS96] without difficulty.

2 Weil representations of metaplectic groups

We begin with a brief review of Weil representations of metaplectic groups.

2.1 Schrödinger model

Let $(\mathbb{W}, \langle \cdot, \cdot \rangle)$ be an $N$-dimensional symplectic space over $\mathbb{R}$. That is, $\mathbb{W}$ is an $N$-dimensional $\mathbb{R}$-vector space and $\langle \cdot, \cdot \rangle : \mathbb{W} \otimes \mathbb{R} \mathbb{W} \to \mathbb{R}$ is a non-degenerate bilinear form satisfying

$$\langle w, w' \rangle = -\langle w', w \rangle, \quad w, w' \in \mathbb{W}.$$ 

We write $Sp(\mathbb{W}) := \{ g \in GL_R(\mathbb{W}) \mid \langle w.g, w'.g \rangle = \langle w, w' \rangle, \forall w, w' \in \mathbb{W} \}$ for its symplectic group. A non-trivial character $\psi : \mathbb{R} \to \mathbb{C}^\times$ determines the metaplectic group

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp(\mathbb{W}) \longrightarrow Sp(\mathbb{W}) \longrightarrow 1$$

of $(\mathbb{W}, \langle \cdot, \cdot \rangle)$. If $\mathbb{Y} \subset \mathbb{W}$ is a maximal isotropic (Lagrangian) subspace, one has $Mp(\mathbb{W}) = Sp(\mathbb{W}) \times \mathbb{C}^1$ with the multiplication law

$$(g, \epsilon)(g', \epsilon') = (gg', \epsilon \epsilon' c_{\psi}(g, g')), \quad c_{\psi}(g, g') := \gamma_{\psi}(L(\mathbb{Y}, \mathbb{Y}, g'^{-1}, \mathbb{Y}, g)).$$

Here, $L(\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3)$ is the Leray invariant of $(\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3)$ [Kud94 §1] and $\gamma_{\psi}(Q)$ denotes the Weil constant of a quadratic form $Q$ (denoted by $\gamma(\psi(Q)/2)$) in [Kud94]. The isomorphism class of $Mp(\mathbb{W})$ is independent of $\mathbb{Y}$.

The Heisenberg group $H(\mathbb{W})$ of $(\mathbb{W}, \langle \cdot, \cdot \rangle)$ is $\mathbb{W} \times \mathbb{R}$ with the multiplication

$$(w, z)(w', z') := (w + w'; z + z' + \langle w, w' \rangle/2).$$

$Sp(\mathbb{W})$ acts on $H(\mathbb{W})$ by $(w, z).g = (w.g, z)$. There exists a unique isomorphism class $\rho_{\psi}^W = \rho_{\psi}$ of irreducible unitary representation of $H(\mathbb{W})$ on which the center $\{ (0; z) \mid z \in \mathbb{R} \}$ acts by $\psi$ (Stone-von-Neumann theorem). Then this extends to an irreducible unitary representation $\rho_{\psi}$ of the metaplectic Jacobi group $J(\mathbb{W}) := H(\mathbb{W}) \rtimes Mp(\mathbb{W})$. Its restriction $\omega_{\mathbb{W}}$ to $Mp(\mathbb{W})$ is the Weil representation. If $\mathbb{W}$ is a direct sum of two symplectic spaces $(\mathbb{W}_i, \langle \cdot, \cdot \rangle)_i$, $(i = 1, 2)$, we have $\rho_{\psi}^W |_{H(\mathbb{W}_1) \times H(\mathbb{W}_2)} \simeq \rho_{\psi}^{W_1} \otimes \rho_{\psi}^{W_2}$ and hence

$$\omega_{\mathbb{W}}|_{Mp(\mathbb{W}_1) \times Mp(\mathbb{W}_2)} \simeq \omega_{\mathbb{W}_1} \otimes \omega_{\mathbb{W}_2}. \hspace{1cm} (2.1)$$

To be explicit, we take any Witt basis $(e'_i := \{ e'_1, \ldots, e'_N \}) \cup (e := \{ e_1, \ldots, e_N \})$ of $(\mathbb{W}, \langle \cdot, \cdot \rangle)$:

$$\langle e_i, e_j \rangle = \langle e'_i, e'_j \rangle = 0, \quad \langle e'_i, e_j \rangle = \delta_{i,j},$$
Thus the Lie algebra \( j \) with respect to \(( e, \epsilon) \), where \( L \) is the realization \( \text{Hom}_{\mathbb{R}}(\mathbb{R}', \mathbb{Y}) \), we write \( \omega \) is characterized by the following formulae.

\[
\begin{align*}
\rho_{\psi}(y', y; z)\phi(x) &= \psi(z + \frac{\langle 2x + y', y \rangle}{2})\phi(x + y'), \quad y' \in \mathbb{Y}', y \in \mathbb{Y}, z \in \mathbb{C}^1, \\
\omega_{\mathbb{W}}\left( \begin{pmatrix} a \\ \epsilon a^{-1} \end{pmatrix}, \epsilon \right)\phi(x) &= \det a^{1/2}\phi(x.a), \quad a \in GL(N, \mathbb{R}), \\
\omega_{\mathbb{W}}\left( \begin{pmatrix} 1_N \\ b \\ 1_N \end{pmatrix}, \epsilon \right)\phi(x) &= \epsilon\psi \left( \frac{xb'x}{2} \right)\phi(x), \quad b = t^b \in \mathbb{M}_N(\mathbb{R}), \\
\omega_{\mathbb{W}}(w_r, \epsilon)\phi(x_1, x_2) &= \epsilon \int_{\mathbb{R}^r} \phi(y, x_2)\psi(-x_1'y) \, dy.
\end{align*}
\]

Here, we write

\[
w_r := \begin{pmatrix} 0_r & \epsilon \gamma_{1_r} \\ \epsilon^{-1} \gamma_{1_r} & 0_r \end{pmatrix},
\]

and \( x_1, x_2 \in \mathbb{R}^r \). Also \( dy \) denotes the Haar measure on \( \mathbb{R}^r \) selfdual with respect to the duality \( (x, y) \mapsto \psi(x'y) \).

The Lie algebra of \( Sp(\mathbb{W}) \) is denoted by \( \mathfrak{sp}(\mathbb{W}) \). Note that its complexification is

\[\mathfrak{sp}(\mathbb{W}_C) := \{ X \in \text{End}_C(\mathbb{W}_C) \mid \langle w.X, w' \rangle = -\langle w, w'.X \rangle, w, w' \in \mathbb{W}_C \},\]

where \( (\mathbb{W}_C = \mathbb{W} \otimes \mathbb{C}, \langle \cdot, \cdot \rangle) \) is the scalar extension of \( (\mathbb{W}, \langle \cdot, \cdot \rangle) \) to \( \mathbb{C} \). The Lie algebra \( \mathfrak{h}(\mathbb{W}) \) of \( \mathcal{H}(\mathbb{W}) \) is given by \( \mathbb{W} \times \mathbb{R} \) with the bracket

\[
[(w, z), (w', z')] = (0; \langle w, w' \rangle), \quad w, w' \in \mathbb{W}, z, z' \in \mathbb{R}.
\]

Thus the Lie algebra \( j(\mathbb{W}) \) of \( \mathcal{J}(\mathbb{W}) \) is the semidirect product \( \mathfrak{h}(\mathbb{W}) \oplus \mathfrak{sp}(\mathbb{W}) \) defined by

\[
[(w, z), X] = (w.X, 0), \quad (w, z) \in \mathfrak{h}(\mathbb{W}), X \in \mathfrak{sp}(\mathbb{W}).
\]

The restriction \( (\rho_{\psi}, S(\mathbb{Y}')) \) of \( \rho_{\psi} \) to the space of Schwartz-Bruhat functions is a smooth representation of \( \mathcal{J}(\mathbb{W}) \), so that we can consider its derived representation. Using the matrix realization

\[
\mathfrak{sp}(\mathbb{W}) = \left\{ \begin{pmatrix} A & B \\ C & -^tA \end{pmatrix} \mid A \in \mathbb{M}_N(\mathbb{R}), B = t^B, C = t^C \in \mathbb{M}_N(\mathbb{R}) \right\}
\]

with respect to \( e' \cup e \), we can deduce the following formulas from (2.2). The character \( \psi \) is of the form \( \psi(x) = e^{d\psi x} \) for some \( d\psi \in \mathbb{iC} \). We write \( E_{j,k} \) for the \((j, k)\)-elementary matrix and \( S_{j,k} := E_{j,k} + E_{k,j} \in \mathbb{M}_N(\mathbb{C}) \).

\[
\rho_{\psi}(e_j) = d\psi x_j, \quad \rho_{\psi}(e'_j) = \frac{\partial}{\partial x_j},
\]

\[
\omega_{\psi}\left( \begin{pmatrix} 0_N & S_{j,k} \\ 0_N & 0_N \end{pmatrix} \right) = d\psi x_j x_k, \quad \omega_{\psi}\left( \begin{pmatrix} 0_N & 0_N \\ S_{j,k} & 0_N \end{pmatrix} \right) = -\frac{1}{d\psi} \frac{\partial^2}{\partial x_j \partial x_k},
\]

\[
\omega_{\psi}\left( \begin{pmatrix} E_{j,k} \\ -E_{k,j} \end{pmatrix} \right) = x_j \frac{\partial}{\partial x_k} + \frac{\delta_{j,k}}{2}.
\]
2.2 Fock model

We introduce the quantum algebra

\[ \Omega_\psi(\mathbb{W}_C) := T(\mathbb{W}_C)/ (w \otimes w' - w' \otimes w - d\psi \langle w, w' \rangle | w, w' \in \mathbb{W}_C), \]

where \( T(\mathbb{W}_C) \) is the tensor algebra of \( \mathbb{W}_C \). This is an associative algebra equipped with a filtration \( \Omega_\psi^0(\mathbb{W}_C) = \mathbb{C} \subset \Omega_\psi^1(\mathbb{W}_C) \subset \cdots \subset \Omega_\psi^g(\mathbb{W}_C) \subset \cdots \) induced from the grading \( T(\mathbb{W}_C) = \bigoplus_{n \in \mathbb{N}} T^n(\mathbb{W}_C) \). It follows from the definition that

\[ \Phi : h(\mathbb{W}_C) \ni (w; z) \longmapsto w + d\psi z \in \Omega_\psi^1(\mathbb{W}_C) \] (2.5)

extends to an isomorphism \( \Phi : \mathfrak{U}(h(\mathbb{W}_C)) \simeq \Omega_\psi(\mathbb{W}_C) \). Here \( \mathfrak{U}(\mathfrak{g}) \) denotes the universal enveloping algebra of a complex Lie algebra \( \mathfrak{g} \). One can easily verify that (2.5) uniquely extends to an isomorphism \( \Phi : j(\mathbb{W}_C) \simeq \Omega_\psi^2(\mathbb{W}_C) \).

Take a polarization \( \mathbb{W}_C = \mathbb{L}' \oplus \mathbb{L} \) satisfying \( \mathbb{L}' \cap \mathbb{W} = \{0\} \), \( \mathbb{L} = \mathbb{L}' \) (complex conjugate). We have Siegel parabolic subalgebras \( q := \{ X \in \mathfrak{sp}(\mathbb{W}_C) | \mathbb{L} X \subset \mathbb{L} \} \), \( \bar{q} := \{ X \in \mathfrak{sp}(\mathbb{W}_C) | \mathbb{L}' X \subset \mathbb{L}' \} \) opposite to each other. Thus one can write

\[ q = \mathfrak{e}_{\mathbb{W},C} \oplus \mathfrak{p}^+_{\mathbb{W}}, \quad \bar{q} = \mathfrak{e}_{\mathbb{W},C} \oplus \mathfrak{p}^-_{\mathbb{W}}, \]

where \( \mathfrak{e}_{\mathbb{W},C} \) is the complexification of the Lie algebra of a maximal compact subgroup \( K_\mathbb{W} \subset S_P(\mathbb{W}) \). Now the Poincaré-Birkhoff-Witt theorem imply \( \Omega_\psi(\mathbb{W}_C) = \Omega_\psi(\mathbb{L}) \oplus \Omega_\psi(\mathbb{W}_C)\mathbb{L}' \), where \( \Omega_\psi(\mathbb{L}) \) denotes the subalgebra of \( \Omega_\psi(\mathbb{W}_C) \) generated by \( \mathbb{L} \). Since \( \mathbb{L} \subset h(\mathbb{W}_C) \) is abelian, \( \Omega_\psi(\mathbb{L}) \) is just the symmetric algebra \( S(\mathbb{L}) \) over \( \mathbb{L} \). The Fock representation \( (r_\psi, S(\mathbb{L})) \) of \( j(\mathbb{W}_C) \) is defined by

\[ r_\psi(X) P := \Phi(X).P, \quad P \in S(\mathbb{L}) = \Omega_\psi(\mathbb{W}_C)/\Omega_\psi(\mathbb{W}_C)\mathbb{L}'. \] (2.6)

For more understanding, we need explicit description of \( \Phi : j(\mathbb{W}_C) \simeq \Omega_\psi^2(\mathbb{W}_C) \) and \( \Omega_\psi(\mathbb{W}_C) \).

Let us choose the Witt basis \( \{ e'_1, \ldots, e'_N; e_1, \ldots, e_N \} \) in (2.1) in such a way that

\[ \mathbb{L}' := \text{span}\left\{ e'_j := \frac{e'_j - ie_j}{\sqrt{2}} \mid 1 \leq j \leq N \right\}, \quad \mathbb{L} := \text{span}\left\{ e_j := \frac{e_j - ie'_j}{\sqrt{2}} \mid 1 \leq j \leq N \right\}. \]

Then we have

\[ \mathfrak{e}_{\mathbb{W}} := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \quad \begin{array}{c} A = -tA, \\ B = tB \end{array} \in \mathbb{M}_N(\mathbb{R}) \right\}, \]

\[ \mathfrak{p}^\pm_{\mathbb{W}} := \left\{ \begin{pmatrix} X & \pm iX \\ \pm iX & -X \end{pmatrix} \quad X = tX \in \mathbb{M}_N(\mathbb{C}) \right\}. \] (2.7)

Notice that \( e' = \{ e'_1, \ldots, e'_N \} \) and \( e = \{ e_1, \ldots, e_N \} \) form another Witt basis of \( (\mathbb{W}_C, \langle \cdot, \cdot \rangle) \). Identifying \( e_j, e'_j \in \Omega^1_\psi(\mathbb{W}_C) \) with

\[ z_j, \quad d\psi \frac{\partial}{\partial z_j}, \quad (1 \leq j \leq N), \]

where \( z_j \) is the \( j \)-th coordinate of \( \mathbb{L}' \) with respect to the basis \( e' \), we have the realization

\[ \Omega_\psi(\mathbb{W}_C) = \mathbb{C}[z_1, \ldots, z_N, \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_N}], \quad S(\mathbb{L}) = \mathbb{C}[z_1, \ldots, z_N]. \]
It is implicit in the isomorphism $\Phi : j(\mathbb{W}_C) \cong \Omega^2_0(\mathbb{W}_C)$ that

$$X^*_{j,k} := e_j e_k, \quad X^-_{j,k} := e'_j e'_k, \quad (1 \leq j \leq k \leq N)$$

$$Z_{j,k} := \frac{1}{2}(e'_j e_k + e_k e'_j) = e'_j e_k - \frac{d\psi}{2}\delta_{j,k}, \quad (1 \leq j, k \leq N)$$

form a basis of $\mathfrak{sp}(\mathbb{W}_C) \subset \Omega^2_0(\mathbb{W}_C)$. More explicitly, we have

$$\left[ \begin{pmatrix} e' \\ e \end{pmatrix}, X^+_{j,k} \right] = \begin{pmatrix} [e', e_j]e_k + e_j[e', e_k] \\ 0 \end{pmatrix} = \begin{pmatrix} (d\psi(\delta_{e,j}e_k + \delta_{e,k}e_j))_\ell \\ 0 \end{pmatrix} = d\psi \begin{pmatrix} 0_N \\ 0_N \end{pmatrix} \begin{pmatrix} E_{j,k} + E_{k,j} \\ 0_N \end{pmatrix} \begin{pmatrix} e' \\ e \end{pmatrix},$$

$$\left[ \begin{pmatrix} e' \\ e \end{pmatrix}, X^-_{j,k} \right] = \begin{pmatrix} [e, e'_j]e'_k + e'_j[e, e'_k] \\ 0 \end{pmatrix} = \begin{pmatrix} -d\psi(\delta_{e,j}e'_k + \delta_{e,k}e'_j)_\ell \\ 0 \end{pmatrix} = -d\psi \begin{pmatrix} 0_N \\ 0_N \end{pmatrix} \begin{pmatrix} 0_N \\ E_{j,k} + E_{k,j} \end{pmatrix} \begin{pmatrix} e' \\ e \end{pmatrix},$$

$$\left[ \begin{pmatrix} e' \\ e \end{pmatrix}, Z_{j,k} \right] = \begin{pmatrix} e'_j \otimes [e', e_k] \\ [e, e'_j] \otimes e_k \end{pmatrix} = \begin{pmatrix} (d\psi(\delta_{e,j}e'_k + \delta_{e,k}e'_j))_\ell \\ -(d\psi(\delta_{e,j}e'_k + \delta_{e,k}e'_j))_\ell \end{pmatrix} = d\psi \begin{pmatrix} E_{k,j} \\ 0_N \end{pmatrix} \begin{pmatrix} 0_N \\ -E_{j,k} \end{pmatrix} \begin{pmatrix} e' \\ e \end{pmatrix}.$$

Comparing these with (2.3), we deduce

$$\Phi \left( \begin{pmatrix} 0 \\ S_{j,k} \\ 0 \end{pmatrix} \right) = \frac{1}{d\psi}X^+_{j,k}, \quad \Phi \left( \begin{pmatrix} 0 \\ S_{j,k} \\ 0 \end{pmatrix} \right) = -\frac{1}{d\psi}X^-_{j,k}, \quad \Phi \left( \begin{pmatrix} E_{j,k} \\ 0 \\ 0 \end{pmatrix} \right) = \frac{1}{d\psi}Z_{j,k}. \tag{2.8}$$

Here, matrices in brackets stands for the matrix representation with respect to the basis $e' \cup e$.

