Configuration space of Moebius Kaleidocycle

HSE International Laboratory of Algebraic Topology and Its Applications
Seminar, 8 Apr. 2022

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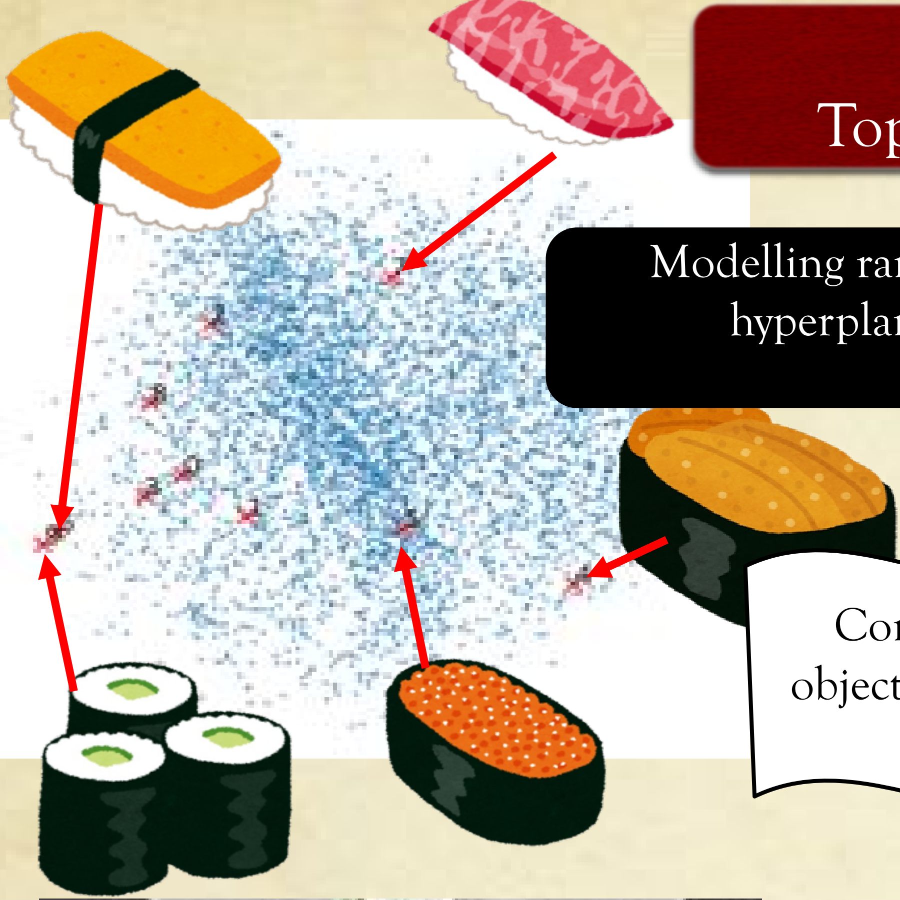
Articles, simulation codes, paper models, 3D printable models are available at
https://github.com/shizuo-kaji/Kaleidocycle

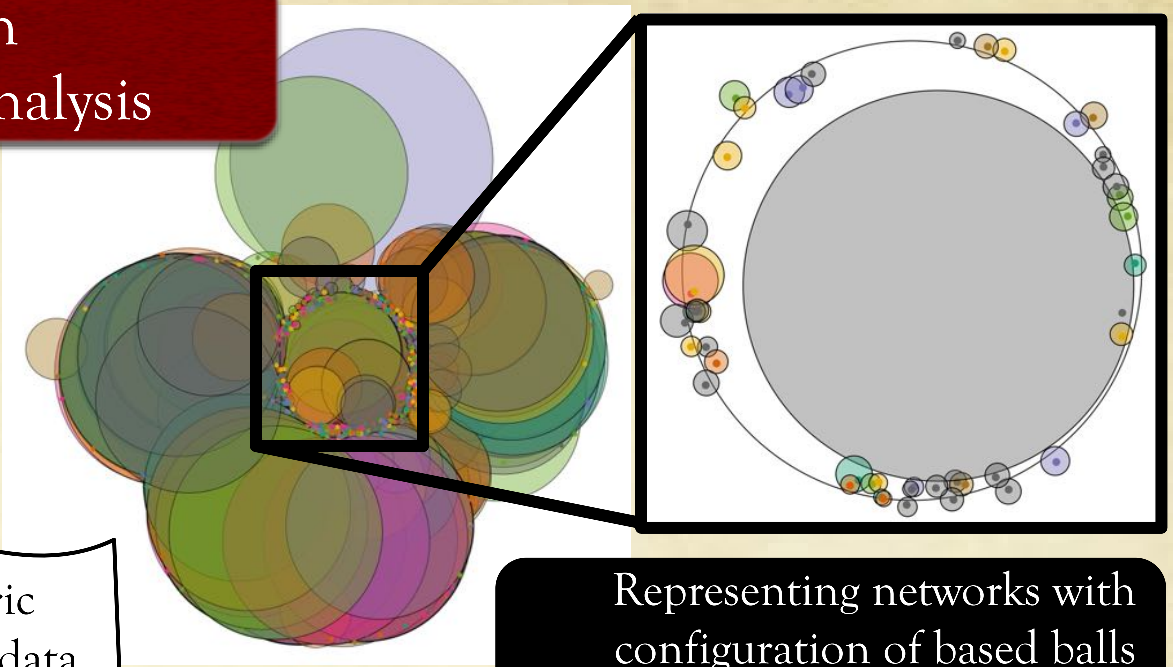

My research on
Shape Generation/Analysis
Shape similarity
Shape generation
Shape deformation
4D visualisation
Shape interpolation
Image manipulation

My research on Topological Data Analysis



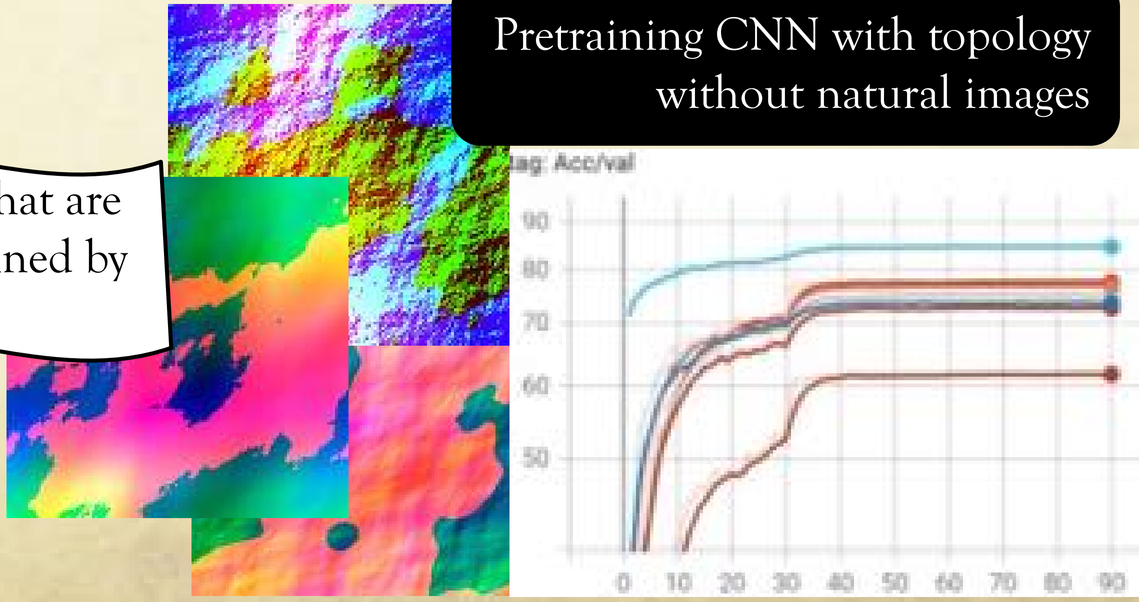
Modelling ranking data with hyperplane arrangement (SIGKDD ‘21)

Configuration of geometric objects as representation of data in machine learning



Representing networks with configuration of based balls (ICML ‘20)

Pretraining CNN with topology without natural images



Homological image features that are complementary to those obtained by convolution

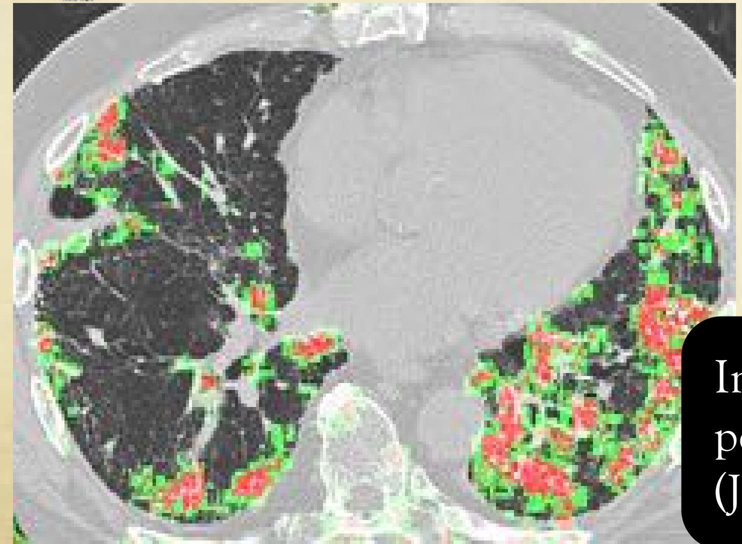


Image segmentation with persistent homology (J Appl Physiol. 2021)

Advert

Cubical Ripser (with T. Sudo and K. Ahara)

- Fast and easy-to-use software for computing persistent homology of images and volume data.

  https://github.com/shizuo-kaji/CubicalRipser_3dim

Tutorial for TDA libraries in Python

- Concrete and simple example task demonstration for various TDA packages that runs on Google Colab (no installation needed!)

  https://github.com/shizuo-kaji/TutorialTopologicalDataAnalysis

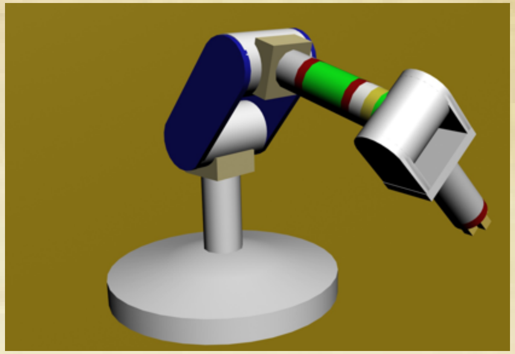
For Japanese

数学セミナー連載(2020/04 – 2021/03) かたちを算する

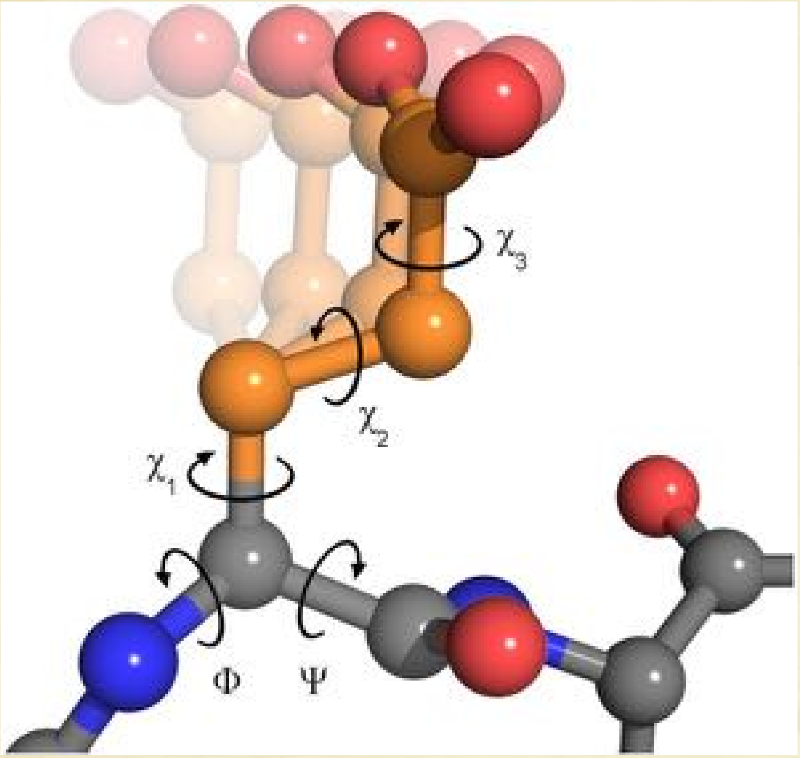
Introduction

Configuration of objects in the Euclidean space

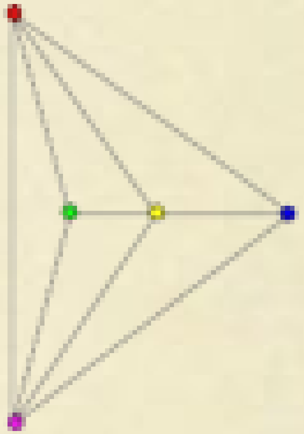
Configuration of objects in R^n

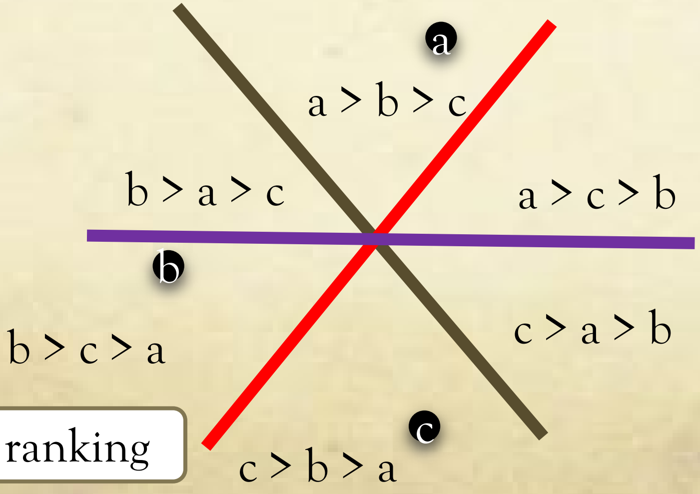


- Linkage (robot arm)
- Fixed-angle chain (protein folding)
- Moduli of polygon
- Graph embedding
- Geometric data representation



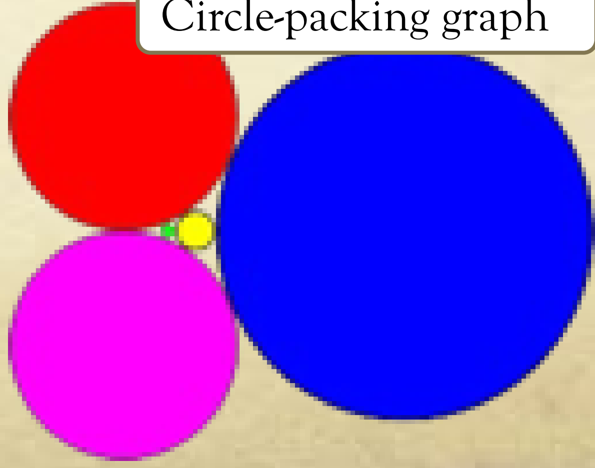
From Harder et al. 2010



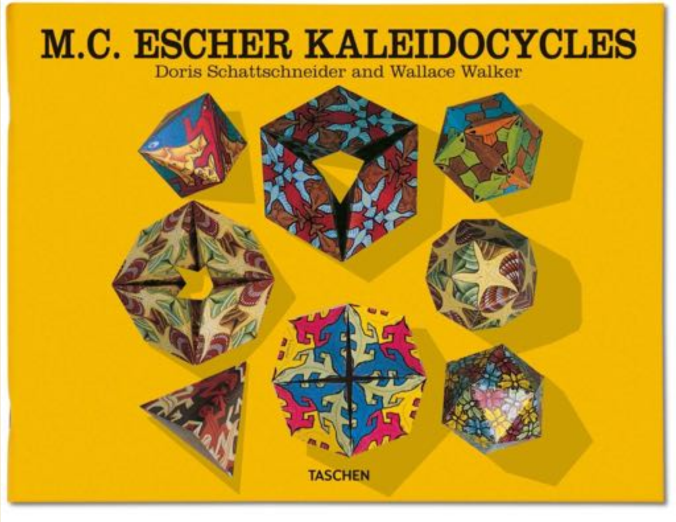


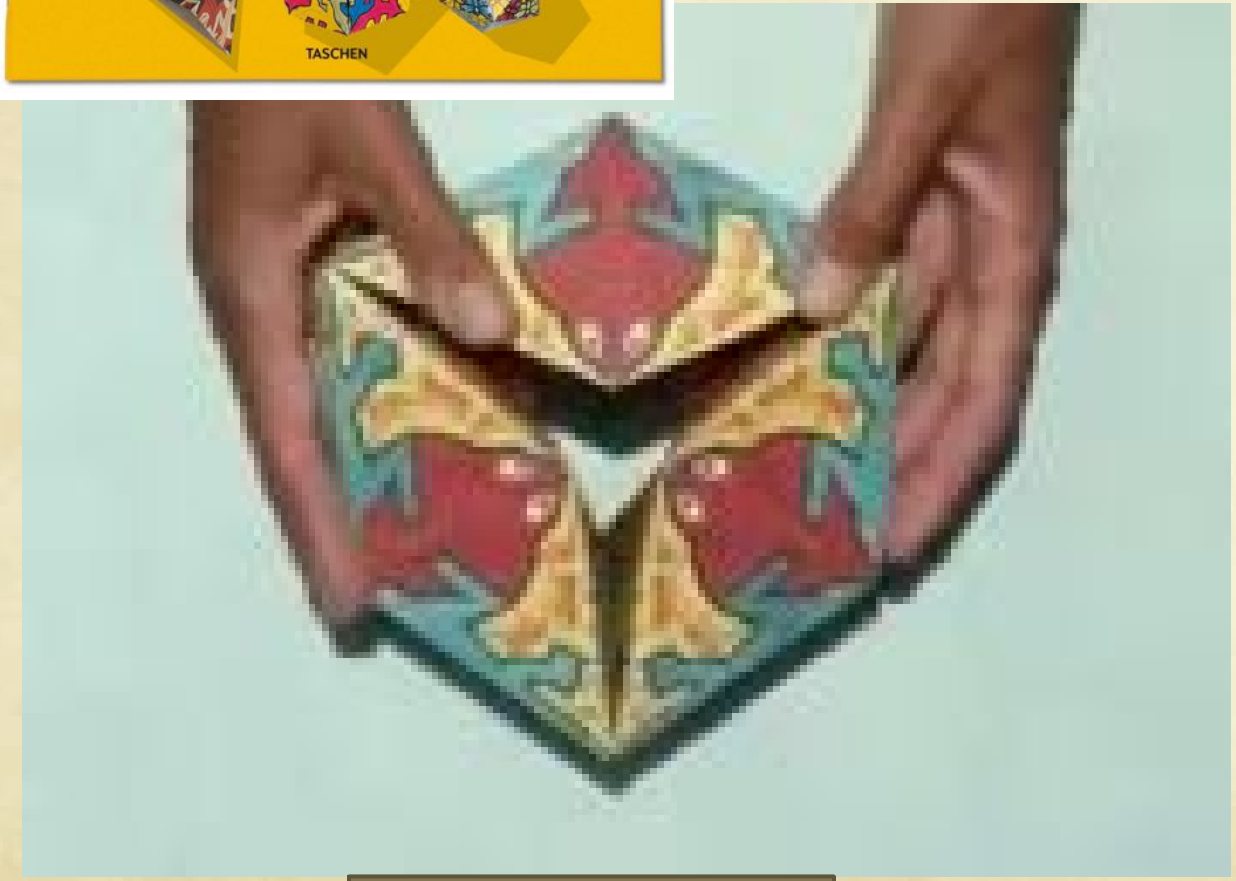
hyperplane arrangement and ranking

Circle-packing graph



Kaleidocycle



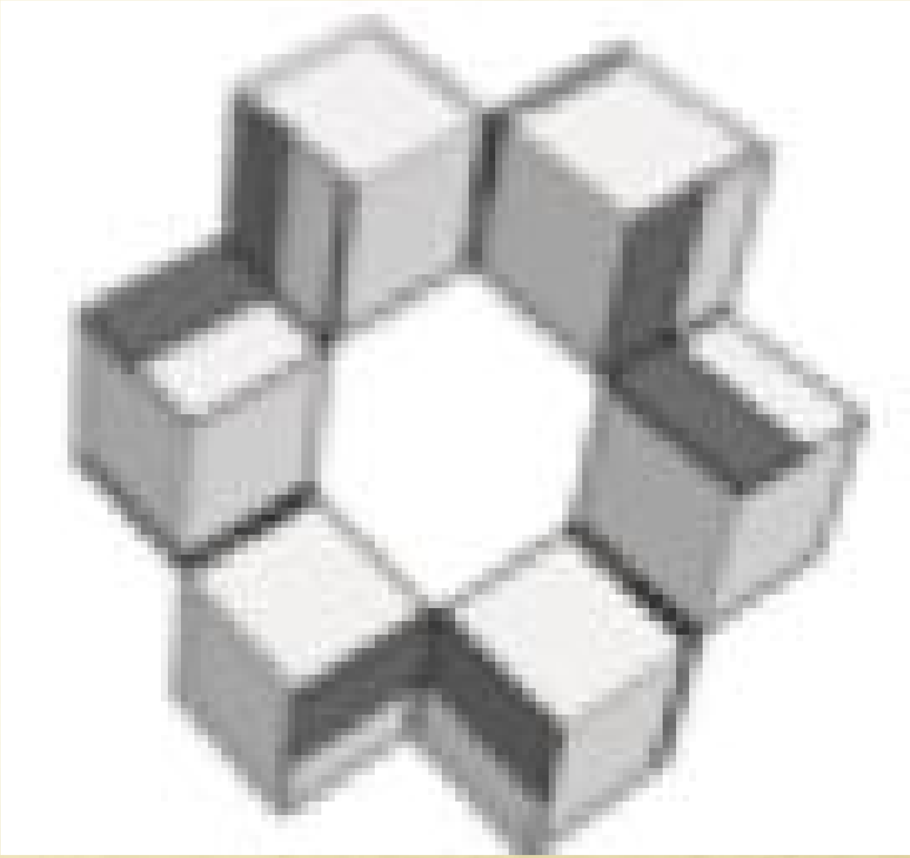


Video from Youtube

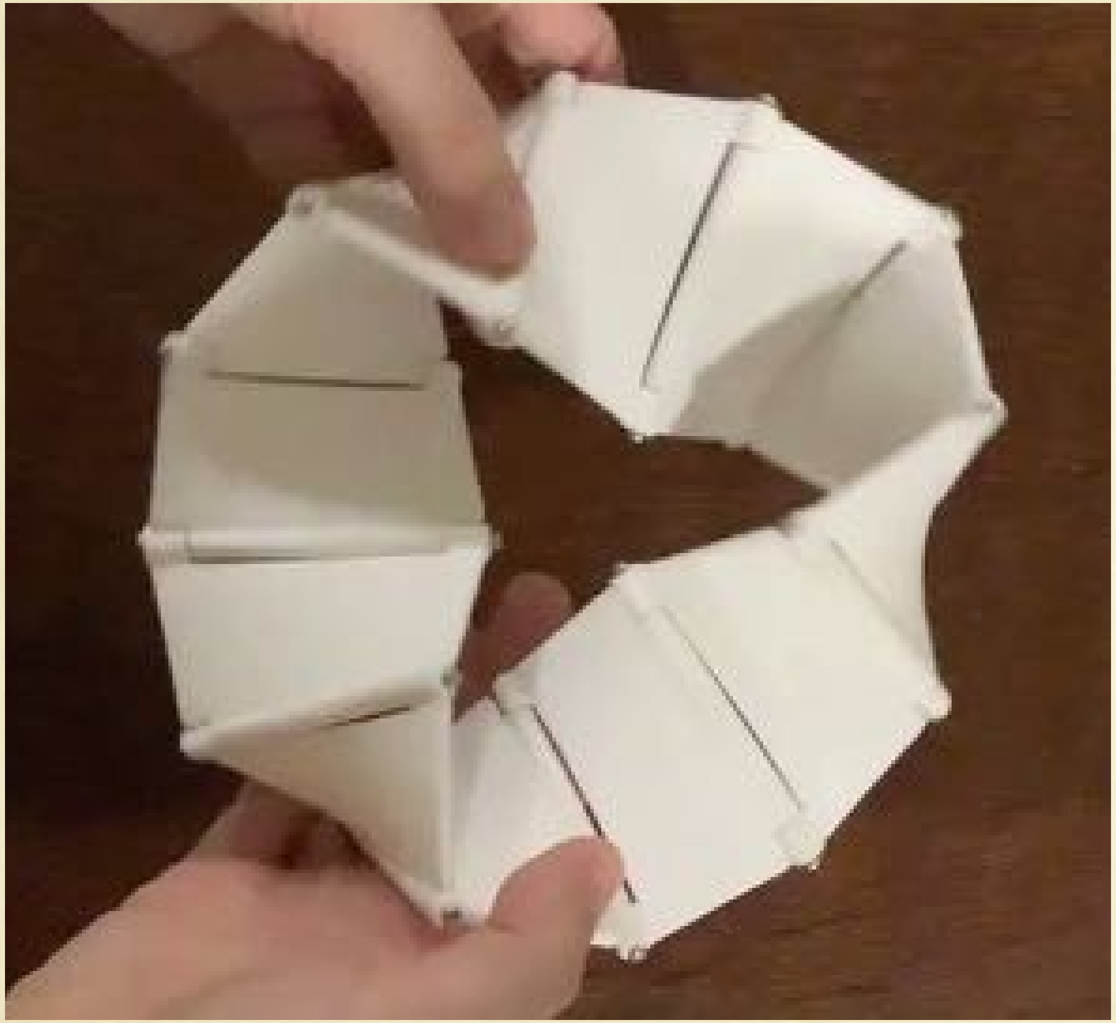
T. Tendo, 2019
The Variety of Kaleidocycles (excerpt)

Variation: Magic Cube

Video from Youtube



Variation: Mobius Kaleidocycle



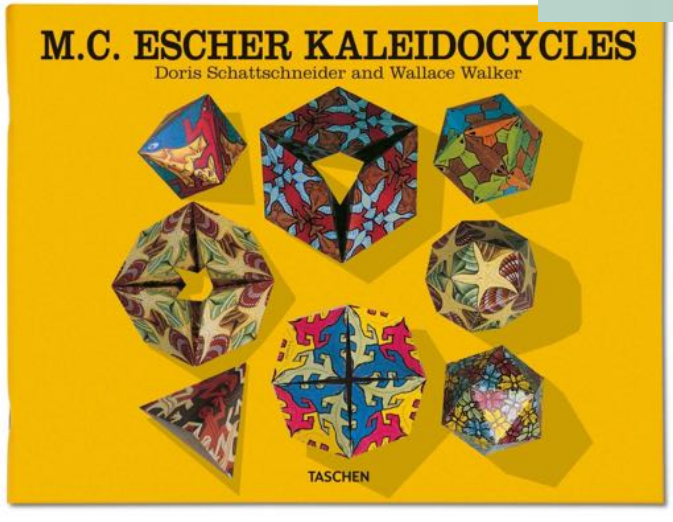
S. Kaji, J. Schoenke, E. Fried, M. Grunwald, ***Moebius Kaleidocycle***, Patent filed, JP2018-033395, Feb. 2018
The results of the joint patent were later published (without permission) in the following paper:
J. Schoenke , E. Fried, ***Single degree of freedom everting ring linkages with nonorientable topology***, PNAS 116 (1), 2019.

Kaleidocycle plays a role in

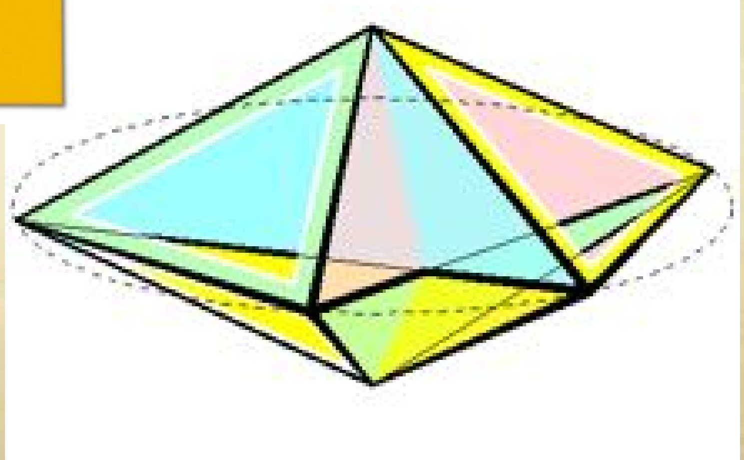
- Recreational maths
  - W. W. Rouse Ball (1939?)
  - "Mathematical recreations and essays"
  - D. Schattschneider and W. M. Walker (1985)
  - "M. C. Escher Kaleidocycles"

- Theory of Polytopes
  - Rigidity theorems
  - Bellows theorem
  - Rigid Origami

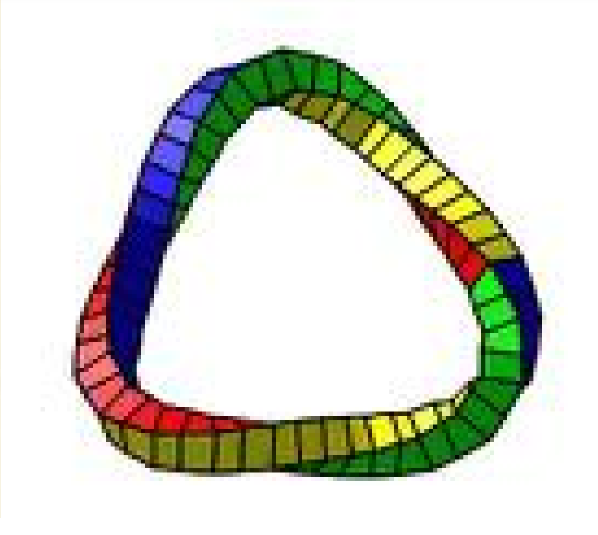
- Kinematics, Robotics
  - Bricard 6R linkage mechanism
    - violating Mobility formula



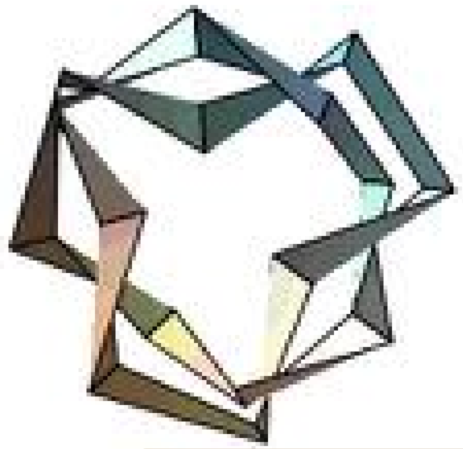
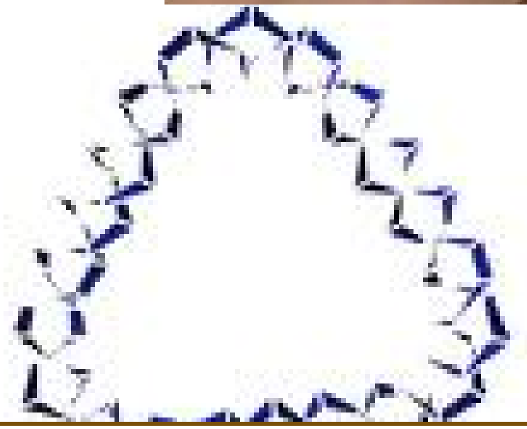
Bricard's octahedron

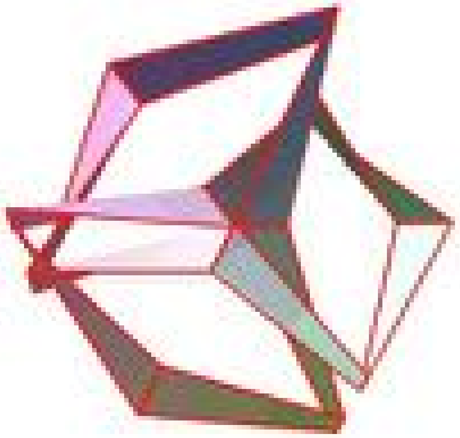


Today's topic Mobius Kaleidocycle







Mathematics： Intersection of topology, geometry, algebraic geometry, integrable system
Engineering：Linkage mechanism with many desirable properties: 1-DoF, constant energy, falling cat
Outreach：fun and tangible object that convey various levels of mathematics

Outline

Kinematic Chain (KC)
a maths model of transformable shape

Study of KC through
its Configuration Space

Kaleidocycle:
a special family of KC
<=>
constant torsional curve

Motion analysis of Kaleidocycle
a flow on its configuration space

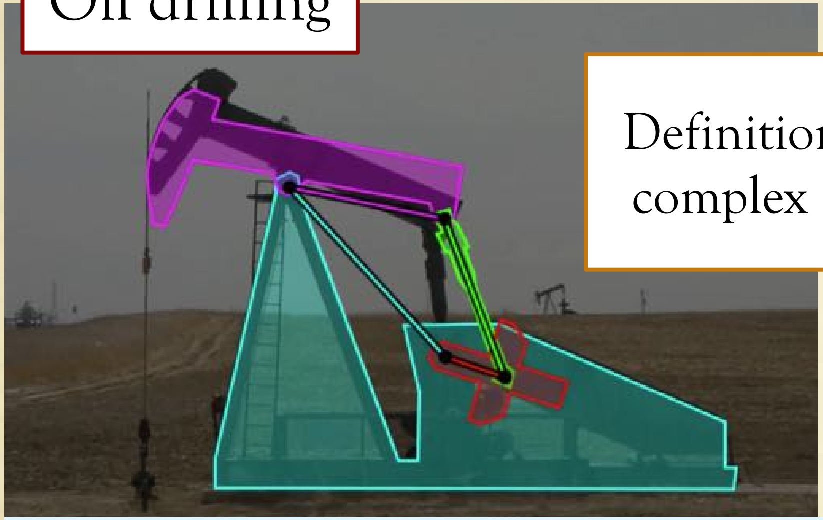
Kinematic Chains (KC)

Kinematic Chain (linkage mechanism)

A kinematic chain is a collection of rigid bodies (links) connected by joints.
It is used to transfer/transform motion.