These show

$$r_\psi \left( \begin{pmatrix} 0 \\ S_{j,k} \\ 0 \end{pmatrix} \right) = \frac{1}{d\psi}z_j z_k, \quad r_\psi \left( \begin{pmatrix} 0 \\ S_{j,k} \\ 0 \end{pmatrix} \right) = -d\psi z_j \frac{\partial^2}{\partial z_j \partial z_k}, \quad r_\psi \left( \begin{pmatrix} E_{j,k} \\ 0 \\ 0 \end{pmatrix} \right) = z_j \frac{\partial}{\partial z_k} + \frac{1}{2} \delta_{j,k}. \tag{2.9}$$

We introduce the extension $\widetilde{K}_\mathbb{W} = K_\mathbb{W} \times \mathbb{C}^1$ of $K_\mathbb{W}$ by $\mathbb{C}^1$ defined by

$$(k_1, z_1)(k_2, z_2) = (k_1 k_2, z_1 z_2 \det(k_1 k_2)^{1/2}), \quad k_i \in K_\mathbb{W}, z_i \in \mathbb{C}^1.$$ 

Here, for $\det(k_1 k_2) = e^{i\theta}$ with $0 \leq \theta < 2\pi$, we write $\det(k_1 k_2)^{1/2} = e^{i\theta/2}$.

**Lemma 2.1.** (1) The space $\mathbb{C} \subset S(\mathbb{L})$ of constants is annihilated by $\mathbb{L}' \subset \mathfrak{h}(\mathbb{W}_C)$, and $\bigoplus_{j=0}^n S^j(\mathbb{L}) = r_\psi(\mathfrak{H}^n(\mathbb{L}))\mathbb{C}$. Here $\mathfrak{H}^n(\mathfrak{g})$ denotes the image of $T^n(\mathfrak{g})$ in $\mathfrak{H}(\mathfrak{g})$.

(2) $r_\psi(\mathbb{L}(\mathbb{L}))$ is the unique (up to isomorphisms) irreducible $\mathfrak{h}(\mathbb{W}_C)$ or $(\mathbb{W}_C)$-module on which the center $\mathbb{C}$ of $\mathfrak{h}(\mathbb{W}_C)$ acts by $d\psi$.

(3) Each $\bigoplus_{j=0}^n S^j(\mathbb{L})$ is a finite dimensional $\mathfrak{h}_\mathbb{W}$-module which exponentiates to a genuine representation of $\widetilde{K}_\mathbb{W}$.
Sketch of the proof. (1) is obvious. The irreducibility in (2) is also clear from the action of \( \mathbb{L} \), \( \mathbb{L}' \) on \( S(\mathbb{L}) \). The uniqueness is no other than the Stone-von Neumann theorem. Since \( \mathfrak{f}_W \subset \mathfrak{g} \), \( \bigoplus_{j=0}^n S^j(\mathbb{L}) \) is stable under \( \mathfrak{f}_W \). Then the final formula in (2.9) shows that this exponentiates to a \( K_W \)-module.

Let us compare \((r_\psi, S(\mathbb{L}))\) with the Weil representation \((\rho_\psi, S(\mathbb{Y}'))\). Set
\[
\mathcal{S}_\psi(\mathbb{Y}') := \{ e^{d_\psi i x/2} P(x) \mid P(x) \in \mathbb{C}[\mathbb{Y}'] \}.
\]
This is a dense subspace in \( L^2(\mathbb{Y}') \).

Lemma 2.2. (1) \( e^{d_\psi i x/2} \) is annihilated by \( \mathbb{L}' \) and
\[
\rho_\psi(\mathfrak{U}^n(\mathbb{L})) e^{d_\psi i x/2} = \{ e^{d_\psi i x/2} P(x) \in \mathcal{S}_\psi(\mathbb{Y}') \mid \text{deg} P \leq n \}.
\]
(2) \( (\rho_\psi, \mathcal{S}_\psi(\mathbb{Y}')) \simeq (r_\psi, S(\mathbb{L})) \) as \( \mathfrak{g}(\mathbb{W}_C) \)-modules.
(3) In particular, it turns out that \( \mathbf{K}_W \) is isomorphic to the inverse image of \( K_W \) in \( M p(\mathbb{W}) \).

Sketch of the proof: (1) immediately follows from the first row of (2.4). It follows that \((\rho_\psi, \mathcal{S}_\psi(\mathbb{Y}'))\) is an irreducible \( \mathfrak{h}(\mathbb{W}_C) \)-module on which the center acts by \( d_\psi \). Then the Stone-von Neumann theorem asserts that it is isomorphic to \( r_\psi \) as a \( \mathfrak{g}(\mathbb{W}_C) \)-module. Now for each \( n \in \mathbb{N} \), \( \{ e^{d_\psi i x/2} P(x) \in \mathcal{S}_\psi(\mathbb{Y}') \mid \text{deg} P \leq n \} \) is a finite dimensional \( \mathfrak{f}_W \)-module which exponentiates to a representation of the inverse image of \( K_W \) in \( M p(\mathbb{W}) \). Since this is isomorphic to the derived representation of the \( \mathbf{K}_W \)-module \((r_\psi, \bigoplus_{j=0}^n S^j(\mathbb{L}))\) (and both \( \mathbf{K}_W \) and the inverse image are connected), we conclude that \( \mathbf{K}_W \) coincides with the inverse image. \( \square \)

3 Weil representation of unitary dual pairs

Let us recall the construction of Weil representations of unitary dual pairs from [HKS96].

3.1 Unitary dual pairs

Let \((V, (\cdot, \cdot))\) be an \( n \)-dimensional hermitian space over \( \mathbb{C} \), that is, an \( n \)-dimensional vector space \( V \) equipped with a map \((\cdot, \cdot) : V \times V \to \mathbb{C} \) satisfying
- \( V \ni v' \mapsto (v, v') \in \mathbb{C} \) is \( \mathbb{C} \)-linear for any \( v \in V \);
- \( (v, v') = (\overline{v}, \overline{v'}) \).

We always assume \((\cdot, \cdot)\) is non-degenerate. We can take a \( \mathbb{C} \)-basis \( \nu = \{v_1, \ldots, v_n\} \) of \( V \) so that \( V \) is identified with the space of \( n \)-dimensional column vectors \( z = \{z_1, \ldots, z_n\} \) and
\[
(\overline{z}, \overline{z'}) = z^* I_{p, q} z', \quad I_{p, q} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}, \quad p + q = n.
\]

Thus the unitary group \( G_V := \{ g \in GL_C(V) \mid (g, v, g, v') = (v, v'), \forall v, v' \in V \} \) of \((V, (\cdot, \cdot))\) is isomorphic to \( U(p, q) = \{ g \in GL(n, \mathbb{C}) \mid g^* I_{p, q} g = I_{p, q} \} \). Similarly, \((W, (\cdot, \cdot))\) denotes a non-degenerate \( n' \)-dimensional skew-hermitian space over \( \mathbb{C} \). Thus \((\cdot, \cdot) : W \times W \to \mathbb{C} \) satisfies
• $W \ni w \mapsto \langle w, w' \rangle \in \mathbb{C}$ is $\mathbb{C}$-linear for any $w' \in W$;
• $(w, w') = -\overline{\langle w', w \rangle}$.

Again we fix a $\mathbb{C}$-basis $w = \{w_1, \ldots, w_{n'}\}$ of $W$ in such a way that $W$ is identified with the space of row vectors $z = (z_1, \ldots, z_{n'})$ and

$$\langle z, z' \rangle = i z \left(\begin{smallmatrix} 1_{p'} & 0 \\ 0 & -1_{q'} \end{smallmatrix}\right) z'^*, \quad p' + q' = n'.$$

The unitary group $G_W := \{g \in GL_{\mathbb{C}}(W) \mid \langle w, g, w', g \rangle = \langle w, w' \rangle, \forall w, w' \in W\}$ of $(W, \langle \cdot, \cdot \rangle)$ is realized as $U(p', q') = \{g \in GL(n', \mathbb{C}) \mid gI_{p'} g^* = I_{p'} \}$ with respect to $w$.

The $\mathbb{R}$-vector space $\mathbb{W} := V \otimes_{\mathbb{C}} W$ with

$$\langle \langle v \otimes w, v' \otimes w' \rangle \rangle := \Re \left( \langle v, v' \rangle \langle w, w' \rangle \right)$$

is a symplectic space of dimension $N := nn'$ over $\mathbb{R}$, where $\Re z$ denotes the real part of $z \in \mathbb{C}$. Then we have a homomorphism

$$\iota_{V,W} = \iota_{W} \times \iota_V : G_V \times G_W \ni (g, g') \mapsto g \otimes g' \in Sp(\mathbb{W}),$$

and $\iota_W(G_V), \iota_V(G_W)$ form a dual reductive pair in $Sp(\mathbb{W})$. Using the $\mathbb{C}$-basis $\mathbf{v} \otimes \mathbf{w} = \{v_i \otimes w_j\}_{1 \leq i \leq n, 1 \leq j \leq n'}$, we often identify $\mathbb{W}$ with the space of complex $n \times n'$-matrices $\mathbb{M}_{n,n'}(\mathbb{C})$. Notice that

$$z.\iota_{V,W}(g, g') = g^{-1} z.g', \quad g \in G_V, g' \in G_W, z \in \mathbb{M}_{n,n'}(\mathbb{C}).$$

To obtain a Weil representation of the unitary dual pair $G_V \times G_W$, we need a splitting of metaplectic extension restricted to $\iota_{V,W}(G_V \times G_W)$. We briefly review the construction of such splittings from [Kud94], [HKS96].

### 3.2 Splitting on doubled dual pairs

The splitting constructed in [Kud94], §§1-3 is applicable only to unitary groups for hyperbolic hermitian space. Reduction to that case is achieved by the doubling trick [HKS96]. We write $(V^-, (-, -)^-) := (V, -(\cdot, \cdot))$ and set $(V^H, (-, -)^H) := (V, (-, -)) \oplus (V^-, (-, -)^-)$. The latter is a hyperbolic hermitian space. In fact,

$$\Delta V := \{(v, v) \in V^H \mid v \in V\}, \quad \nabla V := \{(v, -v) \in V^H \mid v \in V\}$$

are maximal isotropic subspaces of $V^H$ and $V^H = \Delta V \oplus \nabla V$. On the other hand, since $G_{V^-} = G_V$, we have the standard embedding $i_V : G_V \times G_V \hookrightarrow G_{V^H}$. Similar construction and notation apply to $(W, \langle \cdot, \cdot \rangle)$. Then we have

$$\mathbb{W}^H := V^H \otimes_{\mathbb{C}} W = V \otimes_{\mathbb{C}} W^H,$$

which yields the following commutative diagram

$$\begin{array}{ccc}
G_{V^H} & \xrightarrow{\pi_H W} & Sp(\mathbb{W}^H) & \xleftarrow{\iota_{V,H}} & G_{W^H} \\
\iota_V \downarrow & & \iota_W \downarrow & & \iota_V \downarrow \\
G_V \times G_V & \xrightarrow{\iota_W \times \iota_V} & Sp(\mathbb{W}) \times Sp(\mathbb{W}) & \xleftarrow{\iota_V \times \iota_V} & G_W \times G_W
\end{array}$$

(3.1)
It follows from the explicit description of $Mp(\mathbb{W})$ \cite{2.1} that $i_\mathbb{W}: Sp(\mathbb{W}) \times Sp(\mathbb{W}) \to Sp(\mathbb{W}^\mathbb{H})$ lifts to

$$
i_\mathbb{W}: Mp(\mathbb{W}) \times Mp(\mathbb{W}) \ni ((g_1, \epsilon_1), (g_2, \epsilon_2)) \mapsto (i_\mathbb{W}(g_1, g_2), \epsilon_1 \bar{\epsilon}_2) \in Mp(\mathbb{W}^\mathbb{H}).$$

As for the $W$-side, we take a Witt basis $\{w'_j, w'_k\} = \{w'_1, \ldots, w'_{n'}; w_1, \ldots, w_n\}$ of $W^\mathbb{H}$:

$$\langle w'_j, w'_k \rangle = \langle w'_j, w'_k \rangle = 0, \quad \langle w'_j, w'_k \rangle = \delta_{j,k},$$

such that $V = \operatorname{span}_C w', \Delta W = \operatorname{span}_C w$. We have the Siegel parabolic subgroup $P_{\Delta W} := \operatorname{Stab}(\Delta W; G_{\mathbb{W}}) = M_{\Delta W} U_{\Delta W}$

$$M_{\Delta W} = \left\{ m_{\Delta W}(a) = \begin{pmatrix} a & 0_{n'} \\ 0_{n'} & a^{* -1} \end{pmatrix} \mid a \in GL(n', \mathbb{C}) \right\},$$

$$U_{\Delta W} = \left\{ u_{\Delta W}(b) = \begin{pmatrix} 1_{n'} & b \\ 0_{n'} & 1_{n'} \end{pmatrix} \mid b = b^* \in \mathbb{M}_{n'}(\mathbb{C}) \right\},$$

and the Bruhat decomposition $G_{\mathbb{W}} = \bigsqcup_{j=1}^{n'} P_{\Delta W} \cdot w_j \cdot P_{\Delta W}$, where $w_j$ is defined in the same way as in the $Sp(\mathbb{W})$-case \cite{2.1}. Writing $g \in G_{\mathbb{W}}$ as

$$g = \begin{pmatrix} a_1 & b_1 \\ a_1^{* -1} \end{pmatrix} w_r \begin{pmatrix} a_2 & b_2 \\ a_2^{* -1} \end{pmatrix},$$

we set $d(g) := \det(a_1 a_2) \in \mathbb{C}^\times / \mathbb{R}_+, r(g) := r = n' - \dim_C \Delta W g \cap \Delta W$. Fix a character $\xi'$ of $\mathbb{C}^\times$ such that $\xi'|_{\mathbb{R}_+} = \text{sgn}^n$. Then

$$\beta_{V,\xi'}(g) := \gamma(g, 1)^{2n'} (-1)^q \cdot r(g) \xi'(d(g)), \quad g \in G_{\mathbb{W}}.$$

gives a splitting of the restriction $c_V(g_1, g_2) := c_{V \otimes_{\Delta W} \mathbb{W}}(i_{\mathbb{W}}^H(g_1), i_{\mathbb{W}}^H(g_2))$ of the metaplectic 2-cocycle to $i_{\mathbb{W}}^H(G_{\mathbb{W}})$ \cite[Th.3.1]{Kud94}:

$$c_V(g_1, g_2) = \frac{\beta_{V,\xi'}(g_1, g_2)}{\beta_{V,\xi'}^H(g_1, g_2)}; \quad g_1, g_2 \in G_{\mathbb{W}}.$$

Thus $i_{\mathbb{W}}^H: G_{\mathbb{W}} \ni g \mapsto (g, \beta_{V,\xi'}^H(g)) \in Mp(\mathbb{W}^\mathbb{H})$ is an analytic homomorphism.