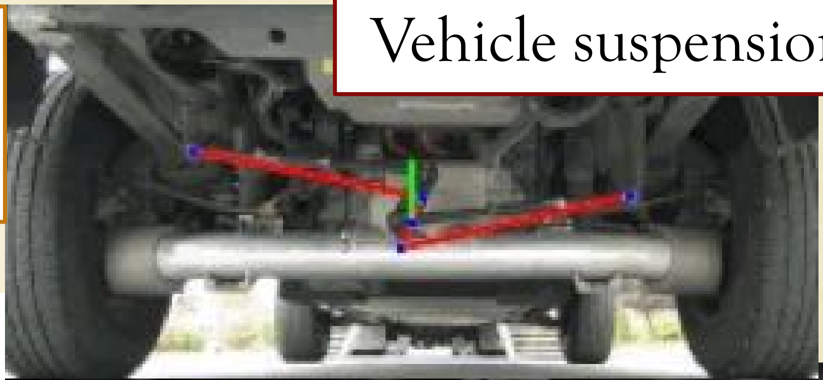
A kinematic chain is closed if the underlying graph contains a cycle.

Oil drilling

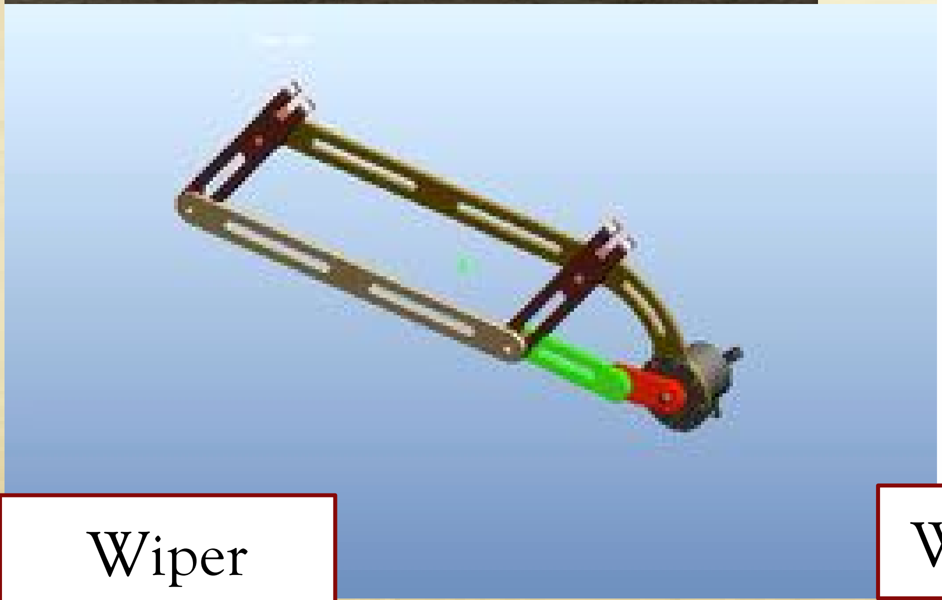


Definition: KC is a realisation of a filtered 1-dim complex as (the 1-skeleton of) the VR complex.

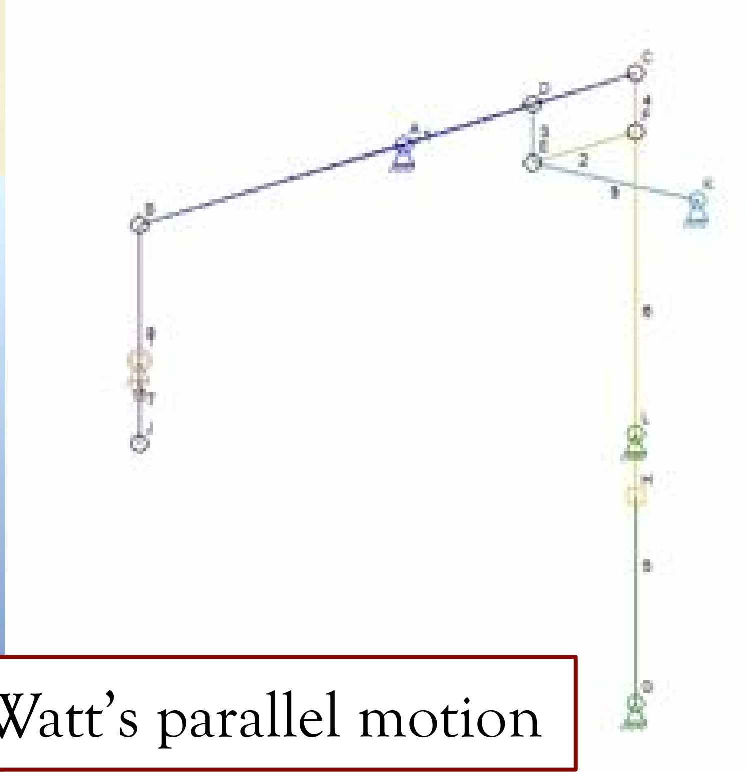
Vehicle suspension



Wiper

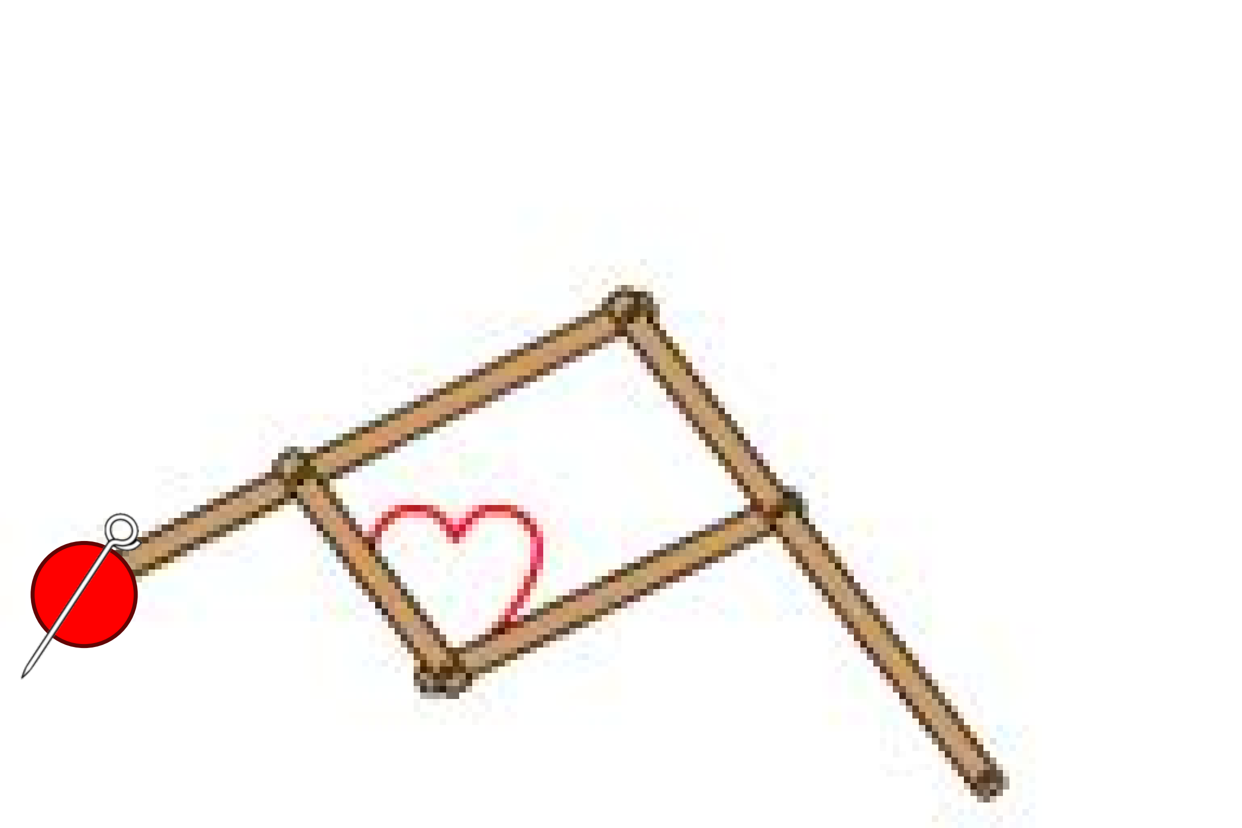


Watt's parallel motion



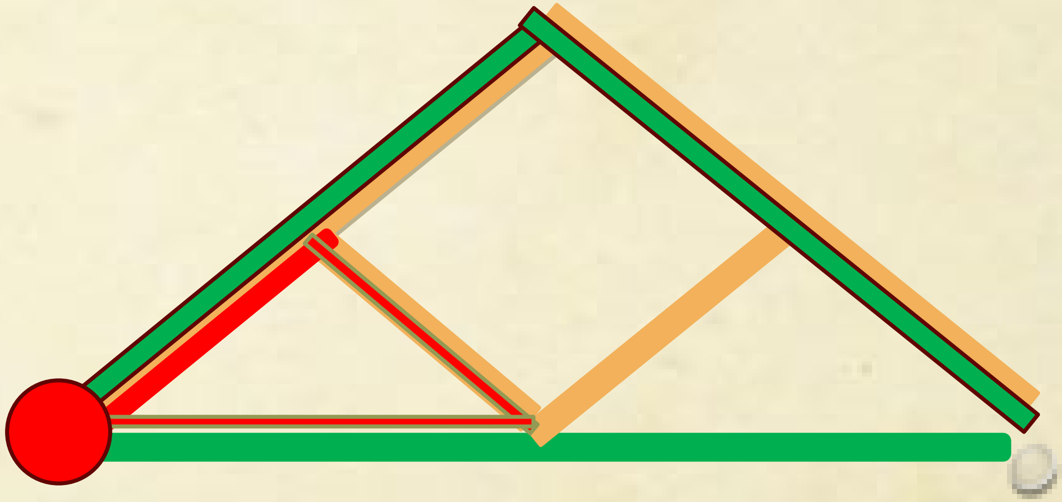
http://dynref.engr.illinois.edu/aml.html

How a pantograph works

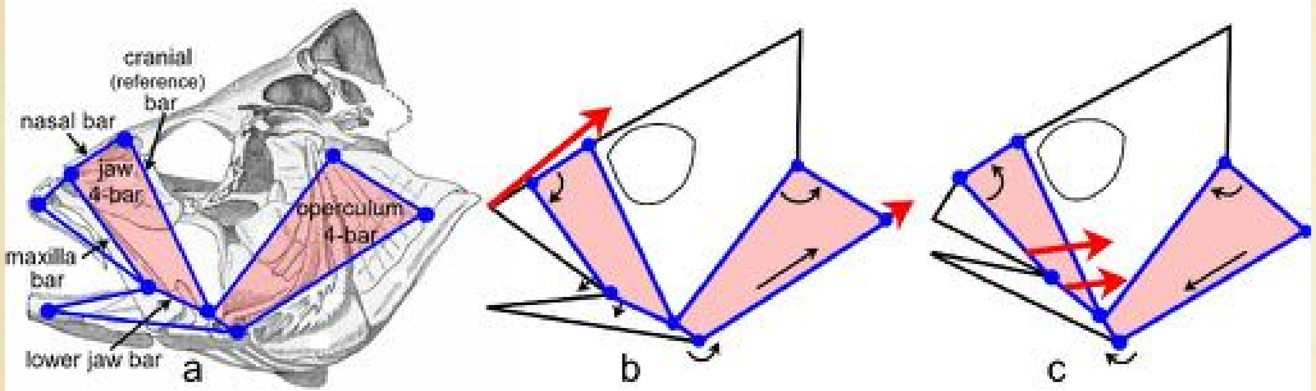


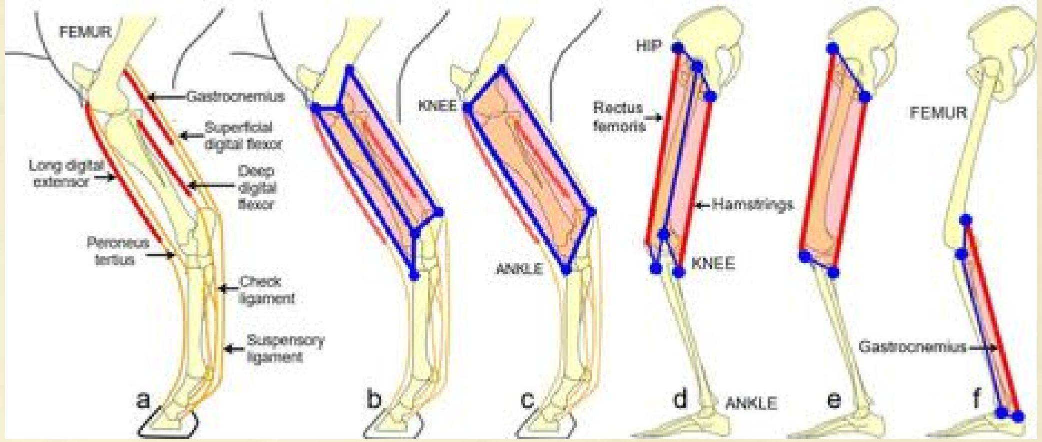
Pantogprah = Copier in Greek

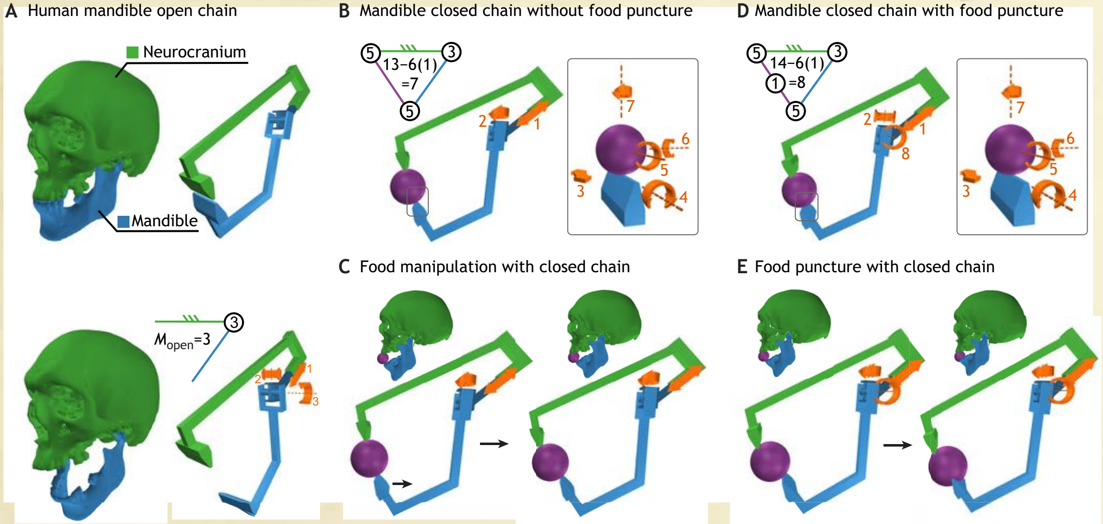
The large green and small red triangles are always similar!