Next comes the $V$-side. Again take a Witt basis $\{v'_j, v'_k\} = \{v'_1, \ldots, v'_{n'}; v_1, \ldots, v_n\}$ of $V^\mathbb{H}$:

$$\langle v'_j, v'_k \rangle = \langle v'_j, v'_k \rangle = 0, \quad \langle v'_j, v'_k \rangle = i \delta_{j,k},$$

such that $\Delta V = \operatorname{span}_C v', \nabla V = \operatorname{span}_C v$. In particular, with respect to this basis, one has the matrix realization

$$G_{\mathbb{V}^\mathbb{H}} = \left\{ g \in GL(2n, \mathbb{C}) \mid g^* \begin{pmatrix} 0_n & i1_n \\ -i1_n & 0_n \end{pmatrix} g = \begin{pmatrix} 0_n & i1_n \\ -i1_n & 0_n \end{pmatrix} \right\}.$$

Let us introduce $n'$-dimensional hermitian space $(\mathbb{H}, \langle \cdot, \cdot \rangle) := w I'_{\mathbb{C}^{2n', \mathbb{H}}}$ and $2n$-dimensional skew-hermitian space $(V^{\mathbb{H}, *}, (v^*, \langle \cdot, \cdot \rangle)^{\mathbb{V}, *}) := v^*(\mathbb{0}_{n'}^{a}, \mathbb{1}_{n'}) v')$, and the corresponding $4N$-dimensional symplectic space

$$i_{\mathbb{W}}^{\mathbb{V}^{\mathbb{H}}} := W^* \otimes_C \mathbb{V}^{\mathbb{H}, *}, \quad \langle \cdot, \cdot \rangle^{\mathbb{V}, *} := \mathbb{R} \langle \langle \cdot, \cdot \rangle, \otimes (\cdot, \cdot)^{\mathbb{H}, *}).$$
We use the basis $w$ to identify $W$ with $\mathbb{C}^n'$. Then we have
\[
\langle v \otimes w, v' \otimes w' \rangle = \Re \left( v^* \left( \begin{array}{cc} 0_n & i 1_n \\ -i 1_n & 0_n \end{array} \right) v' \cdot \left( w I_{p', q'} w'^* \right) \right) \\
= \Re \left( v^* \left( \begin{array}{cc} 0_n & 1_n \\ -1_n & 0_n \end{array} \right) v' w' I_{p', q'} w'^* \right) \\
= \Re \left( w I_{p', q'} w'^* \left( v^* \left( \begin{array}{cc} 0_n & 1_n \\ -1_n & 0_n \end{array} \right) v' \right) \right) \\
= \langle \langle w^* \otimes v^*, w'^* \otimes v'^* \rangle \rangle,
\]
which gives the following commutative diagram:
\[
\begin{array}{ccc}
G_{V^H} \times G_W & \xrightarrow{(g, g') \mapsto (g^*, -1, g^*)} & G_{W^*} \times G_{V^H}^* \\
\iota_{V^H, W}^H & \downarrow & \iota_{W^*, V^H}^H \\
Sp(\mathbb{W}^H) & \xrightarrow{g^t g^{-1}} & Sp(\mathbb{W}^H)
\end{array}
\]
This avails us to transform the construction for $G_{W^H}$ to that for $G_{V^H}$.

We write $c_W(g_1, g_2) := c_{\Delta V \otimes W}(\iota_W(g_1), \iota_W(g_2))$ for the restriction of the metaplectic 2-cocycle $c_{\Delta V \otimes W}$ to $\iota_W(G_{V^H})$. Then, putting $X := \Delta V \otimes W$, (3.2) shows
\[
c_W(g_1, g_2) = \gamma_W(L(X, \iota_W(g_1) X, \iota_W(g_1)^{-1} X)) \\
= \gamma_W(L(t_X, t_X, t_\iota_W(g_2), t_X, t_\iota_W(g_1)^{-1})) \\
= c_X(t_\iota_W(g_1)^{-1}, t_\iota_W(g_2)^{-1}) = c_X(t_\iota_W(g_1^*), t_\iota_W(g_2^*)) \\
= c_W(g_1^*, g_2^*)
\]
Note our convention on the action of $G_{V^H} \times G_W$ at the end of §3.1. Again we take the Siegel parabolic subgroup $P_{\Delta V} := \text{Stab}(\Delta V, G_{V^H}) = M_{\Delta V} U_{\Delta V}$:
\[
M_{\Delta V} = \left\{ m_{\Delta V}(a) := \left( \begin{array}{cc} a & 0_n \\ 0_n & a^{-*} \end{array} \right) \in GL(n, \mathbb{C}) \right\}, \\
U_{\Delta V} = \left\{ u_{\Delta V}(b) := \left( \begin{array}{cc} 1_n \\ b \end{array} \right) \in \mathbb{M}_n(\mathbb{C}) \right\}.
\]
Using the Bruhat decomposition
\[
g = \left( \begin{array}{cc} a_1 & 0_n \\ b_1 & a_1^{-*} \end{array} \right) w_r \left( \begin{array}{cc} a_2 & 0_n \\ b_2 & a_2^{-*} \end{array} \right) \in G_{V^H},
\]
we set $d(g) := \det(a_1 a_2) \in \mathbb{C}^X / \mathbb{R}^X_+,$ $r(g) := r.$ Fixing a character $\xi$ of $\mathbb{C}^X$ satisfying $\xi|_{\mathbb{R}^X} = \text{sgn}^{\cdot^*}$. [Kud94] Th.3.1] combined with the above calculation shows that
\[
\beta_{W^H, \xi}(g) := \beta_{W^*, \xi}(g^*^{-1}) = \left( \gamma_W(1)^{2n'}(-1)^{q'} \right)^{-r(g)} \xi(d(g)), \quad g \in G_{V^H}
\]
splits $c_W(g_1, g_2)$. Hence again we have the analytic homomorphism $\iota_{W^H, \xi}^H : G_{V^H} \ni g \mapsto (g, \beta_{W^H, \xi}(g)) \in Mp(\mathbb{W}^H).$
3.3 Splitting on unitary dual pairs and the Weil representation

We now fix a pair $\xi = (\xi, \xi')$ of characters of $\mathbb{C}^\times$ such that $\xi|_{\mathbb{R}^\times} = \text{sgn}^{n'}$, $\xi'|_{\mathbb{R}^\times} = \text{sgn}^n$. Using the above constructed splittings, define

\[
\bar{i}_{W,\xi} : G_V \xrightarrow{1\text{st}} G_V \times G_V \xrightarrow{i_V} G_{V\oplus H} \xrightarrow{i_{W,\xi}} \text{Mp}(\mathbb{W}^H),
\]

\[
\bar{i}_{V,\xi'} : G_W \xrightarrow{1\text{st}} G_W \times G_W \xrightarrow{i_V} G_{W\oplus W} \xrightarrow{i_{V,\xi'}} \text{Mp}(\mathbb{W}'^H),
\]

where the left arrows are the embeddings to the first components. Since the images of these homomorphisms are contained in $\bar{i}_W(\text{Mp}(\mathbb{W}) \times \{1\})$, these yield homomorphisms

\[
\bar{i}_W : G_V \rightarrow \text{Mp}(\mathbb{W}), \quad \bar{i}_{V,\xi'} : G_W \rightarrow \text{Mp}(\mathbb{W}).
\]

For later use, we recall [HKS96, Lem.1.1]

\[
\bar{i}_{W,-\xi} = \xi(\det)^{-1}\bar{i}_{W,\xi}, \quad \bar{i}_{V,-\xi'} = \xi'(\det)^{-1}\bar{i}_{V,\xi'}.
\] (3.4)

Composing the above homomorphisms with the Weil representation $\omega_W$ of $\text{Mp}(\mathbb{W})$, we obtain the Weil representation $\omega_{V,\xi} = \omega_{V,\xi} \times \omega_{V,\xi'}$ of $G_V \times G_W$:

\[
\omega_{V,\xi} := \omega_W \circ \bar{i}_{W,\xi}, \quad \omega_{V,\xi'} := \omega_W \circ \bar{i}_{V,\xi'}.
\]

Also we have the Weil representations $\omega_{W,\xi} := \omega_W \circ \bar{i}_{W,\xi}$, $\omega_{W,\xi'} := \omega_W \circ \bar{i}_{V,\xi'}$ of $G_{W,\oplus H}$, $G_{W,\oplus W}$, respectively. It follows from (3.4) and $\omega_{W,-} \simeq \omega_W$(contragredient) that [HKS96, Prop.2.2]

\[
\omega_{W,\xi} \circ i_V \simeq \omega_{V,\xi} \otimes \xi(\det)\omega_{V,\xi'}, \quad \omega_{W,\xi'} \circ i_V \simeq \omega_{V,\xi'} \otimes \xi'(\det)\omega_{V,\xi'}.
\] (3.5)

For completeness, we adopt the following convention in this paper. If $V = 0$ or $W = 0$, we regard $G_V \times G_W$ as the trivial group and $\omega_{V,\xi}$ as its trivial representation.

We remark that explicit formulas for $\omega_{V,\xi}$ are not available. All we have at the moment is the following formulas for the doubled representations.

**Lemma 3.1.** *The doubled Weil representations $\omega_{W,\xi}^H$, $\omega_{V,\xi'}^H$ are characterized by the following formulas.*

(a) As for $\omega_{W,\xi}^H$, we identify $\forall V \otimes \mathbb{C} W$ with $M_{n,n'}(\mathbb{C})$ using the basis $\{v'_k \otimes w_j\}$. Then we have

\[
\omega_{W,\xi}^H(m_{\Delta V}(a))\phi(z) = \xi(\det a)\det a^{-n'}\phi(a^{-1}z), \quad a \in GL(n, \mathbb{C})
\] (3.6)

\[
\omega_{W,\xi}^H(u_{\Delta V}(b))\phi(z) = \psi\left(-\frac{\text{tr}b(z, z)}{2l}\right)\phi(z), \quad b = b^* \in M_{n,n'}(\mathbb{C})
\]

\[
\omega_{W,\xi}^H(w_j)\phi(z) = \left(\frac{-1}{\gamma_j(1)^{2n'}}\right)^j \mathcal{F}_{W,j}\phi\left(-\frac{z_1}{z_2}\right),
\] (3.8)

where in (3.8), we have written $z = (z_1, z_2)$ with $z_1 \in M_{j,n'}(\mathbb{C})$, $z_2 \in M_{n-j,n'}(\mathbb{C})$. Also

\[
\mathcal{F}_{W,j}\phi\left(-\frac{z_1}{z_2}\right) := \int_{W,j} \phi\left(z'/z_2\right) \psi(\Im(\text{tr}(z_1, z'))) \, dz'.
\]
denotes the partial Fourier transform. \( \Im(z) \) stands for the imaginary part of \( z \in \mathbb{C} \). The invariant measure \( dz' \) is the selfdual one with respect to this transform.

(b) As for \( \omega_{V,\xi}^\mathbb{H} \), we identify \( V \otimes \mathbb{C} \gamma W \) with \( M_{n,n'}(\mathbb{C}) \) via the basis \( \{ v_k \otimes w'_j \} \). We have

\[
\omega_{V,\xi}^\mathbb{H}(m_{\Delta W}(a))\phi(z) = \xi'(\det a)[\text{det} a^n] \phi(z,a), \quad a \in GL(n', \mathbb{C})
\]

\[
\omega_{V,\xi}^\mathbb{H}(u_{\Delta W}(b))\phi(z) = \psi\left( \frac{\text{tr}(z, b)}{2} \right) \phi(z), \quad b = b^* \in M_{n}(\mathbb{C})
\]

\[
\omega_{V,\xi}^\mathbb{H}(w_j)\phi(z) = \left( \frac{(-1)^q}{2^{n'} \gamma_q(1)^{2n}} \right)^j F_{V,j} \phi(-z_1, z_2).
\]

Here again, we have written \( z = (z_1, z_2) \) with \( z_1 \in M_{n,j}(\mathbb{C}), z_2 \in M_{n,n'-j}(\mathbb{C}) \).

\[
F_{V,j} \phi(z_1, z_2) := \int_{V_j} \phi(z', z_2) \psi(\Re(\text{tr}(z_1, z'))) \, dz'
\]

is the partial Fourier transform. Again the measure \( dz' \) is chosen to be selfdual with respect to this duality.

**Proof.** (b) is no other than the restatement of the result in [Kud94, §5]. (a) follows from (b) by the “taking adjoint” procedure explained in §3.2.

\[ \square \]

## 4 Doubling construction of the Fock model

In this section, we apply the consideration of §2.2 to the doubled representations \( \omega_{W,\xi}^\mathbb{H}, \omega_{V,\xi}^\mathbb{H} \).

### 4.1 Derived representation of the Schrödinger model

First we calculate some explicit formulas for \( \omega_{W,\xi}^\mathbb{H}(g_{V,n}) \) and \( \omega_{V,\xi}^\mathbb{H}(g_{W,n}) \). We realize these complexified Lie algebras as \( g_{V,n} = \mathfrak{gl}(2n, \mathbb{C}), g_{W,n} = \mathfrak{gl}(2n', \mathbb{C}) \) with respect to the basis \( \{ v'; v \}, \{ w'; w \} \), respectively. The character pair \( \xi = (\xi, \xi') \) can be written as \( \xi(z) = (z/\bar{z})^{m'/2}, \xi'(z) = (z/\bar{z})^{m/2} \) with \( m \equiv n', m' \equiv n \) (mod 2). We write \( z_j, z_j^* \) for the \( j \)-th. row and column vectors of \( z \in M_{n,n'}(\mathbb{C}) = \mathfrak{v}\mathfrak{W} \), respectively. We write \( \phi \in S(\mathfrak{v}\mathfrak{W}) \) as \( \phi(z, \bar{z}) \), since we regard \( z, \bar{z} \) as real coordinates of \( z \in \mathfrak{v}\mathfrak{W} \).

**Computation of \( \omega_{W,\xi}^\mathbb{H}(g_{V,n}) \)** For \( X \in M_n(\mathbb{C}) \), we set

\[
m_{\Delta V}(X) := \begin{pmatrix} X & 0 \\ 0 & -X^* \end{pmatrix}, \quad n_{\Delta V}(X) := \begin{pmatrix} 0_n & 0_n \\ X & 0_n \end{pmatrix}.
\]

Thus, the Lie algebras of \( M_{\Delta V}, U_{\Delta V} \) are given by \( m_{\Delta V} = \{ m_{\Delta V}(X) \mid X \in M_n(\mathbb{C}) \} \), \( u_{\Delta V} = \{ n_{\Delta V}(X) \mid X = X^* \in M_n(\mathbb{C}) \} \) in our realization. Then some easy calculation show, for
We also use (3.8) to deduce the following from (4.3)

\[
\omega^H_{V,\xi}(E_{n+j,n+k}) = \frac{1}{2} \omega^H_{V,\xi}(m_{\Delta V}(E_{n+j,k})) - \frac{i}{2} \omega^H_{V,\xi}(m_{\Delta V}(iE_{n+j,k}))
\]

(4.1)

\[
\omega^H_{W,\xi}(E_{n+j,n+k}) = \frac{1}{2} \omega^H_{W,\xi}(m_{\Delta V}(E_{n+j,k})) - \frac{i}{2} \omega^H_{W,\xi}(m_{\Delta V}(iE_{n+j,k}))
\]

(4.2)

\[
\omega^H_{W,\xi}(E_{n+j,k}) = \frac{1}{2} \omega^H_{W,\xi}(m_{\Delta V}(E_{n+j,k})) - \frac{i}{2} \omega^H_{W,\xi}(m_{\Delta V}(iE_{n+j,k}))
\]

(4.3)

We also use (3.8) to deduce the following from (4.3)

\[
\omega^H_{W,\xi}(E_{n+j,n+k}) = - \omega^H_{W,\xi}(w_n) \omega^H_{W,\xi}(E_{n+j,k}) \omega^H_{W,\xi}(w_n)^{-1}
\]

(4.4)

**Computation of \(\omega^H_{V,\xi}(g_{WH,C})\)** Let us carry out the similar calculation for the \(W\)-side. For \(X \in \mathbb{M}_{n'}(\mathbb{C})\), we write

\[
m_{\Delta W}(X) := \begin{pmatrix} X' & -X^* \end{pmatrix}, \quad n_{\Delta W}(X) := \begin{pmatrix} 0_{n'} & X \end{pmatrix}.
\]

Then we obtain, for \(1 \leq j, k \leq n',\)

\[
\omega^H_{V,\xi}(E_{j,k}) = \frac{1}{2} \omega^H_{V,\xi}(m_{\Delta W}(E_{j,k})) - \frac{i}{2} \omega^H_{V,\xi}(m_{\Delta W}(iE_{j,k}))
\]

(4.5)

\[
\omega^H_{V,\xi}(E_{n+k,n+j}) = - \frac{1}{2} \omega^H_{V,\xi}(m_{\Delta W}(E_{j,k})) - \frac{i}{2} \omega^H_{V,\xi}(m_{\Delta W}(iE_{j,k}))
\]

(4.6)
\[ \omega^H_{V, U}(E_{j,n'+k}) = \frac{1}{2} \omega^H_{V, U}(n_\Delta W(E_{j,k} + E_{k,j})) - \frac{i}{2} \omega^H_{V, U}(n_\Delta W(iE_{j,k} - iE_{k,j})) \]

\[ = \pi i \left( \sum_{\ell=1}^{p} z_{\ell,j} \bar{z}_{\ell,k} - \sum_{\ell=p+1}^{n} z_{\ell,j} \bar{z}_{\ell,k} \right) , \]  

Again (3.11) applied to (4.7) yields

\[ \omega^H_{V, U}(E_{j,n'+k}) = - \omega^H_{V, U}(w_n') \omega^H_{V, U}(E_{j,n'+k}) \omega^H_{V, U}(w_n')^{-1} \]

\[ = - \frac{2}{d\psi} \left( \sum_{\ell=1}^{p} \frac{2}{\partial z_{\ell,j} \partial z_{\ell,k}} - \sum_{\ell=p+1}^{n} \frac{\partial^2}{\partial z_{\ell,j} \partial z_{\ell,k}} \right) . \]  

### 4.2 Compatible Cartan decompositions

Recall the C-basis \( v, w \) of \( V, W \). The Lie algebra \( g_V \) of \( G_V \) is realized as

\[ g_V = \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \mid A = -A^* \in \mathbb{M}_p(\mathbb{C}), D = -D^* \in \mathbb{M}_q(\mathbb{C}) , B \in \mathbb{M}_{p,q}(\mathbb{C}) \right\} \]

with respect to \( v \). As usual, we choose the Cartan decomposition \( g_V = \mathfrak{t}_V \oplus \mathfrak{p}_V \) to be

\[ \mathfrak{t}_V := \left\{ \begin{pmatrix} A \\ D \end{pmatrix} \in g_V \right\} , \quad \mathfrak{p}_V := \left\{ \begin{pmatrix} 0_p & B \\ B^* & 0_q \end{pmatrix} \in g_V \right\} . \]

We adopt the analogous realization of \( g_W \) with respect to \( w \) and the Cartan decomposition \( g_W = \mathfrak{t}_W \oplus \mathfrak{p}_W \).

We often regard \( v, w \) as matrices

\[ v = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix} , \quad w = \begin{pmatrix} w_1 \\ \vdots \\ w_n' \end{pmatrix} . \]

Then it follows from

\[ \langle v \otimes w, i_v \otimes w \rangle = \Re \left( i_v(v, v)(w, w) \right) = I_{p,q} \otimes I_{p',q'} \]

that \( \{ v \otimes w; i_v I_{p,q} \otimes I_{p',q'} w \} \) is a Witt basis of \( (\mathcal{W}, \langle \cdot, \cdot \rangle) \). If we realize \( \mathfrak{sp}(\mathcal{W}) \) in \( \mathbb{M}_{2N}(\mathbb{R}) \) with respect to this basis, then \( \iota_W : g_V \hookrightarrow \mathfrak{sp}(\mathcal{W}), \iota_V : g_W \hookrightarrow \mathfrak{sp}(\mathcal{W}) \) are given by

\[ \iota_W(Z) = \begin{pmatrix} X \otimes 1_n & -Y I_{p,q} \otimes I_{p',q'} \\ I_{p,q} Y \otimes I_{p',q'} & \theta_{p,q}(X) \otimes 1_n' \end{pmatrix} , \]

\[ \iota_V(Z') = \begin{pmatrix} 1_n \otimes X' & I_{p,q} \otimes Y' I_{p',q'} \\ -I_{p,q} \otimes I_{p',q'} Y' & 1_n \otimes \theta_{p',q'}(X') \end{pmatrix} \]  

for \( Z = X + iY \in g_V, Z' = X' + iY' \in g_W \). Here \( \theta_{p,q} = \text{Ad}(I_{p,q}) \) denotes the Cartan involution associated to the above fixed Cartan decomposition of \( g_V \). This together with (2.7) gives

\[ \iota_W(\mathfrak{t}_V) = \mathfrak{t}_W \cap \iota_W(g_V), \quad \iota_W(\mathfrak{p}_V) = \mathfrak{p}_W \cap \iota_W(g_V), \]

\[ \iota_V(\mathfrak{t}_W) = \mathfrak{t}_W \cap \iota_V(g_W), \quad \iota_V(\mathfrak{p}_W) = \mathfrak{p}_W \cap \iota_V(g_W) , \]

(4.10)
where
\[ p_W = (p^+_W \oplus p^-_W) \cap \mathfrak{sp}(W) = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A = ^tA \in M_2(\mathbb{R}) \right\}. \]

Actually, what we need is the Cartan decompositions of the doubled Lie algebras compatible with these decompositions. We write \( \mathfrak{v}^-, \mathfrak{w}^- \) for \( \mathfrak{v}, \mathfrak{w} \) viewed as basis of \( V^-, W^- \), respectively. Thus \( (\cdot, \cdot)^- \) is represented by \( -I_{p,q} \) with respect to \( \mathfrak{v}^- \). In the realization with respect to the basis \( \{ \mathfrak{v}, \mathfrak{v}^\perp \} \) and \( \{ \mathfrak{w}, \mathfrak{w}^\perp \} \), we have the decompositions \( \mathfrak{g}_{V^H} = \mathfrak{k}_{V^H} \oplus p_{V^H}, \mathfrak{g}_{W^H} = \mathfrak{t}_{W^H} \oplus p_{W^H} \), where

\[ \mathfrak{g}_{V^H} : = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -Z^* \in M_{2p}(\mathbb{C}) \} , \]
\[ \mathfrak{g}_{W^H} : = \{ \begin{pmatrix} 0_p & X \\ X^* & 0_{2q} \end{pmatrix} \mid X, Y \in M_{p,2q}(\mathbb{C}) \} , \]
\[ \mathfrak{t}_{V^H} : = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -Z^* \in M_{2p}(\mathbb{C}) \} , \]
\[ \mathfrak{p}_{V^H} : = \{ \begin{pmatrix} 0_p & X \\ X^* & 0_{2q} \end{pmatrix} \mid X, Y \in M_{p,2q}(\mathbb{C}) \} . \]

For \( (\mathbb{W}^H, \langle \cdot, \cdot \rangle^H) \), we choose the first Witt basis
\[ \{ (\mathfrak{v}, \mathfrak{v}^\perp) \otimes \mathfrak{w}, i(\mathfrak{v}^- I_{p,q}, - \mathfrak{v}^- I_{p,q}) \otimes I_{p',q'} \mathfrak{w}^- \} \]
\[ = \{ \mathfrak{v} \otimes (\mathfrak{w}, \mathfrak{w}^-); i\mathfrak{v}^- I_{p,q} \otimes (I_{p',q'} \mathfrak{w}^-, -I_{p',q'} \mathfrak{w}^-) \} \]
as in the case of \( \mathbb{W} \). Here, we have identified two basis \( \mathfrak{v}^-, \mathfrak{w}^- \) of \( \mathbb{W}^- \) in the equality \( \mathfrak{z}^H \). In the realization of \( \mathfrak{sp}(\mathbb{W}^H) \) with respect to this basis, we choose the decomposition \( \mathfrak{sp}(\mathbb{W}^H) = \mathfrak{t}_{W^H, C} \oplus \mathfrak{p}_{W^H} \oplus \mathfrak{p}_{W^H} \) in exactly the same way as before. Obviously, this is compatible with the above decompositions of \( \mathfrak{sp}(\mathbb{W}) \times \mathfrak{sp}(\mathbb{W}) \) under the embedding
\[ i_W : \mathfrak{sp}(\mathbb{W}) \times \mathfrak{sp}(\mathbb{W}) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A^- & B^- \\ C^- & D^- \end{pmatrix} \rightarrow \begin{pmatrix} A & A^- \\ B & -B^- \\ C & -C^- \\ -D & D^- \end{pmatrix} \in \mathfrak{sp}(\mathbb{W}^H). \]

Also, we have
\[ \tau_{V^H}(\mathfrak{t}_V) = \tau_{W^H}(\mathfrak{t}_V), \quad \tau_{V^H}(\mathfrak{p}_V) = \tau_{W^H}(\mathfrak{p}_V), \]
\[ \tau_{V^H}(\mathfrak{t}_W) = \tau_{W^H}(\mathfrak{t}_W), \quad \tau_{V^H}(\mathfrak{p}_W) = \tau_{W^H}(\mathfrak{p}_W) \]
\[ \tau_{V^H}(\mathfrak{t}_W) = \tau_{W^H}(\mathfrak{t}_W), \quad \tau_{V^H}(\mathfrak{p}_W) = \tau_{W^H}(\mathfrak{p}_W) \]  \hspace{1cm} (4.11)

On the other hand, the explicit formulas for \( \omega_{W^H, \xi}^H \) in Lemmata \( 4.1 \) \( 4.2 \) are with respect to Witt basis compatible with the polarization \( \mathbb{W}^H = \sqrt{\mathbb{W}} \oplus \Delta \mathbb{W} \). In \( 4.2 \) we may choose the Witt basis

\( 15 \)

\{v^\prime; v\}, \{w^\prime; w\} of \ V^\mathbb{H}, \ W^\mathbb{H}, respectively, as

\[
\left\{ \frac{v - v^-}{\sqrt{2}}; i\frac{(v + v^-)I_{p,q}}{\sqrt{2}} \right\}, \quad \left\{ \frac{w - w^-}{\sqrt{2}}; i\frac{I_{p',q'}(w + w^-)}{\sqrt{2}} \right\}.
\]

(4.12)

Using these, we may form the second Witt basis

\[
\{\xi, \xi\}' = \left\{ \frac{(v - v^-)}{\sqrt{2}} \otimes w; i\frac{(v + v^-)I_{p,q} \otimes I_{p',q'}w}{\sqrt{2}} \right\}
\]

\[
= \left\{ \frac{v \otimes (w - w^-) - vI_{p,q} \otimes I_{p',q'}(w - w^-)}{\sqrt{2}}, \frac{vI_{p,q} \otimes I_{p',q'}(w + w^-) - v \otimes (w + w^-)}{\sqrt{2}} \right\}
\]

(4.13)

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix}
1_N & -1_N & 0_N & 0_N \\
0_N & 0_N & 1_N & 1_N \\
0_N & 0_N & 1_N & -1_N \\
-1_N & -1_N & 0_N & 0_N
\end{pmatrix}
\begin{pmatrix}
v \otimes w \\
v^- \otimes w \\
ivI_{p,q} \otimes I_{p',q'}w \\
-vI_{p,q} \otimes I_{p',q'}w
\end{pmatrix}
\]

compatible with \(\mathbb{W}^\mathbb{H} = \mathbb{V} \oplus \mathbb{W}.\) Note that this basis transformation matrix from the first to the second Witt basis belongs to the maximal compact subgroup \(K_{\mathbb{W}} \subset Sp(\mathbb{W}^\mathbb{H})\) with the Lie algebra \(\mathfrak{f}_{\mathbb{W}}.\) Thus, with respect to \(\{\xi, \xi\}',\) the decomposition \(\mathfrak{sp}(\mathbb{W}^\mathbb{H}_C) = \mathfrak{f}_{\mathbb{W}, C} \oplus p_{\mathbb{W}, C}^+ \oplus p_{\mathbb{W}, C}^-\) takes the same form as in the first basis realization.

Finally, we summarize the calculation of \(\mathfrak{L1}\) in a form suitable for our purpose. With respect to the basis \(\{v, v^-\}, \{w, w^-\},\) elements of \(i_V(g_V \times g_V)_C, i_W(g_W \times g_W)_C\) are written as

\[
\begin{pmatrix}
U & Y \\
X & V
\end{pmatrix}
\begin{pmatrix}
\bar{U} & \bar{X} \\
\bar{Y} & \bar{V}
\end{pmatrix}
\]

\[
\begin{pmatrix}
U' & X' \\
Y' & V'
\end{pmatrix}
\begin{pmatrix}
\bar{U}' & \bar{X}' \\
\bar{Y}' & \bar{V}'
\end{pmatrix}
\]

(4.14)

respectively. These give the basis

\[
U_{j,k} := E_{j,k}, \quad (1 \leq j, k \leq p), \quad V_{j,k} := E_{p+j,p+k}, \quad (1 \leq j, k \leq q),
\]

\[
X_{j,k} := E_{p+j,k}, \quad (1 \leq j \leq q, 1 \leq k \leq p), \quad Y_{j,k} := E_{j,p+k}, \quad (1 \leq j \leq p, 1 \leq k \leq q),
\]

\[
\bar{U}_{j,k} := E_{n+j,n+k}, \quad (1 \leq j, k \leq p), \quad \bar{V}_{j,k} := E_{n+p+j,n+p+k}, \quad (1 \leq j, k \leq q),
\]

\[
\bar{X}_{j,k} := E_{n+j,n+p+k}, \quad (1 \leq j \leq p, 1 \leq k \leq q), \quad \bar{Y}_{j,k} := E_{n+p+j,n+k}, \quad (1 \leq j \leq q, 1 \leq k \leq p),
\]

16
\[ U'_{j,k} := E_{j,k}, \quad (1 \leq j, k \leq p'), \quad V'_{j,k} := E_{p'+j,p'+k}, \quad (1 \leq j, k \leq q'), \]

\[ X'_{j,k} := E_{j,p'+k}, \quad (1 \leq j \leq p', 1 \leq k \leq q'), \quad Y'_{j,k} := E_{p'+j,k}, \quad (1 \leq j \leq q', 1 \leq k \leq p'), \]

\[ \bar{U}'_{j,k} := E_{n'+j,n'+k}, \quad (1 \leq j, k \leq p'), \quad \bar{V}'_{j,k} := E_{n'+j,n'+p'+k}, \quad (1 \leq j, k \leq q'), \]

\[ X'_{j,k} := E_{n'+p'+j,n'+k}, \quad (1 \leq j \leq q', 1 \leq k \leq p'), \quad Y'_{j,k} := E_{n'+j,n'+p'+k}, \quad (1 \leq j \leq p', 1 \leq k \leq q'), \]

of \( i_V(g_V \times g_V) \subset_i W \), \( i_W(g_W \times g_W) \subset_i W \), respectively.