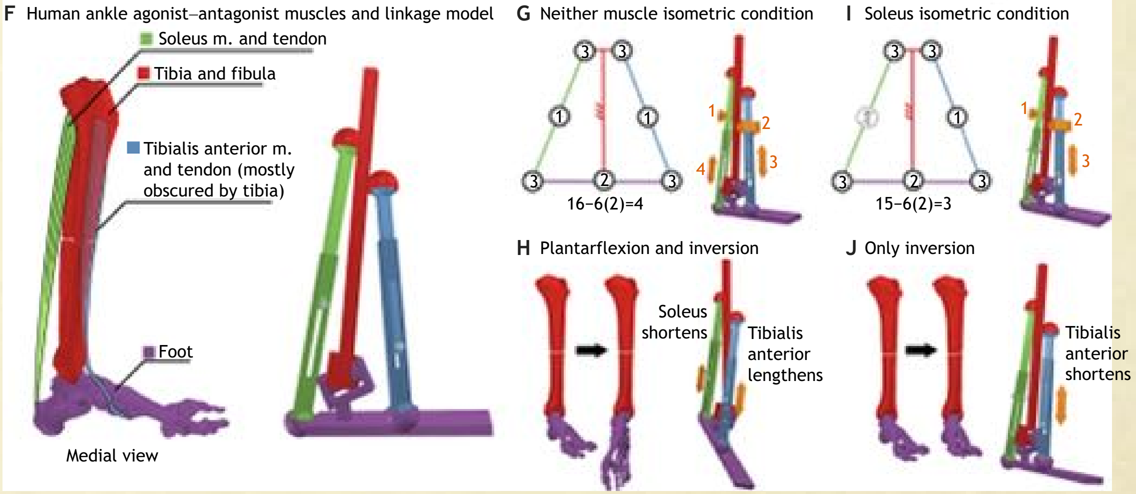


Closed Kinematic Chains in living creatures









- *The significance of closed kinematic chains to biological movement and dynamic stability*
  Stephen M LevinSusan Lowell de SolórzanoSusan Lowell de SolórzanoGraham ScarrGraham Scarr, 2017
- *A mobility-based classification of closed kinematic chains in biomechanics and implications for motor control*
  A. M. Olsen, 2019

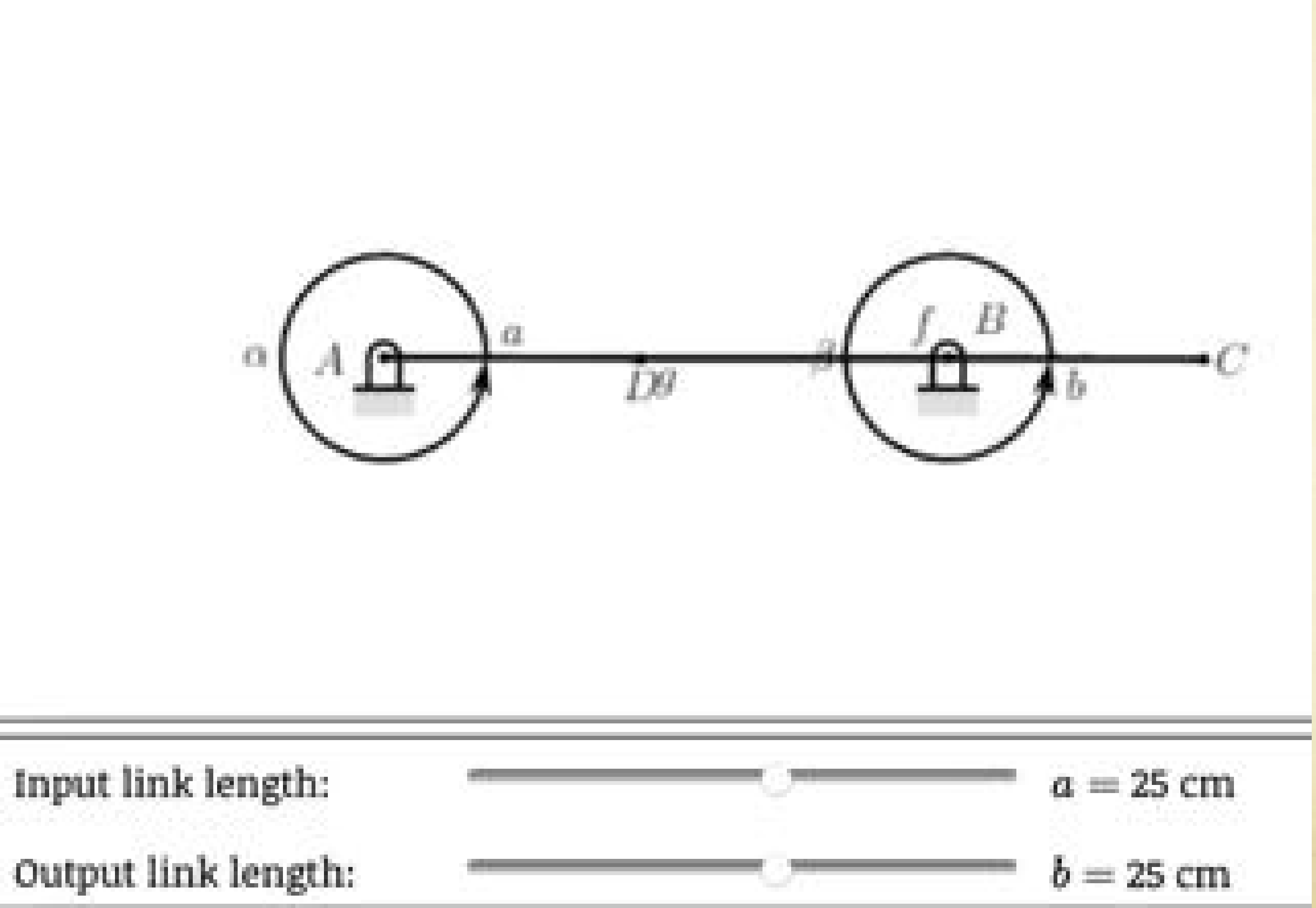
Fundamental Questions of KC

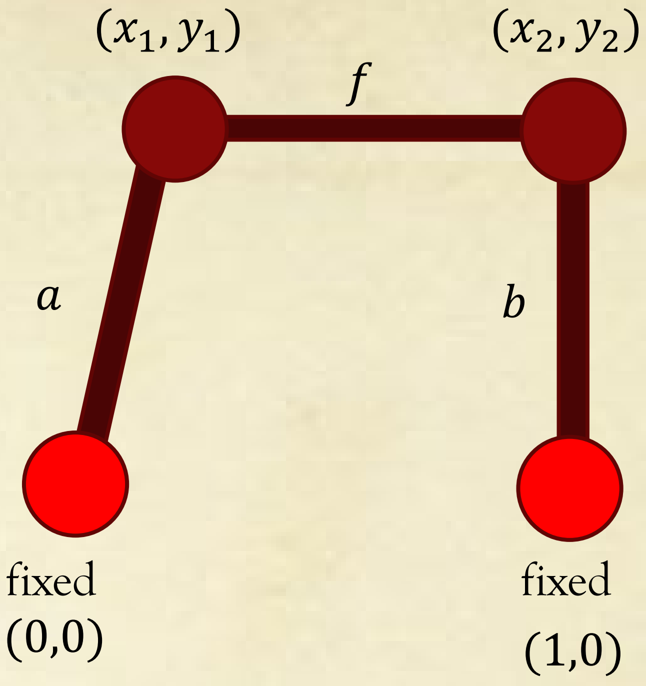
1. How many independent ways to change the state?
2. Is the given state flexible or stiff?
3. Is the motion easy to control? (cf. Miura folding)
4. Is there a continuous transition from the current state to the desired one?

These questions can be formulated and studied through the topology of the ***configuration space*** consisting of all states of a given KC.

The configuration space of KC is defined by equations

http://dynref.engr.illinois.edu/aml.html





States are identified with the **solution to a system of equations**

$$x_1^2 + y_1^2 - a^2 = 0$$
$$(x_2 - 1)^2 + y_2^2 - b^2 = 0$$
$$(x_1 - x_2)^2 + (y_2 - y_1)^2 - f^2 = 0$$

4 variables satisfying 3 equations
We expect 1-dimensional family of solutions

Fundamental example of the configuration space

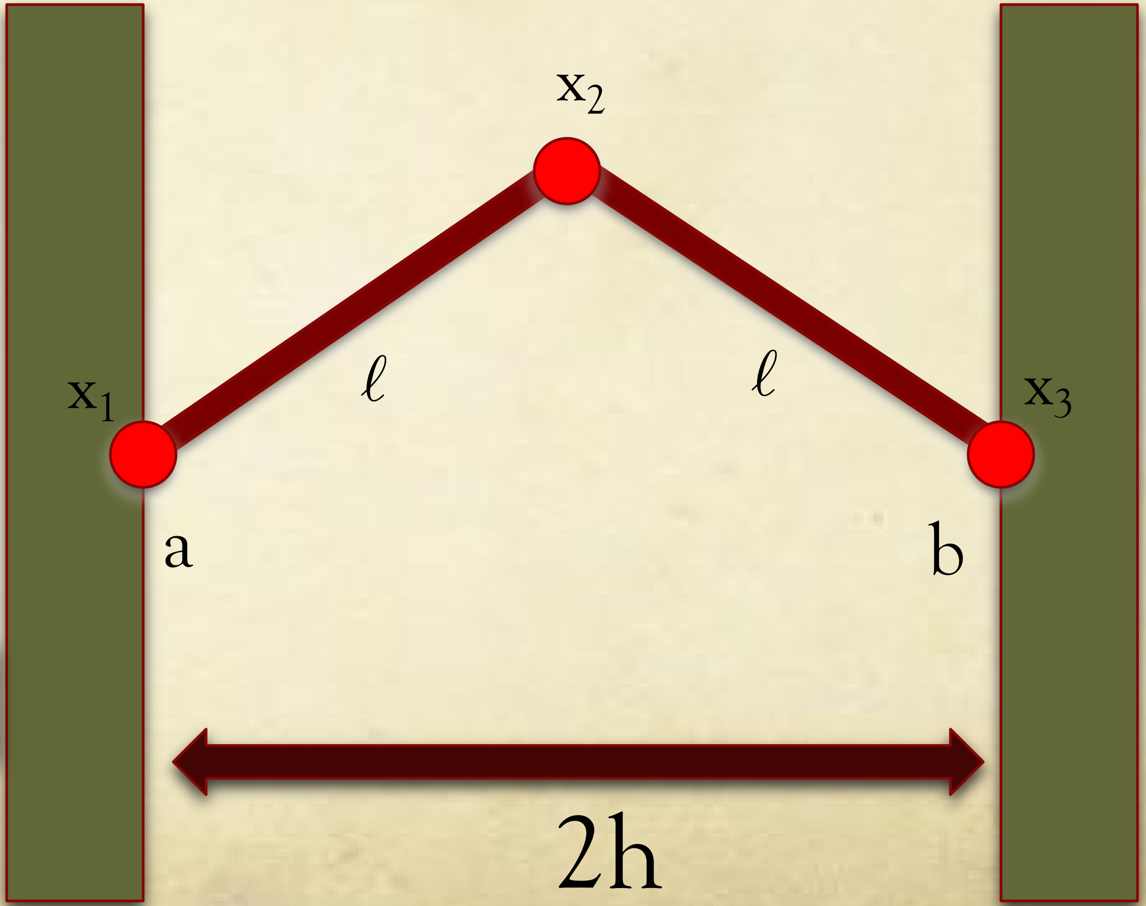
Consider the linkage below with 3 joints in R^3

Let h>0 be a constant and the two ends

$$a = (-h, 0, 0), \quad b = (h, 0, 0)$$

are fixed to the wall. Then

$$M_3(L) = \begin{cases} S^1 & (l > h) \\ * & (l = h) \\ \emptyset & (l < h) \end{cases}$$



Topology changes according to the parameter l

Topology of the configuration space

Topological invariants of the configuration space tells a lot about the mechanics:
here's a small dictionary

- path => motion
- fundamental group => non-trivial cyclic motion
- topological complexity => minimum number of "if-statements" for motion planning
- singular point => bifurcation, locking
- number of connected components => number of mutually unreachable configurations
- dimension => degree of freedom

The topic has been studied by many mathematicians including
Euler, Sylvester, Thurston, Niemann, Kapovich-Millson...
See Demaine & O'Rourke's excellent book "Geometric Folding Algorithms"

Configuration space of a bar Linkage

Caution
Self-collision is ignored

A bar linkage L is a finite simple graph (V,E) whose edges are assigned lengths l

The configuration space $M_m(L)$ is the space of isometric embeddings of L in R^m modulo the global symmetry.

$$\left\{\{x_1,\dots,x_{|V|}\}\subset \mathbb{R}^m \mid |x_{v_1}-x_{v_2}| = l(v_1v_2)\forall v_1v_2\in E\right\}/Euclidean(\mathbb{R}^m)$$

A state is described by the real solution to a system of quadratic equations

Each connected component corresponds to "motion" of the linkage

The most notable result on bar linage is

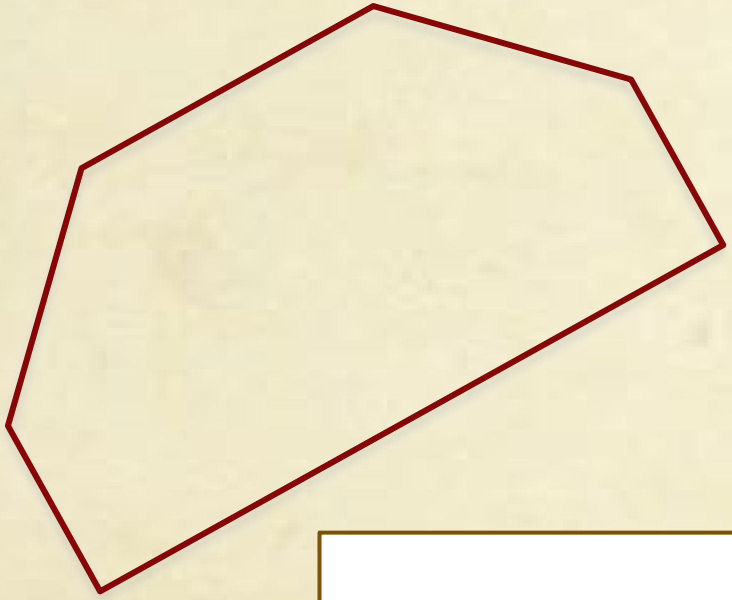
Thm (Kempe's universality theorem, proposed in 19C with a wrong proof, proved in 21C)

Any bounded plane algebraic curve can be traced by some linkage

(or "There is a linkage which can sign your name")

Millennium Prize Problem
phrased in terms of kinematic chain

- Even determining if the configuration space is non-empty is already difficult.
- In fact, it is so even for 1-dimensional KC.