**Lemma 4.1.** The action of \( i_V(g_V \times g_V) \subset_i W \) on the Schrödinger model \( S(N W) \) is given by the followings. For brevity, let us write

\[ \partial(z_{j,k}) := \frac{d\psi}{2} z_{j,k}, \quad \partial^-(z_{j,k}) := -\frac{d\psi}{2} z_{j,k}. \]

(i) As for \( \omega^H_{W,\xi} \), we have

\[ \omega^H_{W,\xi}(U_{j,k}) = \frac{m + n'}{2} \delta_{j,k} + \frac{i}{d\psi} \sum_{\ell=1}^{p'} \partial^-(z_{j,\ell})\partial(z_{k,\ell}) - \frac{i}{d\psi} \sum_{\ell=p'+1}^{n'} \partial(z_{j,\ell})\partial^-(z_{k,\ell}), \]

\[ \omega^H_{W,\xi}(V_{j,k}) = \frac{m + n'}{2} \delta_{j,k} - \frac{i}{d\psi} \sum_{\ell=1}^{n'} \partial(z_{j,\ell})\partial^-(z_{p+k,\ell}) + \frac{i}{d\psi} \sum_{\ell=p'+1}^{n'} \partial^-(z_{j,\ell})\partial(z_{p+k,\ell}), \]

\[ \omega^H_{W,\xi}(\bar{U}_{j,k}) = \frac{m + n'}{2} \delta_{j,k} - \frac{i}{d\psi} \sum_{\ell=1}^{n'} \partial(z_{j,\ell})\partial^-(z_{k,\ell}) + \frac{i}{d\psi} \sum_{\ell=p'+1}^{n'} \partial^-(z_{j,\ell})\partial(z_{k,\ell}), \]

\[ \omega^H_{W,\xi}(\bar{V}_{j,k}) = \frac{m + n'}{2} \delta_{j,k} + \frac{i}{d\psi} \sum_{\ell=1}^{p'} \partial^-(z_{p+k,\ell})\partial(z_{k,\ell}) - \frac{i}{d\psi} \sum_{\ell=p'+1}^{n'} \partial(z_{p+k,\ell})\partial^-(z_{k,\ell}), \]

(ii) As for \( \omega^H_{V,\xi} \), we have

\[ \omega^H_{V,\xi}(U'_{j,k}) = \frac{m' + n}{2} \delta_{j,k} - \frac{i}{d\psi} \sum_{\ell=1}^{p} \partial(z_{\ell,j})\partial^-(\bar{z}_{\ell,k}) + \frac{i}{d\psi} \sum_{\ell=p+1}^{n} \partial^-(z_{\ell,j})\partial(\bar{z}_{\ell,k}), \]

\[ \omega^H_{V,\xi}(V'_{j,k}) = \frac{m' + n}{2} \delta_{j,k} + \frac{i}{d\psi} \sum_{\ell=1}^{p} \partial^-(z_{\ell,p'+j})\partial(\bar{z}_{\ell,p'+k}) - \frac{i}{d\psi} \sum_{\ell=p'+1}^{n} \partial(z_{\ell,p'+j})\partial^-(\bar{z}_{\ell,p'+k}), \]
\[
\omega_{V_{\mathcal{E}}}^{H}(U_{j,k}) = \frac{m'}{2} + n \delta_{j,k} + \frac{i}{d\psi} \sum_{\ell=1}^{p} \partial^{-}(z_{\ell,j}) \partial(\bar{z}_{\ell,k}) - \frac{i}{d\psi} \sum_{\ell=p+1}^{n} \partial(z_{\ell,j}) \partial^{-}(\bar{z}_{\ell,k}),
\]

\[
\omega_{V_{\mathcal{E}}}^{H}(\bar{V}_{j,k}) = \frac{m'}{2} + n \delta_{j,k} - \frac{i}{d\psi} \sum_{\ell=1}^{p} \partial(z_{\ell,j}) \partial^{-}(\bar{z}_{\ell,k}) + \frac{i}{d\psi} \sum_{\ell=p+1}^{n} \partial^{-}(z_{\ell,j}) \partial(\bar{z}_{\ell,k}),
\]

\[
\omega_{V_{\mathcal{E}}}^{H}(X_{j,k}) = -\frac{i}{d\psi} \sum_{\ell=1}^{p} \partial(z_{\ell,j}) \partial^{-}(\bar{z}_{\ell,k}) + \frac{i}{d\psi} \sum_{\ell=p+1}^{n} \partial^{-}(z_{\ell,j}) \partial(\bar{z}_{\ell,k}),
\]

\[
\omega_{V_{\mathcal{E}}}^{H}(Y_{j,k}) = \frac{i}{d\psi} \sum_{\ell=1}^{p} \partial^{-}(z_{\ell,j}) \partial^{-}(\bar{z}_{\ell,k}) - \frac{i}{d\psi} \sum_{\ell=p+1}^{n} \partial(z_{\ell,j}) \partial(\bar{z}_{\ell,k}).
\]

**Proof.** Note that the elements (4.14) become (taking \(\text{Ad}(\sqrt{2}^{-1}(-1_{n}iI_{p,q}')), \text{Ad}(\sqrt{2}^{-1}(i_{p,q}^{}i_{p,q}')),\) respectively)

\[
\begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2} \\
\frac{U + \bar{U}}{2} & \frac{X + \bar{X}}{2} & \frac{Y + \bar{Y}}{2} \\
i(\bar{U} - U) & i(X - \bar{X}) & i(Y - \bar{Y}) \\
i(U - \bar{U}) & i(\bar{X} - X) & i(\bar{Y} - Y)
\end{pmatrix},
\]

in the matrix representation with respect to \(\{v'; v\}, \{w'; w\}\) (4.12), respectively. Thus, for example,

\[
\omega_{V_{\mathcal{E}}}^{H}(U_{j,k}) = \frac{1}{2} \omega_{W_{\mathcal{E}}}^{H}(E_{j,k} + E_{n+j,n+k} + iE_{j,n+k} - iE_{n+j,k})
\]

\[
= m \delta_{j,k} + \frac{1}{2} \sum_{l=1}^{n'} \left( \bar{z}_{j,l} \frac{\partial}{\partial z_{k,l}} - z_{k,l} \frac{\partial}{\partial \bar{z}_{j,l}} \right) + \frac{d\psi i}{4} \left( \sum_{l=1}^{p'} \bar{z}_{j,l} \frac{\partial}{\partial \bar{z}_{j,l}} - \sum_{l=p'+1}^{n'} \bar{z}_{j,l} \frac{\partial}{\partial z_{k,l}} \right)
\]

\[
+ \frac{i}{d\psi} \left( \sum_{l=1}^{p'} \frac{\partial^2}{\partial z_{j,l} \partial \bar{z}_{j,l}} - \sum_{l=p'+1}^{n'} \frac{\partial^2}{\partial \bar{z}_{j,l} \partial z_{k,l}} \right)
\]

\[
= m + n' \delta_{j,k} + \frac{i}{d\psi} \sum_{l=1}^{p'} \left( -\frac{d\psi i}{2} \bar{z}_{j,l} + \frac{\partial}{\partial \bar{z}_{j,l}} \right) \left( \frac{d\psi i}{2} z_{k,l} + \frac{\partial}{\partial z_{k,l}} \right)
\]

\[
- \frac{i}{d\psi} \sum_{l=p'+1}^{n'} \left( \frac{d\psi i}{2} \bar{z}_{j,l} + \frac{\partial}{\partial \bar{z}_{j,l}} \right) \left( -\frac{d\psi i}{2} z_{k,l} + \frac{\partial}{\partial z_{k,l}} \right).
\]
4.3 Explicit formula for the Fock model

We are now ready to construct the Fock model of \( \omega_{V,W} \) from that of doubled representation \( \omega_{V,W}^H \). We have identified \( V \otimes W \) with \( M_{n,n'}(\mathbb{C}) \) using the basis \( (v - v^-) \otimes w / \sqrt{2} = v \otimes (w - w^-) / \sqrt{2} \). If we write \( z \in V \otimes W \) as \( (z_{j,k}) = (x_{j,k}) + i(y_{j,k}) \), \((x_{j,k}), (y_{j,k}) \in M_{n,n'}(\mathbb{C}) \),

\[
    z = \frac{v - v^-}{\sqrt{2}}(z_{j,k})w = \frac{v - v^-}{\sqrt{2}}(x_{j,k})w + \frac{i(v - v^-)}{\sqrt{2}}I_{p,q}(I_{p,q}(y_{j,k})I_{p,q'})I_{p,q}w
\]

shows that \(((x_{j,k}), I_{p,q}(y_{j,k})I_{p,q'}) \in M_{n,n'}(\mathbb{R})^2 \) is the \( e' \)-coordinates of \( z \):

\[
    z = \sum_{j=1}^n \sum_{k=1}^{n'} x_{j,k} \Re e'_{j,k} + \epsilon_j \epsilon'_{j,k} \Im e'_{j,k}.
\]

Here, we adopt the labels \( e' = \{ \Re e'_{j,k}, \Im e'_{j,k}, \epsilon e'_{j,k} \} \), where \( \Re e_{j,k} = \{ \Re e'_{j,k}, \Im e'_{j,k} \}, \epsilon e_{j,k} = \{ \epsilon e'_{j,k} \} \),

\[
    \diamond e'_{j,k} = \frac{\Re e'_{j,k} - i \Im e'_{j,k}}{\sqrt{2}}, \quad \diamond e_{j,k} = \frac{\Re e_{j,k} - i \epsilon e'_{j,k}}{\sqrt{2}}, \quad (\diamond = \Re, \Im).
\]

We identify \( \Re e_{j,k}, \Im e_{j,k} \) with variables \( u_{j,k}, v_{j,k} \), \( 1 \leq j \leq n \), \( 1 \leq k \leq n' \), respectively, so that \( S(\mathbb{H}) = \mathbb{C}[[u_{j,k}, v_{j,k}]] \) (polynomial ring over \( \mathbb{M}_{n,n'}(\mathbb{C})^2 \) on which \( \mathfrak{h}(\mathbb{W}^H \otimes \mathbb{C}) \) acts by (see [2.2]):

\[
    r_\psi(\Re e_{j,k}) = u_{j,k}, \quad r_\psi(\Im e_{j,k}) = v_{j,k},
\]

\[
    r_\psi(\Re e'_{j,k}) = d\psi \frac{\partial}{\partial u_{j,k}}, \quad r_\psi(\Im e'_{j,k}) = d\psi \frac{\partial}{\partial v_{j,k}}.
\]

On the other hand, its action on the Schrödinger model is given by (2.4):

\[
    \rho_\psi(\Re e_{j,k}) = d\psi x_{j,k}, \quad \rho_\psi(\Im e_{j,k}) = d\psi \epsilon_j \epsilon'_{j,k}, \quad \rho_\psi(\Re e'_{j,k}) = \frac{\partial}{\partial x_{j,k}}, \quad \rho_\psi(\Im e'_{j,k}) = \epsilon_j \epsilon'_{j,k} \frac{\partial}{\partial y_{j,k}}.
\]

Using (4.15), we compare (4.16) with (4.17) and obtain the following.

**Lemma 4.2.** We write \( w_{j,k} := \sqrt{2}^{-1}(u_{j,k} + iv_{j,k}), \quad \bar{w}_{j,k} := \sqrt{2}^{-1}(u_{j,k} - iv_{j,k}) \). The \( j(\mathbb{W}^H_{\mathbb{C}}) \)-isomorphism \( S_\psi(\mathbb{W}) \cong S(\mathbb{H}) \) is given by the following.

(i) If either \( 1 \leq j \leq p, 1 \leq k \leq p' \) or \( p < j \leq n, p' < k \leq n' \), we have

\[
    \partial(z_{j,k}) \leftrightarrow i w_{j,k}, \quad \partial(\bar{z}_{j,k}) \leftrightarrow i \bar{w}_{j,k}, \quad \partial^{-}(z_{j,k}) \leftrightarrow d\psi \frac{\partial}{\partial w_{j,k}}; \quad \partial^{-}(\bar{z}_{j,k}) \leftrightarrow d\psi \frac{\partial}{\partial \bar{w}_{j,k}}.
\]

(ii) If either \( 1 \leq j \leq p, p' < k \leq n' \) or \( p < j \leq n, 1 \leq k \leq p' \), we have

\[
    \partial(z_{j,k}) \leftrightarrow i \bar{w}_{j,k}, \quad \partial(\bar{z}_{j,k}) \leftrightarrow i w_{j,k}, \quad \partial^{-}(z_{j,k}) \leftrightarrow d\psi \frac{\partial}{\partial \bar{w}_{j,k}}; \quad \partial^{-}(\bar{z}_{j,k}) \leftrightarrow d\psi \frac{\partial}{\partial w_{j,k}}.
\]
Proof. We show (i). (ii) can be verified in the same way. Comparison of \(4.16\) and \(4.17\) implies that the isomorphism sends
\[
x_{j,k} = \frac{\rho_\psi(\Re e_{j,k})}{d\psi}, \quad y_{j,k} = \frac{\rho_\psi(\Im e_{j,k})}{d\psi}, \quad \frac{\partial}{\partial x_{j,k}} = \rho_\psi(\Re e'_{j,k}), \quad \frac{\partial}{\partial y_{j,k}} = \rho_\psi(\Im e'_{j,k})
\]
to
\[
\frac{1}{\sqrt{2d\psi}}r_\psi(\Re e_{j,k} + i\Im e'_{j,k}) = \frac{1}{\sqrt{2}}\left(\frac{u_{j,k}}{d\psi} + i\frac{\partial}{\partial u_{j,k}}\right),
\]
\[
\frac{1}{\sqrt{2d\psi}}r_\psi(\Im e_{j,k} + i\Re e'_{j,k}) = \frac{1}{\sqrt{2}}\left(\frac{v_{j,k}}{d\psi} + i\frac{\partial}{\partial v_{j,k}}\right),
\]
\[
\frac{1}{\sqrt{2d\psi}}r_\psi(\Re e'_{j,k} + i\Re e_{j,k}) = \frac{1}{\sqrt{2}}\left(iu_{j,k} + dv\psi\frac{\partial}{\partial u_{j,k}}\right),
\]
\[
\frac{1}{\sqrt{2d\psi}}r_\psi(\Im e'_{j,k} + i\Im e_{j,k}) = \frac{1}{\sqrt{2}}\left(iv_{j,k} + dv\psi\frac{\partial}{\partial v_{j,k}}\right),
\]
respectively. (In the case (ii), the image of \(y_{j,k}\) and \(\partial/\partial y_{j,k}\) are multiplied by \(-1\).) Thus we have
\[
z_{j,k} \leftarrow \frac{1}{\sqrt{2}}\left(\frac{u_{j,k} + iv_{j,k}}{d\psi} + i\left(\frac{\partial}{\partial u_{j,k}} + i\frac{\partial}{\partial v_{j,k}}\right)\right) = \frac{w_{j,k}}{d\psi} + i\frac{\partial}{\partial w_{j,k}},
\]
\[
\frac{\partial}{\partial z_{j,k}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{j,k}} - i\frac{\partial}{\partial y_{j,k}}\right) \leftrightarrow \frac{1}{2}\left(i\frac{u_{j,k} - iv_{j,k}}{\sqrt{2}} + dv\psi\frac{\partial}{\partial w_{j,k}}\right) = i\frac{\partial}{\partial w_{j,k}} + \frac{d\psi}{2}\frac{\partial}{\partial w_{j,k}}.
\]
This combined with the definition of \(\partial(\bar{z}_{j,k})\), \(\partial^-(\bar{z}_{j,k})\) gives the result. Notice that the transposition of \(z_{j,k}\) and \(\bar{z}_{j,k}\) is transported to that of \(w_{j,k}\) and \(\bar{w}_{j,k}\). \(\square\)

Putting this into Lem.\([4.1]\) we obtain the following.