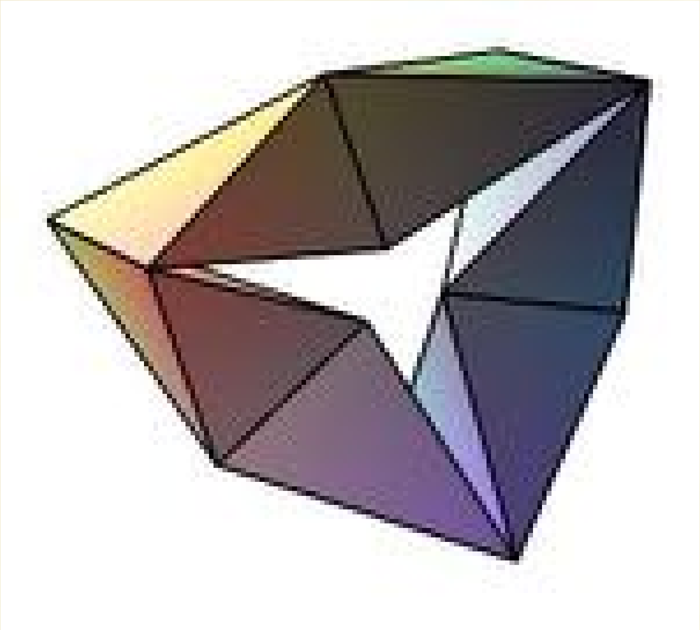
Can we place this closed KC on a line?

This problem is exactly the partition problem, which is NP-complete.
If you find an efficient solution, you will get $1 million!

Kinematic chains consisting of hinges

Kaleidocycle as KC

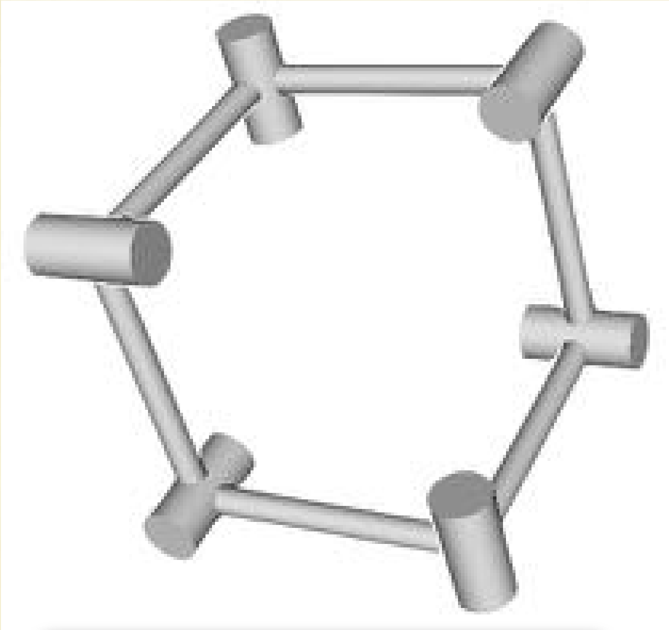
n: number of tetrahedra



Kaleidocycle

2n joints

5n bars



Bricard linkage

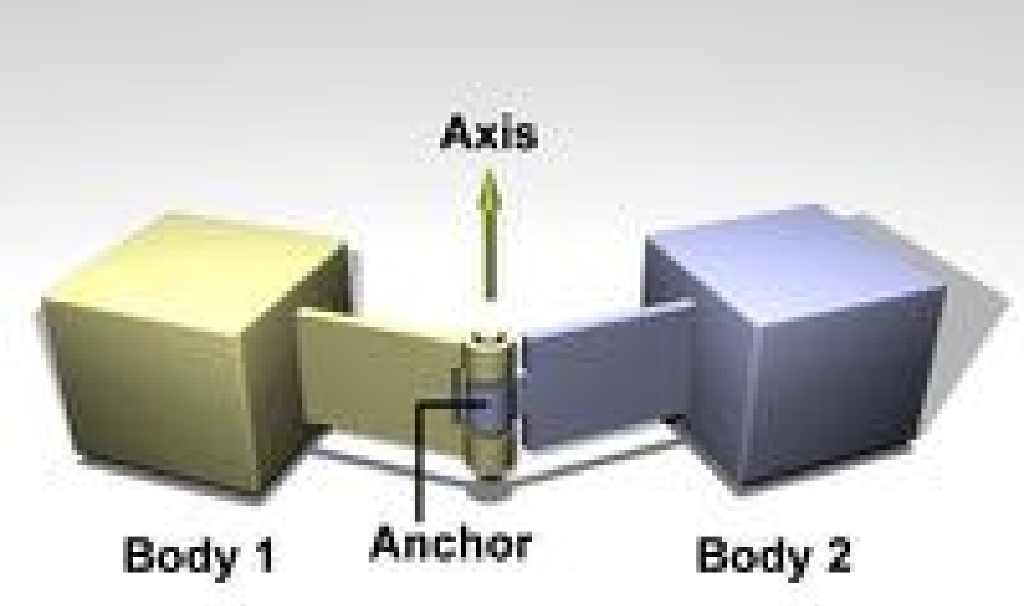
n hinges

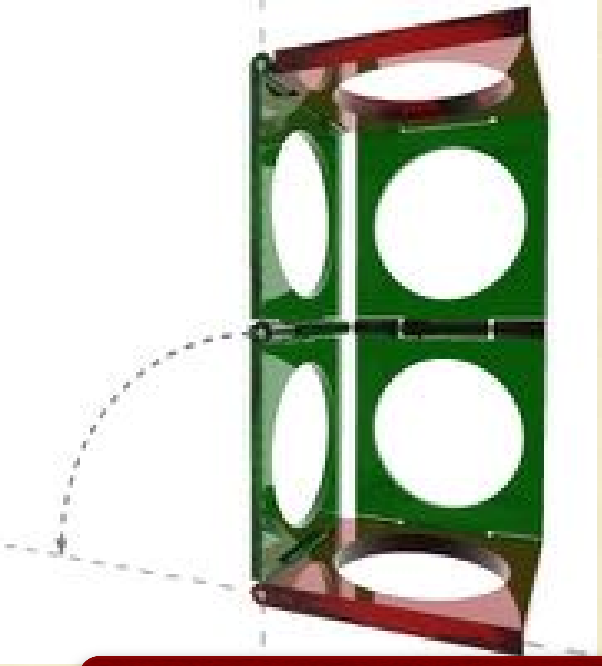
n bars

more convenient to work with

Hinge joint

Hinge is a special type of joint which restricts the relative motion of connected rigid bodies to rotation around its axis.





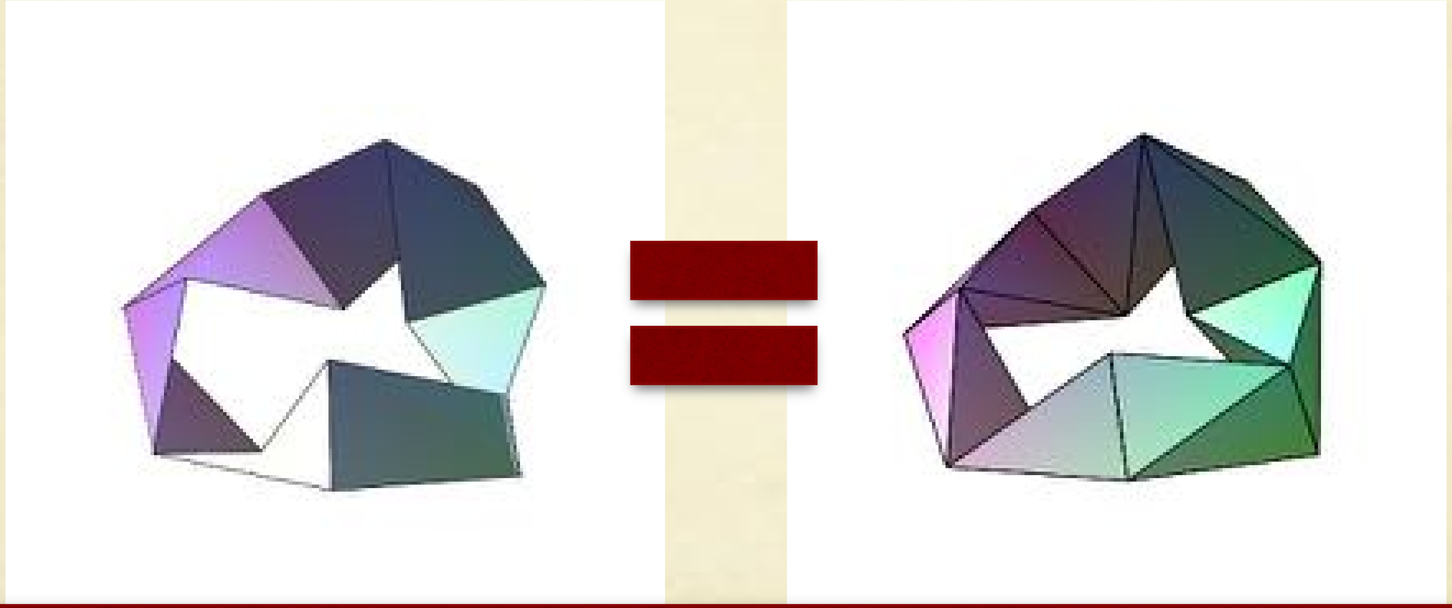
Sarrus linkage with 4 hinges

Note: 2D bar linkages are 3D hinged linkages with parallel hinges.

Remark: various joints (including high-dimensional ones) can be specified by their invariant/fixed subspace.

Our definition of Kaleidocycle

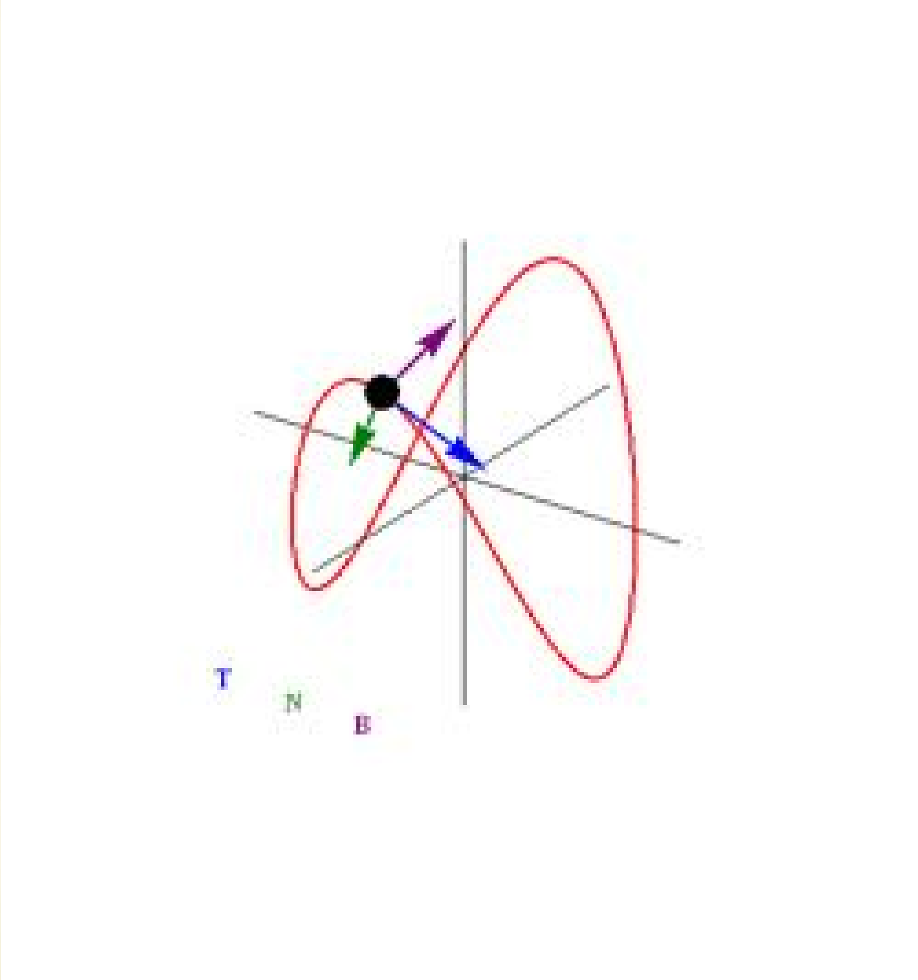
An n-Kaleidocycle (Kn) is a KC consisting of n identical equifacial tetrahedra connected by hinges.



Shape of bodies does not matter (we are ignoring self-collision)
Relation between hinges is essential.

Adapted framed curve

- A framed curve is a space curve with an orthonomal frame assigned at each point.
- A framed curve is said to be adapted if the first vector is tangent to the curve.

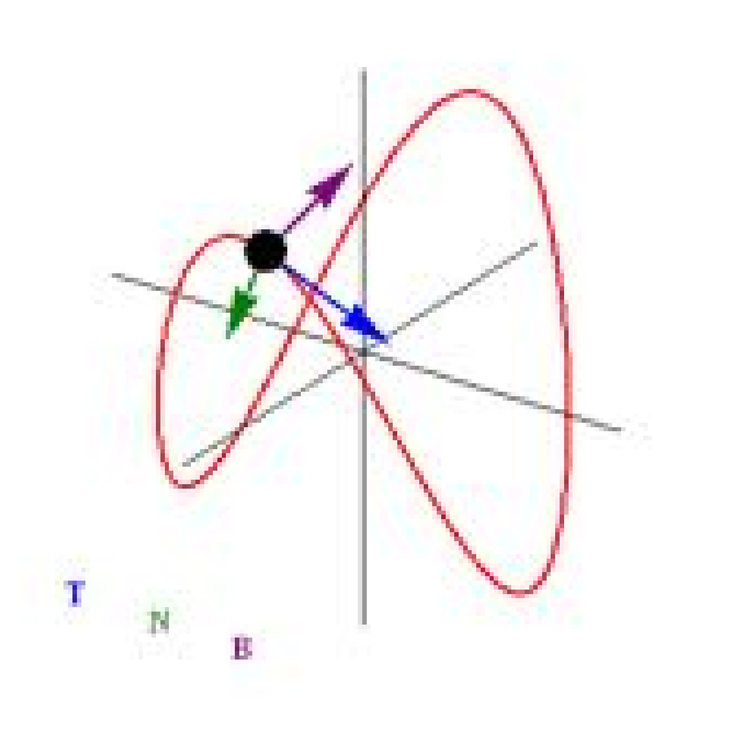


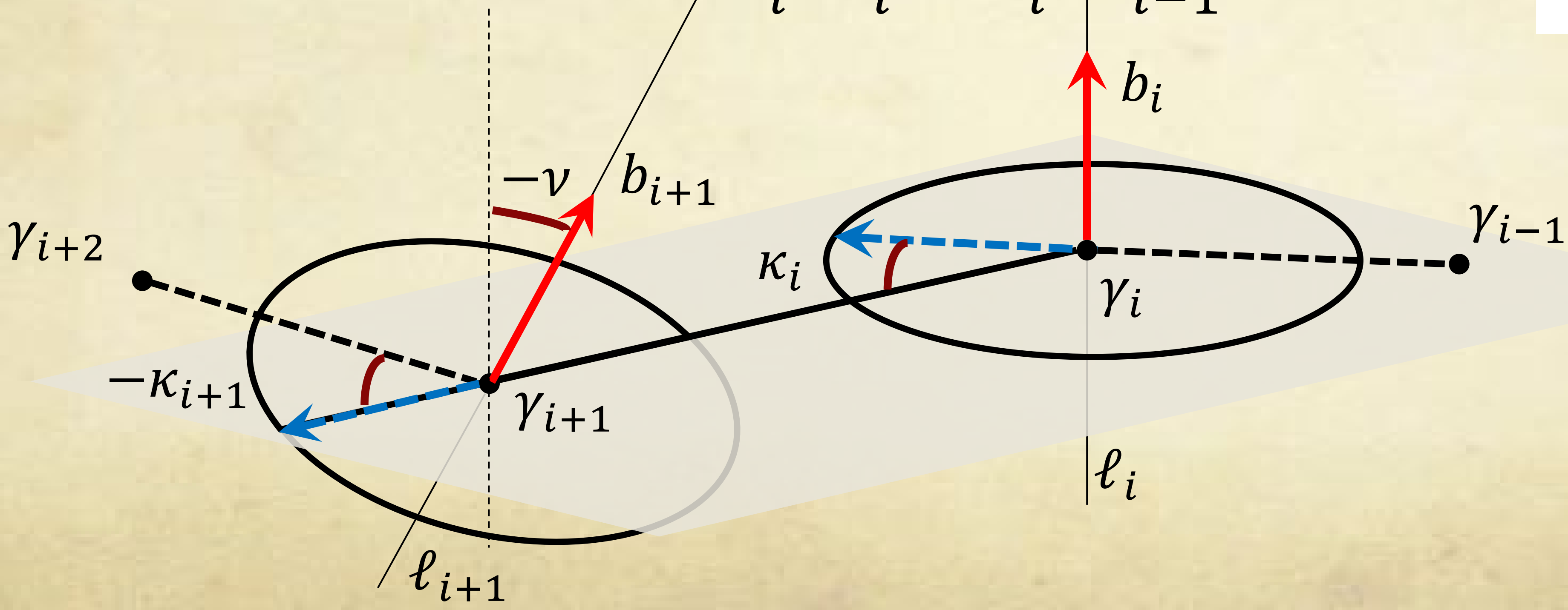
Given a curve, there are canonical ways to assign framings such as Frenet framing and Bishop framing.

Frenet frame of a discrete curve

- Polyline $(\gamma_0, \gamma_1, \ldots, \gamma_n)$
- Tangent $t_{i-1} := \dfrac{\gamma_i - \gamma_{i-1}}{|\gamma_i - \gamma_{i-1}|}$
- Hinge = Binormal $b_i \in S^2 \ (0 \le i \le i)$

s.t. $b_i \cdot t_i = b_i \cdot t_{i-1} = 0$





Frenet frame
(in mathematics)
=
Danavit-Hartenberg
parameter
(in mechanics)

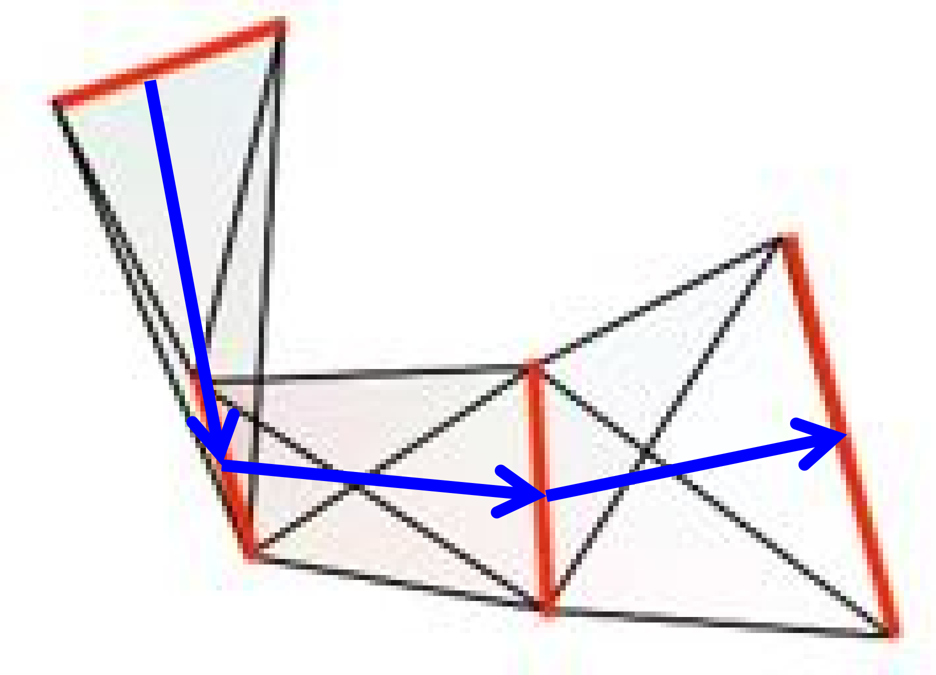
A geometric model of the configuration space

Kaleidocycle ⇔ closed discrete curve

"spine" of tetrahedra ⟺ polygonal curve

hinge ⟺ binormal (up to sign)

Kaleidocycle (congruence of tetrahedra) ⟺ constant torsion

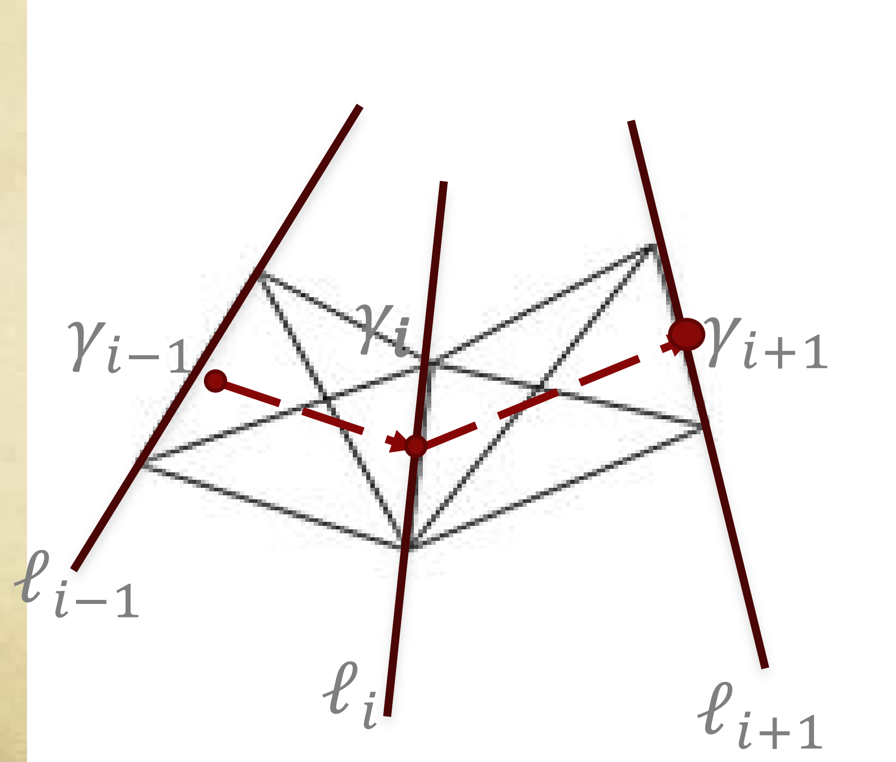


* Fixed-angle chain <=> constant curvature

An algebraic model of the configuration space

Binormals (hinges) reproduce the whole shape

$$t_i := \frac{b_i \times b_{i+1}}{|b_i \times b_{i+1}|}$$



Think of a Kaleidocycle as

an arrangement of lines (hinges) in 3-space

an arrangement of planes in 4D projective space

a sub-variety of the product of the Grassmanians G(4,2)

Note: we ignore the possibility of self-collision, as we can always avoid self-collision by considering thin tetrahedra (by setting the hinge length enough small compared to the length of the centre curve)

The configuration space of Kaleidocycles

Binormals determine a Kaleidocycle up to scaling. They satisfy the following equations

$$\{ b_i \in R^3 \mid 0 \leq i \leq n \} \quad t_i := \frac{b_i \times b_{i+1}}{|b_i \times b_{i+1}|} \quad b_0 = \begin{cases} b_n \ (oriented) \\ -b_n \ (non\ oriented) \end{cases}$$

n: number of hinges

- (the centre polygon should be closed) $$\sum_{i=0}^{n-1} t_i = \sum_{i=0}^{n-1} b_i \times b_{i+1} = 0$$
- (the angles between adjacent hinges are constant)

$$b_i \cdot b_{i+1} = b_{i-1} \cdot b_i, \qquad b_i \cdot b_i = 1 \qquad \text{(for all } i\text{)}$$

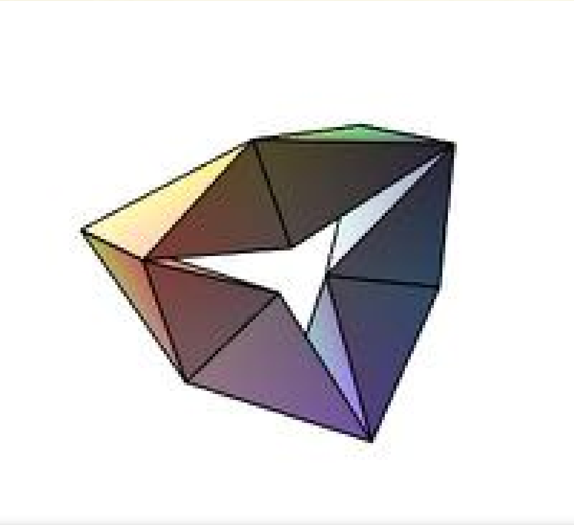
The moduli $M^+(n)$ (resp. M^-(n)) of (resp. non) oriented n-Kaleidocycles is defined to be the space of the real solutions to the above quadratic equations modulo the global symmetry ($O(3) \curvearrowright \{b_i\}$)

Configurations of a Kaleidocycle

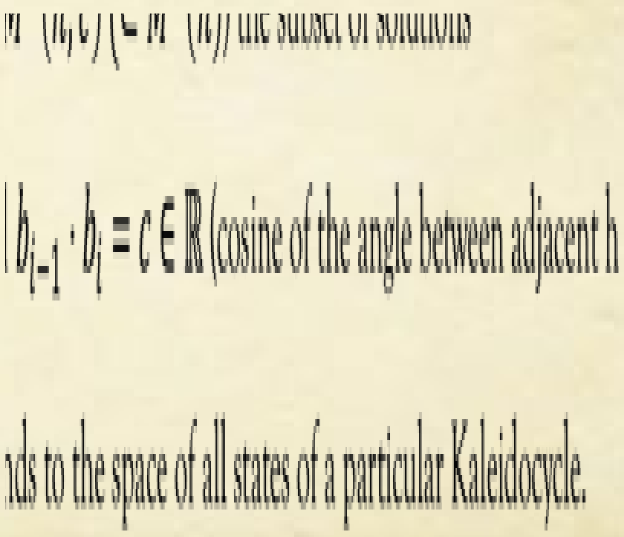
Denote by $M^{\pm}(n;c)$ ($\subset M^{\pm}(n)$) the subset of solutions

with a fixed $b_{i-1} \cdot b_i = c \in \mathbb{R}$ (cosine of the angle between adjacent hinges).

It corresponds to the space of all states of a particular Kaleidocycle.



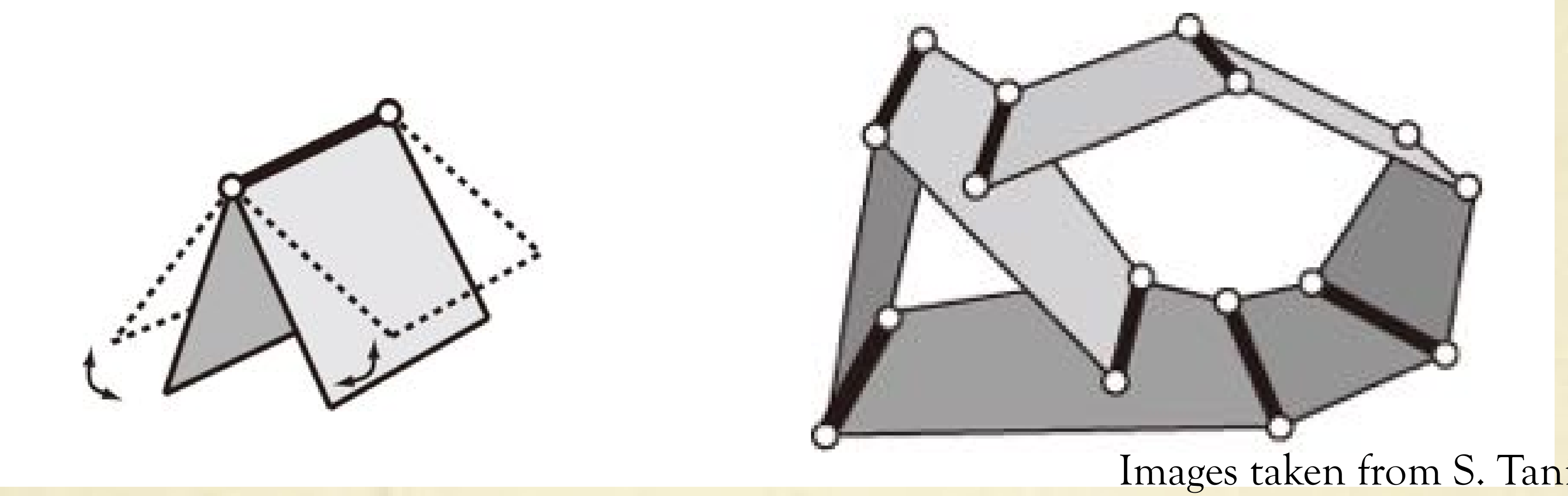
an element of $M^{-}(6;0)$
= a single state of 6-Kaleidocycle



elements of $M^{+}(7;-0.29)$
= motion of 7-Kaleidocycles

Immobility of Panel-hinge body

Kaleidocycles are a special type of panel-hinge body studied in combinatorics and chemistry



Images taken from S. Tanigawa's slides

<u>Molecular conjecture</u> (stated in Tay-Whiteley1984, solved in Kato-Tanigawas2011) provides a combinatorial characterisation for a ***generic*** panel-hinge body to be rigid (that is, <u>the configuration space is 0-dimensional</u>)

Dimension for a generic Kaleidocycle

For a generic c, let us count the dimension of M(n;c) by

Dim of solutions = # of variables – # of equations

Fix two hinges to kill the global symmetry.

We have degree two freedom for each $b_2, b_3, \ldots, b_{n-1} \in S^2$

and there the angle constraints $b_i \cdot b_{i+1} = c$ which contributes -(n-1)

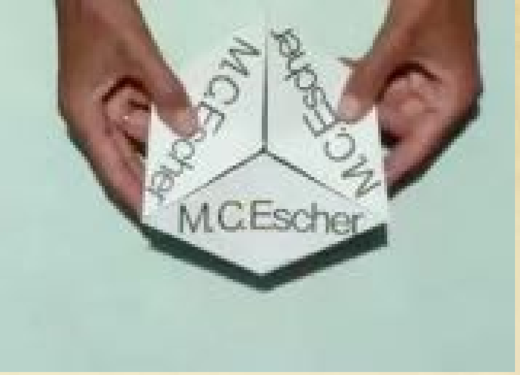
and the closing constraints $\sum b_i \times b_{i+1} = 0$ which contributes -3

In total, the dimension should be

2(n-2) – (n-1) – 3 = n-6

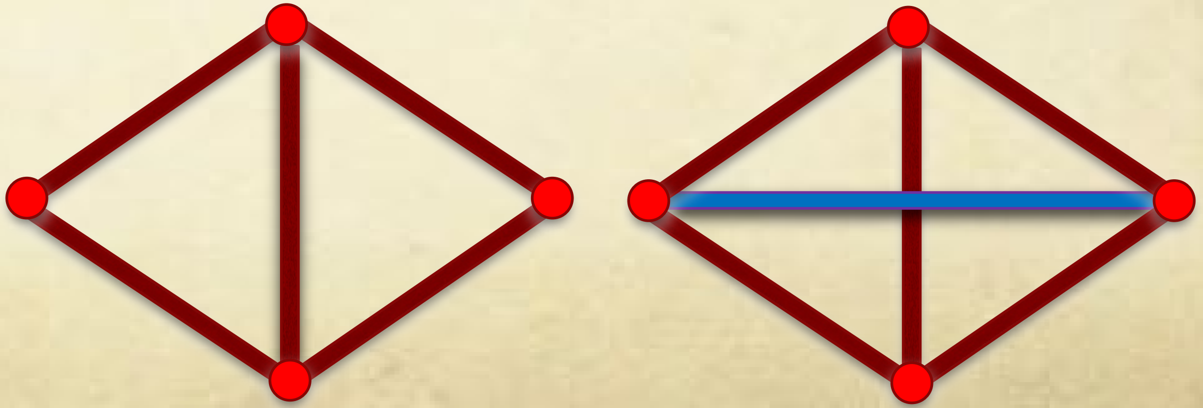
An n-Kaleidocycle has in general n-6 degrees of freedom

Wait! 6-Kaleidocycle is mobile!