Proposition 4.3. (i) \((\omega_{W,\xi}^\| \circ i_V, S(\mathbb{L})) = (\omega_{W,\xi}^\|, \mathbb{C}[\{w_{j,k}\}]) \otimes (\omega_{W,\xi}^\|, \mathbb{C}[\{w_{j,k}\}]), (\omega_{V,\xi}^\| \circ i_W, S(\mathbb{L})) = (\omega_{V,\xi}^\|, \mathbb{C}[\{w_{j,k}\}]) \otimes (\omega_{V,\xi}^\|, \mathbb{C}[\{w_{j,k}\}]). Thus \(\mathcal{P}_{V,W,\xi} := \mathbb{C}[\{w_{j,k}\}]\) is the Fock model of \(\omega_{V,W,\xi}\).

(ii) We have the following explicit formulas for \(\omega_{W,\xi}(U_{j,k}), \omega_{W,\xi}(V_{j,k}), \omega_{W,\xi}(X_{j,k}), \omega_{W,\xi}(Y_{j,k})\).

\[
\omega_{W,\xi}(U_{j,k}) = \frac{m + q' - p'}{2}\delta_{j,k} - \sum_{\ell=1}^{p'} w_{k,\ell} \frac{\partial}{\partial w_{j,\ell}} + \sum_{\ell=p'+1}^{n'} w_{j,\ell} \frac{\partial}{\partial w_{k,\ell}},
\]
\[
\omega_{W,\xi}(V_{j,k}) = \frac{m + q' - q}{2}\delta_{j,k} + \sum_{\ell=1}^{p'} w_{p+j,\ell} \frac{\partial}{\partial w_{p+k,\ell}} - \sum_{\ell=p'+1}^{n'} w_{p+k,\ell} \frac{\partial}{\partial w_{p+j,\ell}},
\]
\[
\omega_{W,\xi}(X_{j,k}) = -i \frac{d\psi}{d\psi} \sum_{\ell=1}^{p'} w_{p+j,\ell} w_{k,\ell} - d\psi i \sum_{\ell=p'+1}^{n'} \frac{\partial^2}{\partial w_{p+j,\ell} \partial w_{k,\ell}},
\]
\[
\omega_{W,\xi}(Y_{j,k}) = -d\psi i \sum_{\ell=1}^{p'} \frac{\partial^2}{\partial w_{j,\ell} \partial w_{p+k,\ell}} - i \frac{d\psi}{d\psi} \sum_{\ell=p'+1}^{n'} w_{j,\ell} w_{p+k,\ell},
\]

20
For brevity, let us write $w = \frac{m' + p - q \delta_{j,k}}{2} + \sum_{\ell=1}^{p} w_{\ell,j} \frac{\partial}{\partial w_{\ell,j}} - \sum_{\ell=p+1}^{n} w_{\ell,k} \frac{\partial}{\partial w_{\ell,k}}$.

$$
\omega_{\nu',\xi'}(U_{j,k}') = \frac{m' + p - q \delta_{j,k}}{2} + \sum_{\ell=1}^{p} w_{\ell,j} \frac{\partial}{\partial w_{\ell,j}} - \sum_{\ell=p+1}^{n} w_{\ell,k} \frac{\partial}{\partial w_{\ell,k}}.
$$

$$
\omega_{\nu',\xi'}(V_{j,k}') = \frac{m' + q - p \delta_{j,k}}{2} - \sum_{\ell=1}^{p} w_{\ell,p' + k} \frac{\partial}{\partial w_{\ell,p' + k}} + \sum_{\ell=p+1}^{n} w_{\ell,p' + j} \frac{\partial}{\partial w_{\ell,p' + j}}.
$$

$$
\omega_{\nu',\xi'}(X_{j,k}') = \frac{i}{\sqrt{2}} \sum_{\ell=1}^{p} w_{\ell,j} w_{\ell,p' + k} + d\psi \sum_{\ell=p+1}^{n} \frac{\partial^2}{\partial w_{\ell,j} \partial w_{\ell,p' + k}}.
$$

$$
\omega_{\nu',\xi'}(Y_{j,k}') = d\psi \sum_{\ell=1}^{p} \frac{\partial^2}{\partial w_{\ell,p' + j} \partial w_{\ell,k}} + \frac{i}{\sqrt{2}} \sum_{\ell=p+1}^{n} w_{\ell,p' + j} w_{\ell,k}.
$$

**Proof.** (i) For brevity, let us write $\nu_+ := \{v_1, \ldots, v_p\}$, $\nu_- := \{v_{p+1}, \ldots, v_n\}$, $w_+ := \{w_1, \ldots, w_{p'}\}$, $w_- := \{w_{p'+1}, \ldots, w_{n'}\}$. We may write

$$
\mathbb{R}_e' = \frac{1}{\sqrt{2}} \left( \begin{array}{c} v_+ - v_- \\ \mathbb{R} \end{array} \right) \otimes \left( \begin{array}{c} w_+ \otimes w_- \\ -w_- \otimes w_+ \end{array} \right),
$$

$$
\mathbb{R}_e' = \frac{i}{\sqrt{2}} \left( \begin{array}{c} v_+ - v_- \\ \mathbb{R} \end{array} \right) \otimes \left( \begin{array}{c} w_+ \otimes w_- \\ -w_- \otimes w_+ \end{array} \right),
$$

$$
\mathbb{R}_e' = \frac{i}{\sqrt{2}} \left( \begin{array}{c} v_+ + v_- \\ \mathbb{R} \end{array} \right) \otimes \left( \begin{array}{c} w_+ \otimes w_- \\ -w_- \otimes w_+ \end{array} \right),
$$

$$
\mathbb{R}_e' = -\frac{1}{\sqrt{2}} \left( \begin{array}{c} v_+ + v_- \\ \mathbb{R} \end{array} \right) \otimes \left( \begin{array}{c} w_+ \otimes w_- \\ -w_- \otimes w_+ \end{array} \right).
$$

where $v_-^+$ denotes $v_-$ viewed as a subset of $V^-$, and so force. Then (4.15) becomes

$$
\mathbb{R}_e' = \left( \begin{array}{c} v_+ \otimes w_+ \\ -v_- \otimes w_- \\ -v_- \otimes w_+ \end{array} \right),
$$

$$
\mathbb{R}_e' = i \left( \begin{array}{c} v_+ \otimes w_+ \\ -v_- \otimes w_- \\ -v_- \otimes w_+ \end{array} \right),
$$

$$
\mathbb{R}_e' = i \left( \begin{array}{c} v_+ \otimes w_+ \\ -v_- \otimes w_- \\ -v_- \otimes w_+ \end{array} \right),
$$

$$
\mathbb{R}_e' = -\left( \begin{array}{c} v_+ \otimes w_+ \\ -v_- \otimes w_- \\ -v_- \otimes w_+ \end{array} \right).
$$

Hence, by definition (Lem 4.2),

$$
\tilde{w} = \sqrt{2i} \left( \begin{array}{cc} 0 & -v_- \otimes w_+ \\ -v_- \otimes w_- & 0 \end{array} \right),
$$

$$
\tilde{w} = \sqrt{2i} \left( \begin{array}{cc} v_- \otimes w_+ & 0 \\ 0 & v_- \otimes w_- \end{array} \right).
$$

This together with the definition of $\omega_{\nu',\xi'}$ (4.3) proves the claimed decompositions.

(ii) is a direct consequence of Lemmas 4.1, 4.2. Note that the formulas for $(\omega_{w,-\xi} \times \omega_{-\xi'}, \mathbb{C}[\langle \tilde{w}_{j,k} \rangle])$ can be obtained by applying (3.3) to the above formulas. ∎

## 5 K-type correspondence

Let us describe the Howe correspondence for K-types. We first review the definition of the correspondence [How89], [AB95].
5.1 Joint harmonics

We write $V_+ := \text{span}_C\{v_1, \ldots, v_p\}$, $V_- := \text{span}_C\{v_{p+1}, \ldots, v_n\}$ and $(\cdot, \cdot)_\pm$ for the restriction of $(\cdot, \cdot)$ to $V_\pm$, respectively. Thus $(V, (\cdot, \cdot))$ is the direct sum of definite hermitian spaces $(V_\pm, (\cdot, \cdot)_\pm)$. We have the similar decomposition $(W, (\cdot, \cdot)) = (W_+, (\cdot, \cdot)_+) \oplus (W_-, (\cdot, \cdot)_-)$. Then $K_V = G_{V_+} \times G_{V_-}$, $K_W = G_{W_+} \times G_{W_-}$. We fix decompositions $\xi = \xi_+ \cdot \xi_-, \xi' = \xi'_+ \cdot \xi'_-$ such that $\xi_\pm \mid_R = \text{sgn dim} W_\pm, \xi'_\pm \mid_R = \text{sgn dim} V_\pm$, respectively.

Consider the seawal dual pairs

\[ G_V \quad G_V \times G_W \quad G_V \times G_V \quad G_W \]

\[ K_V \quad \times \quad G_W \quad \times \quad G_W \quad K_W \]

in $Sp(\mathbb{W})$. The Weil representations $\omega_{V,W,\xi}(\xi,\xi') \otimes \omega_{W,-,\xi,-,\xi'}$ of $(G_V \times G_V) \times K_W$ and $\omega_{V,W,\xi}(\xi',\xi) \otimes \omega_{W,-,\xi',\xi}$ of $K_V \times (G_V \times G_W)$ share the same Fock model $\mathcal{P}_{V,W,\xi}$. Since $\nu_V(\mathfrak{g}_{V,+}) \oplus \nu_V(\mathfrak{g}_{V,-}) = \nu_V(\mathfrak{g}_{W,C})$ preserves the decomposition (5.1), we have a decompositions $g_{V,C} = \mathfrak{g}_{V,C} \oplus p_{V,V_+}^+ \oplus p_{V,V_-}^-$ with $p_{V,V_+}^+ := \nu_V^{-1}(\mathfrak{p}_{V,+}^+)$. The decompositions $g_{W,C} = \mathfrak{g}_{W,C} \oplus p_{W,V_+}^+ \oplus p_{W,V_-}^-$ is defined similarly. (4.9) with $(V, W)$ replaced by $(V, W_\pm), (V_\pm, W)$ shows

\[ p_{V,V_+}^+ = p_{V,V_-} = \text{span}_C\{X_{j,k}\}_{1 \leq j \leq q}, \quad p_{W,V_+}^+ = p_{W,V_-}^+ = \text{span}_C\{Y_{j,k}\}_{1 \leq j \leq q}, \]

\[ p_{V,W_+}^+ = p_{V,W_-}^+ = \text{span}_C\{x_{q+1}, \ldots, q\}, \quad p_{W,W_+}^+ = p_{W,W_-}^+ = \text{span}_C\{y_{q+1}, \ldots, q\}. \]

By definition, the spaces of $K_W$ and $K_V$-harmonics are given by

\[ \mathcal{H}_V(K_W) := \{ P \in \mathcal{P}_{V,W,\xi} \mid \omega_{V,+,\xi}(p_{V,V_+}^+) = 0 \}, \]

\[ \mathcal{H}_W(K_V) := \{ P \in \mathcal{P}_{V,W,\xi} \mid \omega_{V,-,\xi}(p_{W,V_-}^+) = 0 \}, \]

respectively. Their intersection $\mathcal{J}_{V,W,\xi} := \mathcal{H}_V(K_W) \cap \mathcal{H}_W(K_V)$ is called the space of joint harmonics. Proposition (ii) for $(V, W_\pm), (V_\pm, W)$ in place of $(V, W)$ shows that $\mathcal{J}_{V,W,\xi}$ consists of $P \in \mathcal{P}_{V,W,\xi} \setminus K_W$ killed by

\[ \sum_{\ell=1}^{q-1} \frac{\partial^2}{\partial w_{j,\ell} \partial w_{p+k,\ell}}, \quad \sum_{\ell=p+1}^{n-1} \frac{\partial^2}{\partial w_{j,\ell} \partial w_{p+k,\ell}}, \quad (1 \leq j \leq p, 1 \leq k \leq q), \]

\[ \sum_{\ell=1}^{q-1} \frac{\partial^2}{\partial w_{j,\ell} \partial w_{p+k,\ell}}, \quad \sum_{\ell=p+1}^{n-1} \frac{\partial^2}{\partial w_{j,\ell} \partial w_{p+k,\ell}}, \quad (1 \leq j \leq p', 1 \leq k \leq q'). \]

Fact 5.1 ([How89] §3). (1) $\mathcal{J}_{V,W,\xi}$ is stable under $\omega_{V,W,\xi}(K_V \times K_W)$.

(2) We write $\mathcal{R}(K_V, \mathcal{J}_{V,W,\xi})$ for the set of $K_V$-types which appear as irreducible direct summands of $\mathcal{J}_{V,W,\xi}$. Similarly we have $\mathcal{R}(K_W, \mathcal{J}_{V,W,\xi})$ in the $W$-side. Then $\mathcal{J}_{V,W,\xi}$ is multiplicity-free as a $K_V \times K_W$-module, so that it gives a bijection

\[ \mathcal{R}(K_V, \mathcal{J}_{V,W,\xi}) \ni (\tau_V, \tau_W, K_V) \longleftrightarrow (\tau_\xi(\tau_V, K_W) \equiv \tau_W) \in \mathcal{R}(K_W, \mathcal{J}_{V,W,\xi}). \]
(3) For a K_v-type τ_v, we write deg_{W, ξ}(τ_v) for the minimum degree of polynomials in the τ_v-isotypic subspace in \( \mathcal{H}_{V,W,ξ}(τ_v) \). (We set \( \deg_{W, ξ}(τ_v) := \infty \) if \( τ_v \) does not appear in \( \mathcal{H}_{V,W,ξ} \).) A K_v-type \( τ_v \) of \( \pi_V \) ∈ \( \mathcal{H}(G_V, ξ_W, ξ_ξ) \) (see introduction) is of minimal \((W, ξ)\)-degree if \( \deg_{W, ξ}(τ_v) \) is minimal among \( \deg_{W, ξ}(τ) \) (\( τ \) is a K_v-type of \( \pi_V \)). Similar definition applies to the W-side.

(i) Suppose \( τ_v \) is a K_v-type of \( \pi_V \) ∈ \( \mathcal{H}(G_V, ξ_W, ξ_ξ) \) of minimal \((W, ξ)\)-degree. Then \( τ_v \) ∈ \( \mathcal{H}(K_V, \mathcal{F}_V,W,ξ) \).

(ii) Furthermore, \( ξ(τ_v, K_W) \) is a K_W-type of minimal \((V, ξ')\)-degree in the Howe correspondence \( ξ(\pi_V, W) \) ∈ \( \mathcal{H}(G_W, ξ_V, ξ_ξ') \) of \( \pi_V \).

Similar assertion holds in the W-side.

Because of the assertion (3), the K-type correspondence (2) plays an important role in the explicit description of the Howe correspondence \([AB95], [Pau98], [Pau00]\). In the rest of this section, we describe this correspondence in our setting.