- Because the equations are redundant due to its high symmetry
- Such a system is said to be **over-constrained**
- Fowler-Guest2005 gives a method to analyse over-constrained systems using representation theory

Simple example of over-constrained system
Adding the blue bar (one quadratic equation)
has no impact on the configuration space
(which is a point)



Mobius Kaleidocycle

= Kaleidocycle with a min/max twist angle

The space of the shape of tetrahedron (determined by the twisting angle acos(c))

configuration space of all non-orientable n-Kaleidocycles

solutions to

$$\sum b_i \times b_{i+1} = 0$$
$$b_i \cdot b_{i+1} = c \ \ (const)$$
$$b_i \cdot b_i = 1$$
$$b_0 = -b_n$$

subspace corresponding to the Mobius Kaleidocycle

$$b_i \cdot b_{i+1} = c_n$$

map c

1

C_n: maximum

Im(c)

-1

inverse image

a point = a state of a Kaleidocycle

a path = motion of a Kaleidocycle

Degree of Freedom of Mobius Kaleidocycles

Mobius Kaleidocycles have a single degree-of-freedom regardless of n

Conjecture (K-Schoenke, patented in 2018)

There is $c_n > 0$ for each n≧7 such that

$M^{+}(n; c)$ is non-empty iff $-c_n \leq c \leq 1$ (n:odd)

$M^{-}(n; c)$ is non-empty iff $-1 \leq c \leq c_n$ (n:odd)

$M^{-}(n; c)$ is non-empty iff $-c_n \leq c \leq c_n$ (n:even)

Moreover,

$M^{\pm}(n; \pm c_n) \simeq S^1$ (when $\pm c_n$ is the boundary value)

for each point, it is numerically checked that the ε-ball intersects exactly at two points

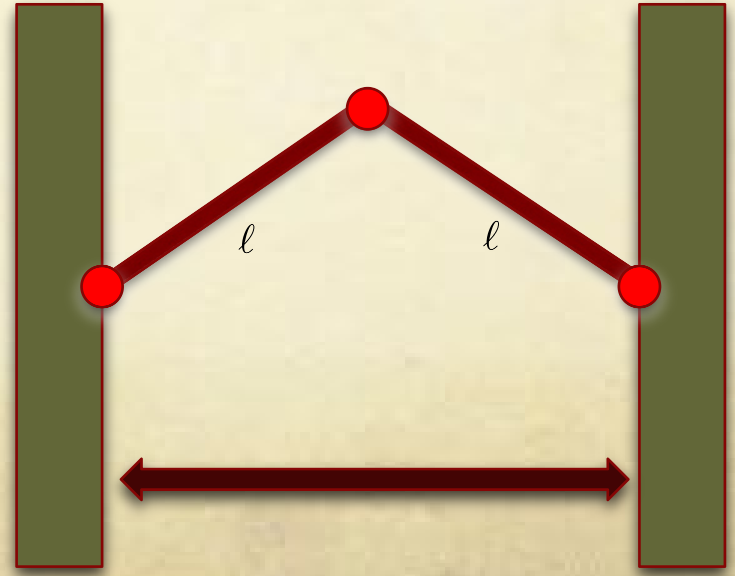
this S^1 is the characteristic “everting” motion

It is very rare for a linkage to have a degenerate but non-trivial configuration space

Under-constrained system

- They are **under-constrained** system: the dimension of solutions is less than expected. This is peculiar to **real** solutions.
  (as opposed to over-constrained systems, which exist for complex solutions as well)
- Under-constrained linkages have not been studied well.
  In fact, (to the best of my knowledge) **Mobius Kaleidocycles are the only example which are under-constrained and have a non-trivial configuration space.**

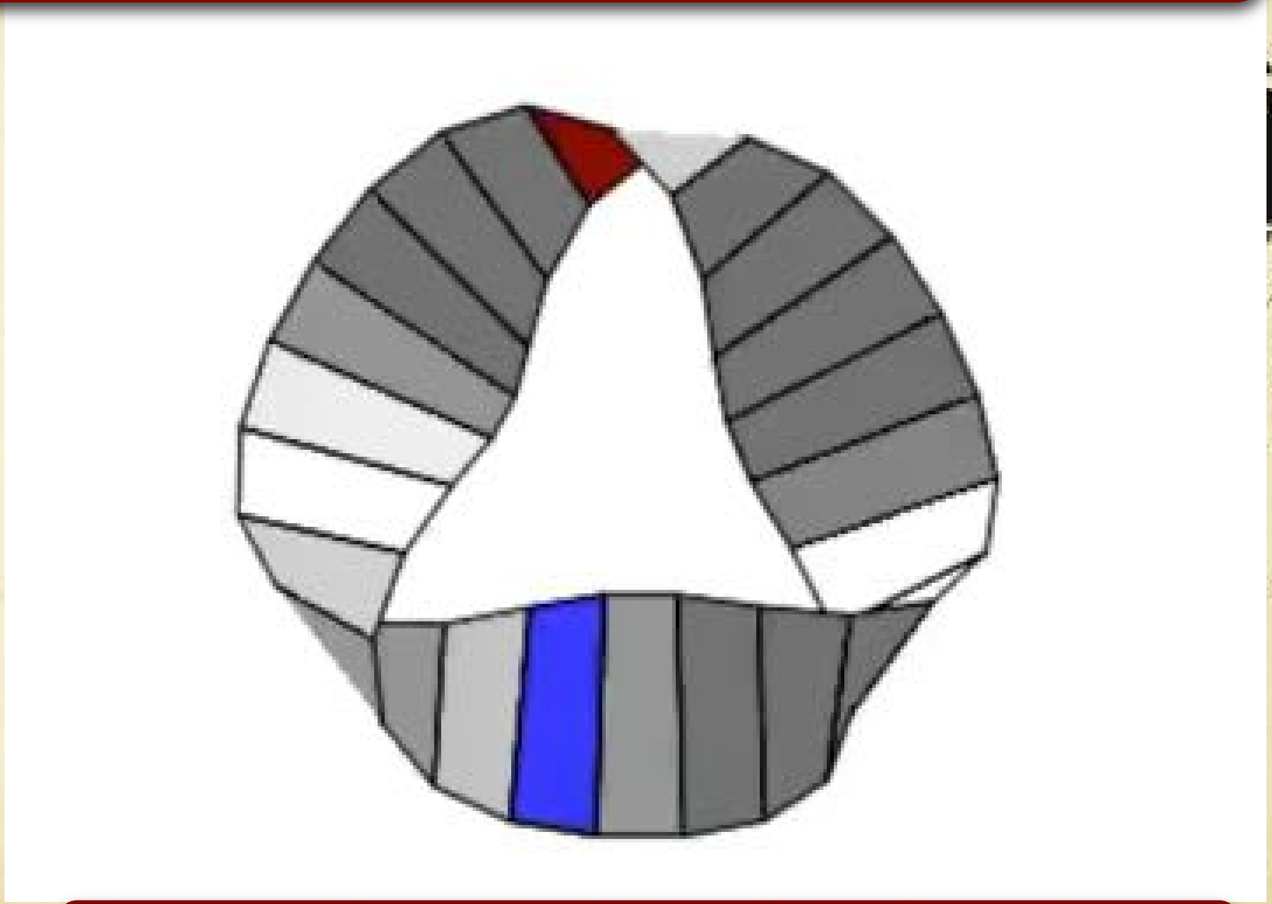
$$M_3(L) = \begin{cases} S^1 & (l > h) \\ * & (l = h) \\ \emptyset & (l < h) \end{cases}$$



A flow on the configuration space that Governs the Motion of Kaleidocycles

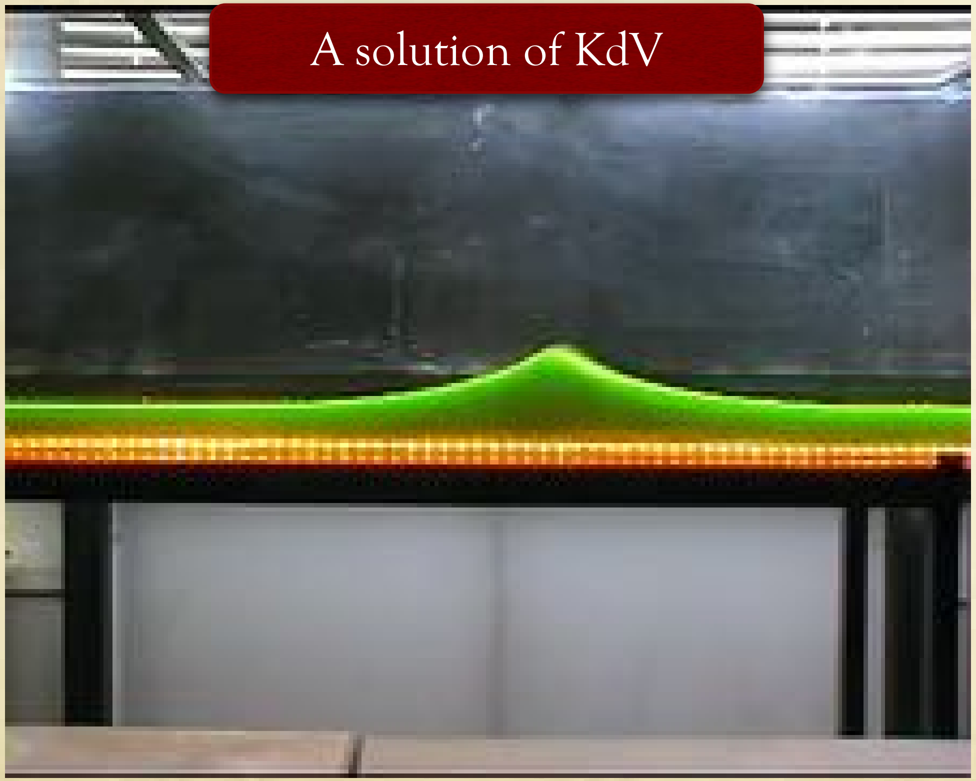
The “everting” motion of Kaleidocycles

Goal: describe this motion as a 1-dim path in the configuration space



It looks like a soliton of some kind...

Idea: construct a flow on the moduli space that generates the everting motion



Physics Department of the University of Burgundy
Vidéo by Julien FATOME, Stéphane PITOIS et Guy MILLOT

- Classically, the modified KdV equation is known to generate an isoperimetric deformation of smooth space curves.
- We consider a semi-discrete version of (discrete space & continuous time)

KdV: $u_t + 6uu_x + u_{xxx} = 0$

mKdV: $u_t + 6u^2u_x + u_{xxx} = 0$

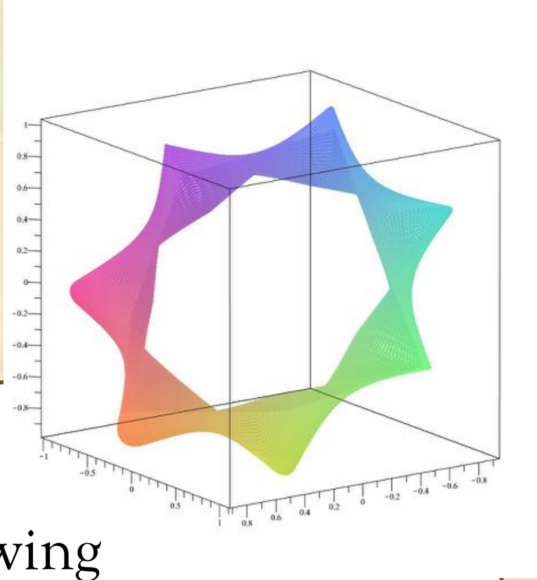
Motion = Curve deformation

CKC	closed discrete curve
Kaleidocycle	constant torsion and speed
motion	path in configuration space
infinitesimal motion	flow on the conf. sp.

We construct a flow by some (semi-discrete) integrable equations.

Curve deformation and Kaleidocycle's motion

1. arc length and torsion are preserved (⇔ bars are rigid)
2. velocity at each vertex lies in the osculating plane (➔ writhe is preserved)
3. equidistant (⇔ speed of motion is uniform at all vertices)



Theorem (K-Kajiwara-Park)

1. To meet the above assumptions, $\kappa_i = \cos(\angle t_i t_{i-1})$ should satisfy the following

$$\kappa_i = \frac{1}{2}(\theta_{i+1} - \theta_{i-1}) \qquad \frac{d}{dt}(\theta_{i+1} \mp \theta_i) = C \sin\left(\frac{\theta_{i+1} \pm \theta_i}{2}\right)$$

2. In an appropriate limit, they yield the potential mKdV and the sine-Gordon equations

$$\theta_T + \frac{1}{2}(\theta_X)^3 + \theta_{XXX} = 0.$$

$$\theta_{XT} - \sin(\theta) = 0.$$

The 1-DoF motion of a Mobius Kaleidocycle is generated by this flow.

An explicit solution was constructed recently by Shigetomi

An orbit in $M^{-}(9; 0.58)$

b_i

t_i

Motion of a Mobius Kaleidocycle

- The semi-discrete mKdV equation generates a 1-dim orbit in $M^{+}(n;c)$ when n is odd and $M^{-}(n;c)$ when n is even for any c, where $\dim(M^{\pm}(n;c)) = n-6$ for a generic c.
- Similarly, the semi-discrete sine-Gordon equation generates a 1-dim orbit in $M^{-}(n;c)$ for any n and c.

Note the duality defined by $b_i \mapsto (-1)^i b_i$

$$M^{-}(2m+1,c) \simeq M^{+}(2m+1,-c)$$
$$M^{-}(2m,c) \simeq M^{-}(2m,-c)$$

- If the 1-DoF conjecture is true, these orbits match the whole configuration space S^1

Conservation laws

Integrable systems have conserved quantities

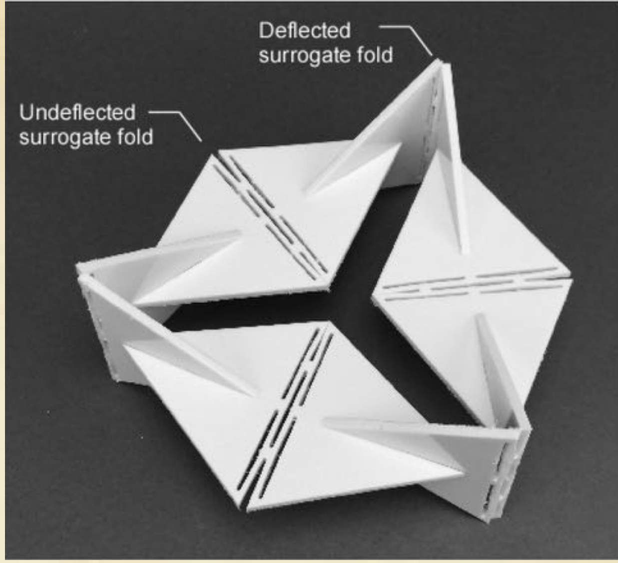
Corollary to Theorem

For the deformation governed by the sine-Gordon equation, we have

$$\frac{d}{dt}\sum_i \cos\frac{\theta_{i+1}-\theta_i}{2} = 0 \qquad \frac{d}{dt}\sum_i \kappa_i = 0$$

for an oriented extreme Kaleidocycles,
the value itself seems to be zero

Conservation laws: Bending energy



The bending (elastic) energy $\int_\gamma \kappa^2 ds$

can be discretised in many different ways but consider

$$E_{bend} = \sum_i \log\left(1 + \tan^2\left(\frac{\kappa_i}{2}\right)\right)$$

Safsten et al. 2016 analysed the potential energy of K6 whose hinges are attached torsional springs.

Theorem (K-Kajiwara-Park-Shigetomi)
E_{bend} is constant under the deformation by the sine-Gordon (mKdV) equation.

Global quantities are mysteriously preserved!

Open problem

Variational problem of bending energy

Conjecture
When non-oriented or odd n,
the minimisers of E_{bend} attain the extremum of the torsion.

Bending energy is determined by the curvature alone.

So the mysterious 1-DoF Kaleidocycles are characterised in two different ways;
one by the curvature and the other by the torsion.