### 5.2 Highest weight vectors

The calculation in this subsection is essentially due to Kashiwara-Vergne [KV78]. Let us write \( L^n, R^n \) for the representation of \( GL(m, \mathbb{C}) \), \( GL(n, \mathbb{C}) \) on the polynomial ring \( \mathbb{C}[M_{m,n}(\mathbb{C})] \) defined by

\[ L^n(g)R^n(g')P(w) = P(g^{-1}.w.g'), \quad g \in GL(m, \mathbb{C}), g' \in GL(n, \mathbb{C}), P(w) \in \mathbb{C}[M_{m,n}(\mathbb{C})]. \]

We write \( b_n = h_n \oplus n_n \) and \( \bar{b}_n = h_n \oplus \bar{n}_n \) for the upper and lower triangular Borel subalgebra of \( gl(n, \mathbb{C}) \), respectively. Let \( d_\ell(w_{j,k}), d_\ell^{-1}(w_{j,k}) \in \mathbb{C}[M_{m,n}(\mathbb{C})] \) be the minor determinants

\[
\det \begin{pmatrix} w_{1,1} & \cdots & w_{1,\ell} \\ \vdots & \ddots & \vdots \\ w_{\ell,1} & \cdots & w_{\ell,\ell} \end{pmatrix}, \quad \det \begin{pmatrix} w_{m-\ell+1,n-\ell+1} & \cdots & w_{m-\ell+1,n} \\ \vdots & \ddots & \vdots \\ w_{m-n,\ell+1} & \cdots & w_{m,n} \end{pmatrix},
\]

respectively. Then elementary properties of the determinant shows that the \( h_m \oplus h_n \)-action on these are given by

\[
L^n(\text{diag}(x_1, \ldots, x_m))d_\ell(w_{j,k}) = -(x_1 + \cdots + x_\ell)d_\ell(w_{j,k}), \\
R^n(\text{diag}(x_1', \ldots, x_m'))d_\ell^{-1}(w_{j,k}) = (x_1' + \cdots + x_\ell')d_\ell^{-1}(w_{j,k}),
\]

\[
L^n(\text{diag}(x_1, \ldots, x_m))d_\ell^{-1}(w_{j,k}) = -(x_{m-\ell+1} + \cdots + x_m)d_\ell^{-1}(w_{j,k}), \\
R^n(\text{diag}(x_1', \ldots, x_m'))d_\ell(w_{j,k}) = (x_{n-\ell+1}' + \cdots + x_m')d_\ell(w_{j,k}),
\]

and

\[
L^n(\bar{n}_m)d_\ell(w_{j,k}) = R^n(n_n)d_\ell(w_{j,k}) = 0, \quad L^n(n_m)d_\ell^{-1}(w_{j,k}) = R^n(\bar{n}_n)d_\ell^{-1}(w_{j,k}) = 0.
\]
This latter formula follows from

\[
L^n(\exp tE_{j,k})d_{e}(w_{j,k}) = d_{e}\begin{pmatrix} w_{1,*} \\ \vdots \\ w_{j,*} - tw_{k,*} \\ \vdots \\ w_{m,*} \end{pmatrix} = d_{e}(w_{j,k}), \quad \text{if } j > k,
\]

\[
R^n(\exp tE_{j,k})d_{e}(w_{j,k}) = d_{e}(w_{*1}, \ldots, w_{*k} + tw_{*j}, \ldots, w_{*n}) = d_{e}(w_{j,k}), \quad \text{if } j < k,
\]

and so on. Next we set, for a decreasing series \(a = (a_1, \ldots, a_r)\) in \(\mathbb{Z}_{>0}\),

\[
\Delta_a(w_{j,k}) := d_1(w_{j,k})^{a_1-a_2} \cdots d_{r-1}(w_{j,k})^{a_{r-1}-a_r}d_r(w_{j,k})^{a_r},
\]

\[
\Delta^-_a(w_{j,k}) := d_1(w_{j,k})^{a_1-a_2} \cdots d_{r-1}(w_{j,k})^{a_{r-1}-a_r}d_r(w_{j,k})^{a_r}.
\]

It follows immediately from (5.2) and (5.3), respectively, that

\[
L^n(\text{diag}(x_1, \ldots, x_m))\Delta_a(w_{j,k}) = (-a_1x_1 - \cdots - a_rx_r)\Delta_a(w_{j,k}),
\]

\[
R^n(\text{diag}(x'_1, \ldots, x'_m))\Delta_a(w_{j,k}) = (a_1x'_1 + \cdots + a_rx'_r)\Delta_a(w_{j,k}),
\]

\[
L^n(\text{diag}(x_1, \ldots, x_m))\Delta^-_a(w_{j,k}) = (-a_rx_{m-r+1} - \cdots - a_1x_m)\Delta^-_a(w_{j,k}),
\]

\[
R^n(\text{diag}(x'_1, \ldots, x'_m))\Delta^-_a(w_{j,k}) = (a_rx_{m-r+1} + \cdots + a_1x_m)\Delta^-_a(w_{j,k}),
\]

and

\[
L^n(\bar{n}_m)\Delta_a(w_{j,k}) = R^n(n_n)\Delta_a(w_{j,k}) = 0,
\]

\[
L^n(n_m)\Delta^-_a(w_{j,k}) = R^n(n_n)\Delta^-_a(w_{j,k}) = 0.
\]

We now go back to the Fock model \((\omega_{V,W,\xi}, \mathcal{P}_{V,W,\xi})\). Comparison of the explicit formulas

\[
L^n(E_{j,k}) = -\sum_{\ell=1}^n w_{k,\ell}\frac{\partial}{\partial w_{j,\ell}}, \quad R^n(E_{j,k}) = \sum_{\ell=1}^n w_{k,\ell}\frac{\partial}{\partial w_{j,\ell}}
\]

of the derived representations of \(L^n, R^n\) with Prop. 4.3 immediately yields the following.

**Lemma 5.2.** The restriction of \((\omega_{V,W,\xi}, \mathcal{P}_{V,W,\xi})\) to \(K_V \times K_W\) is given by

\[
\omega_{V,\xi}|_{\mathcal{G}_{V,+}} = \text{det}^{(m+q'-p')/2}R'_{(w_{j,k})}(w_{p,q'+j}) \otimes R_q'(w_{p,k+j}),
\]

\[
\omega_{V,\xi}|_{\mathcal{G}_{V,-}} = \text{det}^{(m+p'-q)/2}R_{(w_{p+k,j})}(w_{p+j,p+k}) \otimes L_q'(w_{p'+j,k}),
\]

\[
\omega_{V,\xi'}|_{\mathcal{G}_{W,+}} = \text{det}^{(m'+p-q)/2}R_{(w_{j,k})}(w_{p'+j+k}) \otimes L_{q'}'(w_{p'+k,j}),
\]

\[
\omega_{V,\xi'}|_{\mathcal{G}_{W,-}} = \text{det}^{(m'+q-p)/2}L_{(w_{p+k,j})}(w_{p,k+j}) \otimes L_{q'}'(w_{p'+k,j}).
\]

Here, for example, \(L_p'(w_{j,k})\) denotes the \(L^p\) on the space \(\mathbb{C}[(w_{j,k})]\).

Take four decreasing series \(a = (a_1, \ldots, a_r), b = (b_1, \ldots, b_s), c = (c_1, \ldots, c_t), d = (d_1, \ldots, d_u)\) in \(\mathbb{Z}_{>0}\) whose lengths satisfy \(r \leq \min(p, p'), s \leq \min(p, q'), t \leq \min(q, p'), u \leq \min(q, q')\). We set

\[
\Delta_{abcd}(w_{j,k}) := \Delta_a(w_{j,k})\Delta^-_b(w_{j,p'+k})\Delta^-_c(w_{p+j,k})\Delta_d(w_{p+j,p'+k}) \in \mathcal{P}_{V,W,\xi}.
\]

The following lemma follows from the above lemma combined with (5.4), (5.5).
Lemma 5.3. Let $b_V = t_V \oplus n_V$ (resp. $b_W = t_W \oplus n_W$) be the lower (resp. upper) triangular Borel subalgebra of $\mathfrak{k}_V$ (resp. $\mathfrak{k}_W$) in the realization with respect to $\mathfrak{v}$ (resp. $\mathfrak{w}$).

(i) $\Delta_{\text{abcd}}$ is a $b_V \oplus b_W$-highest weight vector in $\mathcal{P}_{\mathfrak{v},\mathfrak{w},\xi}$ of $t_V$-weight

\[
\left( \frac{m + q' - p'}{2}, \ldots, \frac{m + q' - p'}{2} ; \frac{m + p' - q'}{2}, \ldots, \frac{m + p' - q'}{2} \right)
\]

\[
+ (a_1, \ldots, a_r, 0, \ldots, 0; -d_1, \ldots, -d_u, 0, \ldots, 0)
\]

\[
+ (0, 0, 0, b_s, \ldots, b_1; 0, 0, 0, c_t, \ldots, c_1),
\]

and $t_W$-weight

\[
\left( \frac{m' + p - q}{2}, \ldots, \frac{m' + p - q}{2} ; \frac{m' + q - p}{2}, \ldots, \frac{m' + q - p}{2} \right)
\]

\[
+ (a_1, \ldots, a_r, 0, \ldots, 0; d_1, \ldots, d_u, 0, \ldots, 0)
\]

\[
+ (0, 0, -c_t, \ldots, -c_1; 0, 0, -b_s, \ldots, -b_1).
\]

(ii) $\Delta_{\text{abcd}}$ belongs to $\mathcal{J}_{\mathfrak{v},\mathfrak{w},\xi}$ if and only if $r + s \leq p$, $t + u \leq q$, $r + t \leq p'$, $s + u \leq q'$ hold.

(iii) Any $b_V \oplus b_W$-highest weight vector in the $\mathfrak{k}_V \oplus \mathfrak{k}_W$-module $\mathcal{J}_{\mathfrak{v},\mathfrak{w},\xi}$ is of the form $\Delta_{\text{abcd}}$.

Proof. Only (iii) needs some explanation, but this is proved in the same way as in [KV78 Prop.III.6.1]. We omit the details.

This gives the main result of this section.

Theorem 5.4 (K-type correspondence). (i) Write the $b_V$-highest weight of a $K_V$-type $\tau_V$ as

\[
\left( \frac{m + q' - p'}{2}, \ldots, \frac{m + q' - p'}{2} ; \frac{m + p' - q'}{2}, \ldots, \frac{m + p' - q'}{2} \right)
\]

\[
+ (a_1, \ldots, -a_r, 0, \ldots, 0, b_s, \ldots, b_1; -d_1, \ldots, -d_u, 0, \ldots, 0, c_t, \ldots, c_1),
\]

for some $a_1 \geq \cdots \geq a_r$, $b_1 \geq \cdots \geq b_s$, $c_1 \geq \cdots \geq c_t$, $d_1 \geq \cdots \geq d_u \in \mathbb{Z}_{\geq 0}$. Then $\tau_V \in \mathcal{R}(K_V, J_{\mathfrak{v},\mathfrak{w},\xi})$ if and only if $r + t \leq p'$, $s + u \leq q'$. In that case, the $b_W$-highest weight of $\theta_{\xi}(\tau_V, K_W)$ is given by

\[
\left( \frac{m' + p - q}{2}, \ldots, \frac{m' + p - q}{2} ; \frac{m' + q - p}{2}, \ldots, \frac{m' + q - p}{2} \right)
\]

\[
+ (a_1, \ldots, a_r, 0, \ldots, 0, -c_t, \ldots, -c_1; d_1, \ldots, d_u, 0, \ldots, 0, -b_s, \ldots, -b_1).
\]

(ii) Conversely, a $K_W$-type $\tau_W$ with the $b_W$-highest weight (5.7) belongs to $\mathcal{R}(K_W, J_{\mathfrak{v},\mathfrak{w},\xi})$ if and only if $r + s \leq p$, $t + u \leq q$. In that case, $\theta_{\xi}(\tau_W, K_V)$ has the $b_V$-highest weight (5.6).

Corollary 5.5. Let $(V, (\cdot, \cdot))$ be an $n$-dimensional hermitian space and $\tau_V$ be a $K_V$-type. Fix a character $\xi$ of $\mathbb{C}^\times$ such that $\xi|_{\mathbb{R}^\times} = \xi'|_{\mathbb{R}^\times} = \text{sgn}^n$. Then there exists a unique (up to isometry) $n$-dimensional skew-hermitian space $(W, (\cdot, \cdot))$ such that $\tau_V$ is $(W, \xi)$-harmonic. (The same assertion holds in the $W$-side.)

Proof. This can be shown in the same way as [Pau98 Prop.1.4.10].

25
6 Example: Howe correspondence for limit of discrete series

As a consequence of our calculation, we shall compute the Howe correspondence between limit of discrete series representation of unitary dual pairs of the same size.

Let us recall Vogan’s classification of limit of discrete series representations of unitary groups [Vog84] §2. We realize the group $G_V = U(p, q)$ with respect to the basis $\nu$ (§3.1). A limit of discrete series representation of $G_V$ is of the form $X(\gamma)$ in the notation of [loc.cit.]. Here $\gamma = (T_V, \lambda, \Lambda, \Psi)$ is a Langlands datum in Vogan’s sense such that

\[(i) \quad T_V \text{ is the fundamental Cartan subgroup of } G_V \text{ with the Lie algebra } t_V \text{ (see Lem.5.3). We take a basis } \{e_1, \ldots, e_p, \tilde{e}_1, \ldots, \tilde{e}_q\} \text{ of } t_{V, C} \text{ as }
\]
\[e_i(\text{diag}(t_1, \ldots, t_n)) := t_i, \quad \tilde{e}_i(\text{diag}(t_1, \ldots, t_n)) := t_{p+i}
\]
and identify $t_{V, C}$ with $\mathbb{C}^n$ by this basis.

\[(ii) \quad \lambda \in t_{V, C}^* \text{ is of the form } \lambda = (a_1^p, \ldots, a_r^q; a_1^q, \ldots, a_r^q)/2, \quad a_1 > a_2 > \cdots > a_r \in n + 1 + 2\mathbb{Z}. \text{ Here } a^p \text{ denotes the } p-\text{copy } (a, \ldots, a) \text{ of } a \text{ and } \sum_{j=1}^r p_j = p, \sum_{j=1}^r q_j = q.
\]

\[(iii) \quad \Psi \text{ is a positive system in } R(g_{V, C}, t_{V, C}), \text{ the root system of } t_{V, C} \text{ in } g_{V, C}, \text{ satisfying the following conditions: }
\]
\[(a) \quad \alpha^\vee(\lambda) \geq 0, \alpha \in \Psi;
\]
\[(b) \quad \text{If a simple root } \alpha \text{ for } \Psi \text{ satisfies } \alpha(\lambda) = 0, \text{ it must be non-compact } \alpha \in R(p_{V, C}, t_{V, C}).
\]

\[(iv) \quad \Lambda \text{ is a character of } T_V \text{ given by } \Lambda = \lambda + \rho(\Psi_{\text{ncpt}}) - \rho(\Psi_{\text{cpt}}). \text{ Here } \Psi_{\text{ncpt}} := \Psi \cap R(p_{V, C}, t_{V, C}), \Psi_{\text{cpt}} := \Psi \cap R(t_{V, C}, t_{V, C}) \text{ and } \rho(\Sigma) \text{ is the half of the sum of roots in } \Sigma.
\]

$X(\gamma)$ has the unique minimal $K_V$-type with the $\tilde{b}_V$-highest weight $\Lambda$, where $\tilde{b}_V$ denotes the upper-triangular Borel subalgebra in $t_{V, C}$ (cf. Lem.5.3). Moreover, $X(\gamma)$ and $X(\gamma')$ are isomorphic if and only if the Langlands data $\gamma$ and $\gamma'$ are $K_V$-conjugate. Thus in the above, we may assume

$$
\Psi_{\text{cpt}} = \{e_i - e_j \mid 1 \leq i < j \leq p\} \cup \{\tilde{e}_i - \tilde{e}_j \mid 1 \leq i < j \leq q\}.
$$

Since a limit of discrete series $X(\gamma)$ is determined by $(\lambda, \Psi)$ in $\gamma$, we write $\pi(\lambda, \Psi)$ for this.