Open problem

Coulomb & dipole energies

Imagine the centres γ of hinges are electrically charged. The potential of the system is

$$E_{clmb} := \sum_{i<j} \frac{1}{|\gamma_i - \gamma_j|^\alpha} \qquad \alpha \in \mathbb{R}$$

Note that these depend on the global shape of the curve

Imagine the hinges are dipoles. The potential of the system is

$$E_{dipl} := \sum_{i<j} \frac{b_i \cdot b_j}{|\gamma_i - \gamma_j|^3} - \frac{3(b_i \cdot (\gamma_i - \gamma_j))(b_j \cdot (\gamma_i - \gamma_j))}{|\gamma_i - \gamma_j|^5}$$

E_{clmb} and E_{dipl} are almost constant under deformation

Problem: What is the correct discritisation of the these energies that are strictly conserved?

Topology makes it interesting!

Topological constraint has a large impact on the algebraic system defining the Kaleidocycles.

Topological invariants of a smooth curve

We assume the arc length of the strip is 2π.

There are two conformal invariants called twist and the writhe:

$$Tw := \int_0^{2\pi} |\tau(s)| ds, \quad \tau(s) = \frac{1}{2\pi} \dot{b}(s) \cdot (\dot{\gamma}(s) \times b(s))$$

$$Wr := \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{(\dot{\gamma}(s_1) \times \dot{\gamma}(s_2)) \cdot (\dot{\gamma}(s_1) - \dot{\gamma}(s_2))}{|\dot{\gamma}(s_1) - \dot{\gamma}(s_2)|^3} ds_1 ds_2$$

These invariants are also studied in
"Topological Fluid Mechanics".
See, for exameple,
Scheeler et al, "*Complete measurement of helicity and its dynamics in vortex tubes*"
Science 357, 2017

Calugareanu-White's theorem
#half twists := 2(Twist + Writhe)
is an integer and an isotopy invariant

This type of results have been studied mainly in the context of DNA supercoiling

Twist and writhe for a discrete curve

$$Tw = \frac{1}{2\pi}\sum_{i=1}^{n} \arccos(b_{i-1}\cdot b_i)$$

$$Wr := \frac{1}{4\pi}\sum_{i\neq j} \frac{\langle t_i \times t_j, t_i - t_j\rangle}{|t_i - t_j|^3}$$

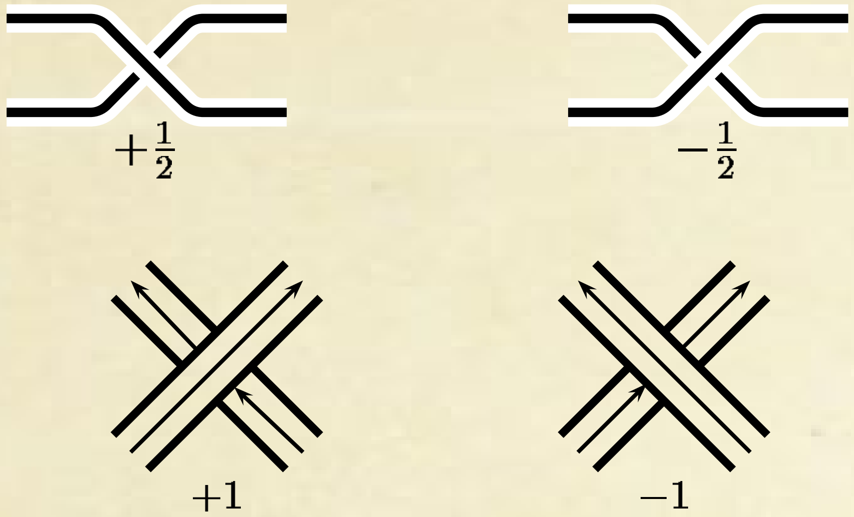
Kaleidocycle with a large Tw

For a Kaleidocycle in motion
Tw remains unchanged by definition
so Wr should remain unchanged as well

Calugareanu-White's theorem
#half-twists := 2(Tw + Wr)
is an integer and an isotopy invariant

This puts a very strong topological constraint for a curve to be closed

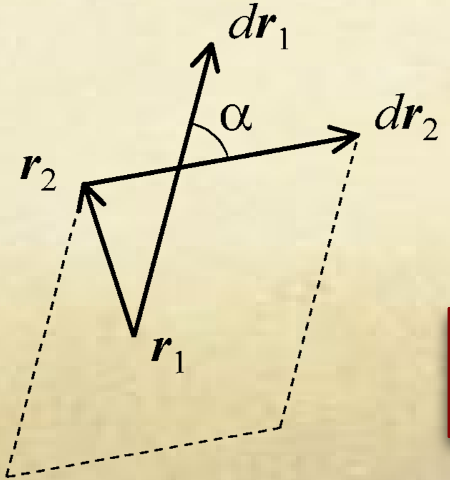
Computing Wr and Tw



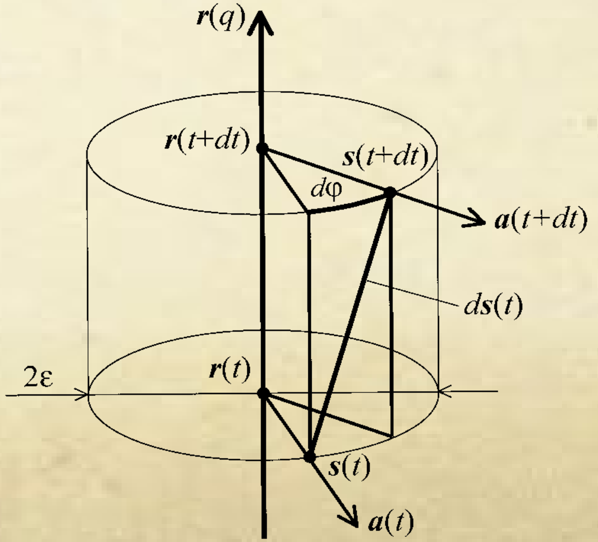
Twist is the average of ← over all projections
It is the difference of b(2π) and
b(0) parallel transported along the curve

Writhe is the average of ← over all projections.
It depends only on the centre curve

The integration formula for Writhe and Tw are understood by the pictures below: Think of the angles with which one sees an intersection for a pair of infinitesimal curve portions.



Klenin-Langowski2000



Open problem

Twist and Writhe

- Twist is the difference between b(2π) and b(0) parallel-transported along the curve.
  It measures how the strip revolves around its centre curve.

- Writhe is the number of self-intersections of the projected centre curve averaged over all the projections.
  It depends only on the centre curve and measures how the curve winds around itself.

- None of them is an isotopy invariant but the sum is.
  The sum is the number of half-twists of the strip.
  If the number is odd, the strip is topologically a Mobius strip.

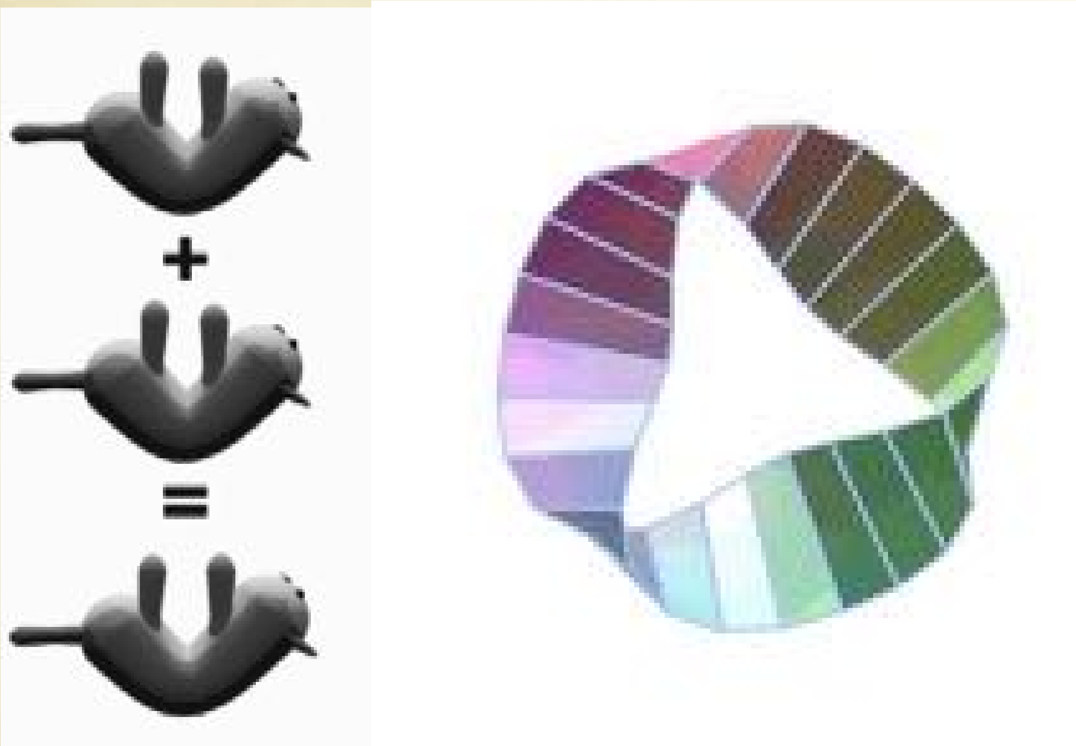
- Mobius Kaleidocycles have the minimum Tw, so the maximum Wr among Kaleidocycles with positive half-twists

Conjecture

the number of half-twists of a Kaleidocycle takes values between 3 and n-3.
(In particular, there is no Kaleidocycle with a single half-twist)

Falling cat

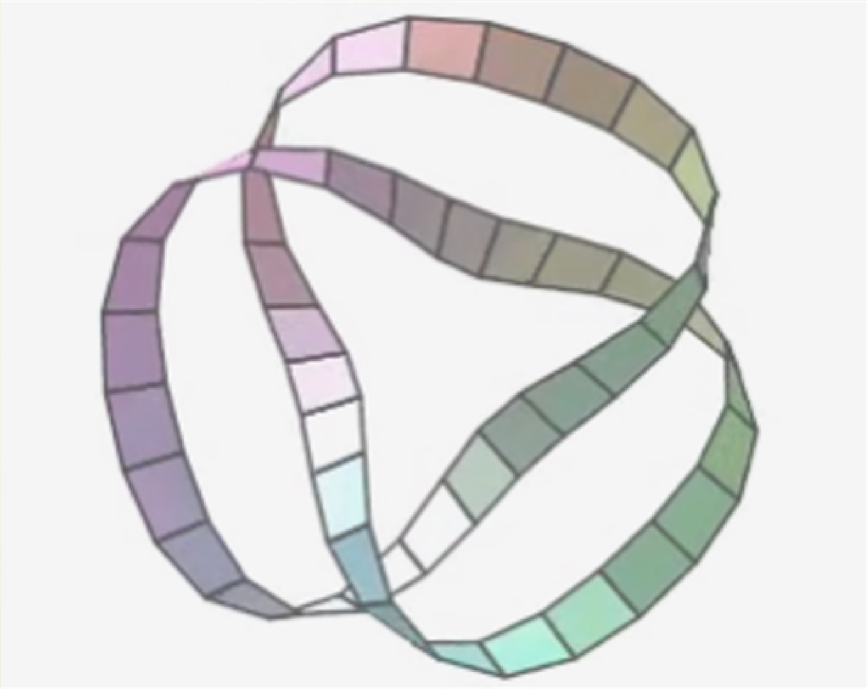
The global orientation seems to evolve without any external force
(that is, keeping zero angular momentum)
Possible application to space antenna



c.f. Montgomery, Gauge theory of the falling cat

Knot

Kaleidocycles can be made knotted.
Which knot types are realizable with what Tw and Wr?



c.f. Naokawa, Extrinsically flat Möbius strips on given knots in 3-dimensional spaceform

Summary

Kaleidocycle provides links among
linkage, algebraic geometry (systems with geometric constraints),
discrete differential geometry, and integrable systems.

closed hinged linkage ⟷ discrete closed framed curve

motion ⟷ length and torsion preserving deformation

for closed curves, topology poses strong constraints

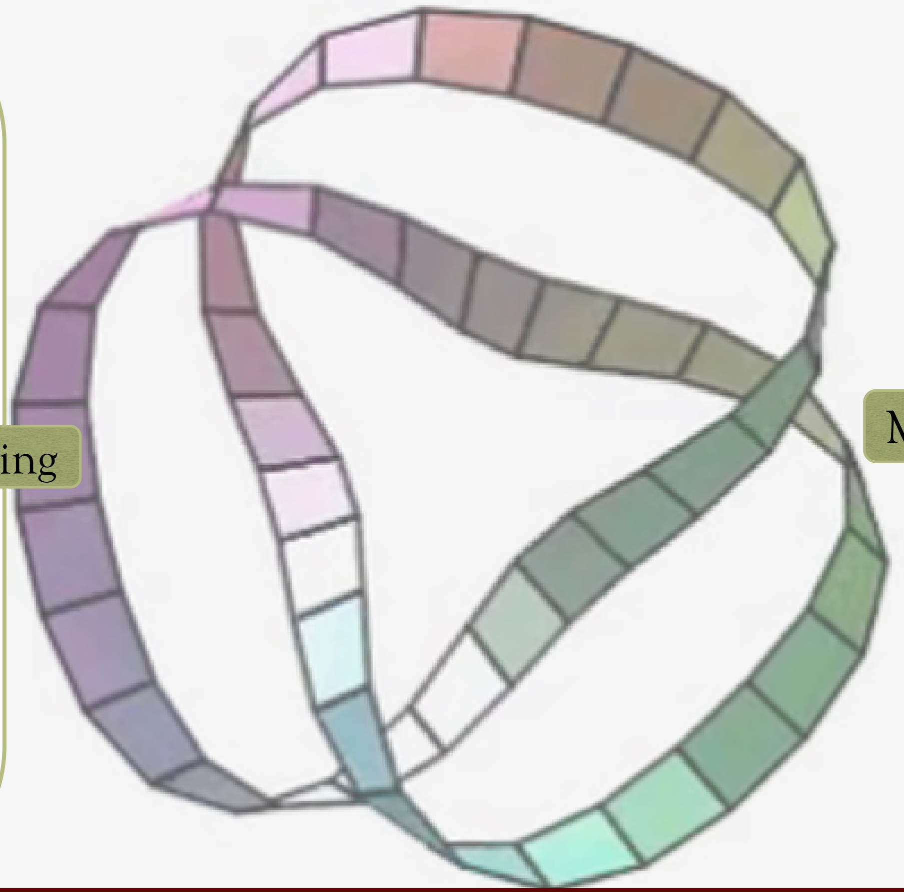
everting motion of Kaleidocycle ⟷ flow on the configuration space defined by a semi-discrete integrable systems

Applied Applied Maths

From "pure" vs "applied" to "pure" + "applied"

- Configuration space of a linkage
- Discovery of a linkage with 1 DoF
- 1 DoF is very important in engineering.
- Everting motion
- Physically interesting properties

Engineering



Maths.

- Topology of a real algebraic variety
- 1-dimensional singular fibre
- A topological constraint (linking number) forces 1 DoF.
- Orbit governed by a semi-discrete integrable system
- Conservation law

Pure maths and application on two sides of the same band, cycling together make an ideal feedback loop

Many open problems!

Open problem

- Prove that Mobius Kaleidocycles have a single-degree-of-freedom
  - This is a problem in real algebraic geometry: the dimension of the real algebraic set
  - Techniques in symbolic algebra such as quantifier elimination and Groebner basis are not feasible due to the large number of variables (currently, only works for $n \leqq 5$)
  - Homotopy continuation would be helpful but it does not give a solution for general n
- Find conserved quantities during the eversion
- Find an analytic expression for Kaleidocycles
  - Recently, a partial answer was given by S. Shigetomi (a PhD candidate at Kyushu University)
- Characterise Mobius Kaleidocycles in terms of a variational problem of Kirchhoff rod (Langer-Singer's theory)
- Symplectic structure on the moduli space? (c.f. Kapovich-Millson's form on the moduli of polygons)
- What is the limit of $n \to \infty$?
- Generalise Mobius Kaleidocycles to find a wider family of under-constrained linkage systems
- Prove there exists no Kaleidocycle with a half twist (Mobius Kaleidocycles have 3-half twists)
- (Stiefel) Realisation problem of persistent homology: find a Euclidean realisation of a complex which has the specified persistent homology.
- Find a (Mobius) Kaleidocycle in nature