As before, we fix a character pair $\xi = (\xi, \xi')$ such that

$$
\xi(z) = \left(\frac{z}{\bar{z}}\right)^{m/2}, \quad \xi'(z) = \left(\frac{\bar{z}}{z}\right)^{m'/2}, \quad m \equiv m' \equiv n \pmod{2}.
$$

Let $\pi(\lambda, \Psi)$ be a limit of discrete series representation of $G_V$. Then we can write

\[
\lambda = -\left(\frac{m}{2}, \ldots, \frac{m}{2} - \frac{q}{2}, -\frac{q}{2}, \ldots, -\frac{q}{2}\right) + \frac{1}{2} \left(\frac{a_1^{k_1}, \ldots, a_r^{k_r}, -b_s^{k_1}, \ldots, -b_s^{k_t}; a_1^{\ell_1}, \ldots, a_r^{\ell_r}, -b_s^{\ell_1}, \ldots, -b_s^{\ell_t}}{p, q}\right)
\]

(6.1)
with \( a_1 > \cdots > a_r, b_1 > \cdots > b_s \in 2\mathbb{N} + 1 \). Writing
\[
k := \sum_{i=1}^{r} k_i, \quad \ell := \sum_{i=1}^{s} \ell_i, \quad \bar{k} := \sum_{i=1}^{r} \bar{k}_i, \quad \bar{\ell} := \sum_{i=1}^{s} \bar{\ell}_i,
\]
we set
\[
p' := k + \bar{\ell}, \quad q' := \bar{k} + \ell.
\]

(6.2)

Let us write \((W, \langle \cdot, \cdot \rangle)\) for the skew-hermitian space of signature \((p', q')\) realized as in \(\text{(5.1)}\) Also let \(t_W\) be as in Lem.\(\text{5.3}\) and take a basis \(\{\varepsilon'_1, \ldots, \varepsilon'_p; \varepsilon''_1, \ldots, \varepsilon''_q\}\) of \(t_W\) as in (i) above. We set
\[
\lambda' := \left( \frac{m'_1}{2}, \ldots, \frac{m'_1}{2}, \frac{q'}{2}, \ldots, \frac{q'}{2} \right)
\]
\[
\lambda' := \left( \frac{m'_1}{2}, \ldots, \frac{m'_1}{2}, \frac{q'}{2}, \ldots, \frac{q'}{2} \right) + \frac{1}{2} \left( a^{k_1}, \ldots, a^{k_r}, -b^{k_s}, \ldots, -b^{k_s}; a^{\bar{k}_1}, \ldots, a^{\bar{k}_s}, -b^{\bar{k}_s}, \ldots, -b^{\bar{k}_s} \right).
\]

\(\Psi' \in R(g_{W;C}, t_{W;C})\) denotes the positive system defined by
\[
\Psi' \ni \varepsilon'_i - \varepsilon'_j \iff \begin{cases} 
(i) & 1 \leq i \leq k, 1 \leq j \leq \bar{k} \text{ and } e_i - \varepsilon_j \in \Psi \\
(ii) & 1 \leq i \leq k, \bar{k} < j \leq q' \text{ or } e_i - \varepsilon_j \in \Psi \\
(iii) & k < i \leq p', \bar{k} < j \leq q' \text{ and } e_i - \varepsilon_j \in \Psi
\end{cases}
\]

(6.4)

Now we can state the following. (Compare this with \([\text{Pau98}, \text{Lem.5.2.5, Th.6.1}]\).)

**Proposition 6.1.** (1) For a limit of discrete series representation \(\pi(\lambda, \Psi)\) of \(G_V\) as above, we have \(\theta_\varepsilon(\pi(\lambda, \Psi)^\vee, W') = 0\) for any \(n\)-dimensional skew-hermitian space whose signature is not \((p', q')\). Here \(\pi^\vee\) denotes the contragredient of \(\pi\).
(2) For \(W\) of signature \((p', q')\) as above, we have \(\theta_\varepsilon(\pi(\lambda, \Psi)^\vee, W) = \pi(\lambda', \Psi')\).

**Proof.** The proof consists of two steps. First we restrict the possible correspondence by considering the \(K\)-type correspondence, and then show that the expected correspondence actually occurs by the refined induction principle of Adams-Barbasch \([\text{AB95}, \text{Prop.3.25}],[\text{Pau98}, \text{Th.4.6.6}]\). Since these follow completely the same argument as in \([\text{Pau98}]\) and our purpose is only to compare our setting with hers, it suffices to compute the \(K\)-type correspondence between the minimal \(K\)-types of \(\pi(\lambda, \Psi)^\vee\) and \(\pi(\lambda', \Psi')\).

We start with the calculation of the \(b_V\)-highest weight \(\Lambda\) of the minimal \(K_V\)-type in \(\pi(\lambda, \Psi)\). Taking the decomposition \(R(\Psi_V, t_V) = R^1_V \sqcup R^2_V \sqcup R^3_V\):
\[
R^1_V := \{ \pm (e_i - \varepsilon_j) \mid 1 \leq i \leq k, 1 \leq j \leq \bar{k} \}
\]
\[
R^2_V := \left\{ \pm (e_i - \varepsilon_j) \mid 1 \leq i \leq k, \bar{k} < j \leq q \text{ or } \begin{array}{l}
1 \leq i \leq \bar{k}, 1 \leq j \leq \bar{k} \\
\end{array} \right\}
\]
\[
R^3_V := \{ \pm (e_i - \varepsilon_j) \mid k < i \leq p, \bar{k} < j \leq q \},
\]
we set \(A := \Psi \cap R^1_V, B := \Psi \cap R^2_V, C := \Psi \cap R^3_V,\) so that \(\Psi_{\text{ncp}} = A \sqcup B \sqcup C.\) The condition (iii) (a) on \(\Psi\) forces
\[
B = \{ e_i - \varepsilon_j \mid 1 \leq i \leq k, \bar{k} < j \leq q \} \cup \{ \bar{e}_j - e_i \mid k < i \leq p, 1 \leq j \leq \bar{k} \}.
\]
For $X = A, B, C$, we write $X_+(i) := \# \{ j \mid e_i - e_j \in X \}$, $X_-(j) := \# \{ j \mid e_j - e_i \in X \}$. It follows from definition that

\[
A_+(i) = \# \{ 1 \leq j \leq \bar{k} \mid e_i - e_j \in \Psi \}, \quad (1 \leq i \leq k),
A_-(j) = \# \{ 1 \leq i \leq \bar{k} \mid e_j - e_i \in \Psi \}, \quad (1 \leq j \leq \bar{k}),
\]

\[
B_+(i) = \begin{cases} 
\bar{\ell} & \text{if } 1 \leq i \leq k, \\
0 & \text{if } k < i \leq p,
\end{cases}
B_-(j) = \begin{cases} 
\ell & \text{if } 1 \leq j \leq \bar{k}, \\
0 & \text{if } \bar{k} < j \leq q,
\end{cases}
\]

\[
C_+(i) = \# \{ \bar{k} < j \leq q \mid e_i - e_j \in \Psi \}, \quad (k < i \leq p),
C_-(j) = \# \{ k < i \leq p \mid e_j - e_i \in \Psi \}, \quad (\bar{k} < j \leq q).
\]

Using the vectors

\[
A_+ := (A_+(1), \ldots, A_+(k)), \quad A_- := (A_-(1), \ldots, A_-(\bar{k})),
C_+ := (C_+(k + 1), \ldots, C_+(p)), \quad C_- := (C_-(\bar{k} + 1), \ldots, C_-(q)),
\]

we can write

\[
\rho(\Psi_{\text{ncpt}}) = \frac{1}{2}(2A_+ - \bar{k} \sigma_{\bar{k}}; 2A_- - k \sigma_k, 0) + \frac{1}{2}(\bar{\ell} \sigma_{\bar{k}}; \ell, -k)
\]

\[
+ \frac{1}{2}(0, 2C_+ - \bar{\ell}; 0, 2C_- - \ell)
\]

\[
= \left( A_+ + \frac{\bar{\ell} - \bar{k}}{2}, C_+ - \frac{q}{2}; A_- + \frac{\ell - k}{2}, C_- - \frac{p}{2} \right).
\]

Let $\rho_k = 2^{-1}(k - 1, k - 3, \ldots, 1 - k)$ be the half sum of the standard positive roots of $\mathfrak{g}l(k, \mathbb{C})$. Then (6) shows

\[
\rho(\Psi_{\text{cpt}}) = (\rho_p, \rho_q) = \left( \frac{p - k}{2} + \rho_k, \frac{\ell - p}{2} + \rho_\ell \sigma_\ell; \frac{q - k}{2} + \rho_{\bar{k}} \sigma_{\bar{k}}, \frac{\bar{\ell} - q}{2} + \rho_{\bar{\ell}} \right),
\]

and hence

\[
\rho(\Psi_{\text{ncpt}}) - \rho(\Psi_{\text{cpt}}) = \left( A_+ - \rho_k + \frac{\bar{\ell} - \bar{k} + k - p}{2}, C_+ - \rho_\ell + \frac{p - \ell - q}{2}; A_- - \rho_{\bar{k}} + \frac{\ell - k + \bar{k} - q}{2}, C_- - \rho_{\bar{\ell}} + \frac{q - \bar{\ell} - p}{2} \right)
\]

\[
= \left( A_+ - \rho_k - \bar{k} + \frac{k}{2}; C_+ - \rho_\ell - \frac{\ell}{2}; A_- - \rho_{\bar{k}} - \bar{k} + \frac{k}{2}, C_- - \rho_{\bar{\ell}} - \frac{\bar{\ell}}{2} \right) - \left( \frac{p - q}{2}, \frac{q - p}{2}; \frac{p - q}{2}, \frac{q - p}{2} \right).
\]

Thus we conclude

\[
\Lambda = \left( \frac{p' - q' - m}{2}, \frac{p' - q' - m}{2}; \frac{q' - p' - m}{2}, \frac{q' - p' - m}{2} \right)
\]

\[
+ \frac{1}{2} \left( a_1^{k_1}, \ldots, a_r^{k_r}, -b_1^{\ell_1}, \ldots, -b_1^{\ell_1}; a_1^{\bar{k}_1}, \ldots, a_r^{\bar{k}_r}, -b_1^{\bar{\ell}_1}, \ldots, -b_1^{\bar{\ell}_1} \right)
\]

\[
+ \left( A_+ - \rho_k - \bar{k} + \frac{k}{2}; C_+ - \rho_\ell - \frac{\ell}{2}; A_- - \rho_{\bar{k}} - \bar{k} + \frac{k}{2}, C_- - \rho_{\bar{\ell}} - \frac{\bar{\ell}}{2} \right)
\]

\[
- \left( \frac{p - q + p' - q'}{2}, \frac{q - p + q' - p'}{2}; \frac{q - p + q' - p'}{2}, \frac{q - p + q' - p'}{2} \right).
\]

28
Consequently, the $b_{V;C}$-highest weight of the minimal $K_V$-type of $\pi(\lambda, \Psi)^r$ is given by

\[
\Lambda^r = \left(\frac{m+q'-p'}{2}, \frac{m+q'-p'}{2}, \frac{m+p'-q'}{2}, \frac{m+p'-q'}{2}\right) + \frac{1}{2} \left(-a_{r}^{i_1}, \ldots, -a_{r}^{i_r}, b_{s}^{i_1}, \ldots, b_{s}^{i_s}, -a_{r}^{i_1}, \ldots, -a_{r}^{i_r}, b_{s}^{i_1}, \ldots, b_{s}^{i_s}\right) - \left(A_+ - \rho_k - \frac{\ell}{2}, C_+ - \rho_{e} - \frac{\ell}{2}; A_+ - \rho_k - k + \frac{\ell}{2}, C_+ - \rho_{e} - \frac{\ell}{2}\right) + \left(\frac{p-q+p'-q'}{2}, \frac{q-p+p'-q'}{2}, \frac{p-q+q'-p'}{2}\right).
\]

(6.5)

Next we calculate $\Lambda' = \Lambda^r + \rho(\Psi'_{\text{ncpt}}) - \rho(\Psi'_{\text{cpt}})$. Again we introduce $A' := \Psi' \cap R_{1_W}^i$, $B' := \Psi' \cap R_{2_W}^i$, $C' := \Psi' \cap R_{3_W}^i$ with

\[
R_{1_W}^i := \{\pm(e'_i - e'_j) \mid 1 \leq i \leq k, 1 \leq j \leq \bar{k}\},
\]
\[
R_{2_W}^i := \left\{\pm(e'_i - e'_j) \left| 1 \leq i \leq k, \bar{k} < j \leq q' \right. \text{ or } k < i \leq p', 1 \leq j \leq \bar{k}\right\},
\]
\[
R_{3_W}^i := \{\pm(e'_i - e'_j) \mid k < i \leq p', k < j \leq q'\}.
\]

This time, $X'_{+}(i) := \# \{j \mid e_i - e_j \in X\}$, $X'_{-}(j) := \# \{j \mid e_j - e_i \in X\}$, $(X = A, B, C)$ are given by

\[
A'_{+}(i) = \# \{1 \leq j \leq \bar{k} \mid e_i - e_j \in \Psi\} = A_{+}(i), \quad (1 \leq i \leq k),
\]
\[
A'_{-}(j) = \# \{1 \leq i \leq k \mid e_j - e_i \in \Psi\} = A_{-}(j), \quad (1 \leq j \leq \bar{k}),
\]
\[
B'_{+}(i) = \left\{\ell \mid 1 \leq i \leq k, B'_{-}(j) = \left\{\bar{\ell} \mid 1 \leq j \leq \bar{k}, 0 \mid k < i \leq p', 0 \mid k < j \leq q', C'_{+}(i) = \# \{\bar{k} < j \leq q' \mid e_{i-k+k} - e_{j-k+k} \in \Psi\} = \# \{k < j \leq p \mid e_{i-k+k} - e_j \in \Psi\} = C_{+}(i - k + \bar{k}), \quad (k < i \leq p'),
\right\}
\]
\[
C'_{-}(j) = \# \{k < i \leq p' \mid e_{j-k+k} - e_{i-k+k} \in \Psi\} = \# \{\bar{k} < i \leq q \mid e_{j-k+k} - e_i \in \Psi\} = C_{-}(j - \bar{k} + k), \quad (\bar{k} < j \leq q').
\]

These show

\[
\rho(\Psi'_{\text{ncpt}}) = \frac{1}{2}(2A'_{+} - \bar{k}, 0; 2A'_{-} - k, 0) + \frac{1}{2}(\ell, -\bar{k}; \ell, -k) + \frac{1}{2}(0, 2C'_{+} - \bar{\ell}; 0, 2C'_{-} - \ell) = \left(A_{+} + \frac{\ell - \bar{k}}{2}, C_{+} - \frac{q'}{2}; A_{+} + \frac{\ell - k}{2}, C_{+} - \frac{p'}{2}\right),
\]
\[
\rho(\Psi'_{\text{cpt}}) = \rho(p', p_{q'}) = \left(\frac{p' - \bar{k}}{2} + \rho_{k}, \frac{\ell - p'}{2} + \rho_{\ell}; \frac{q' - \bar{k}}{2} + \rho_{\ell}, \frac{\ell - q'}{2} + \rho_{\ell}\right),
\]

29
and hence we obtain

\[
\Lambda' = \left( \frac{m' + p - q}{2}, \frac{m' + p - q}{2}, \frac{m' + q - p}{2}, \frac{m' + q - p}{2} \right) \\
+ \frac{1}{2} \left( a_1^{k_1}, \ldots, a_r^{k_r}, -b_1^{k_1}, \ldots, -b_r^{k_r}; a_1^{\bar{k}_1}, \ldots, a_r^{\bar{k}_r}, -b_1^{\bar{k}_1}, \ldots, -b_r^{\bar{k}_r} \right) \\
+ \left( A_+ - \rho_k - \bar{k} + \frac{k}{2}, C_- - \rho_{\bar{k}} - \bar{\ell} + \frac{\ell}{2}; A_- - \rho_k - k + \frac{k}{2}, C_+ - \rho_{\bar{k}} - \bar{\ell} + \frac{\ell}{2} \right) \\
- \left( \frac{p - q + p' - q'}{2}, \frac{p - q + q' - p'}{2}; \frac{q - p + q' - p'}{2}, \frac{q - p + p' - q'}{2} \right).
\]

This evidently corresponds to (6.5) under the correspondence in Th. 5.4. 

References


