Cobordism of algebraic knots defined by Brieskorn–Pham type polynomials

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§1. Introduction

- Algebraic knot
- Cobordism of knots
- Cobordism vs Isotopy
- Problem

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§3. Proofs
Algebraic knot

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\( K_f \) is a \((2n - 1)\)-dim. closed manifold embedded in \( S_{\varepsilon}^{2n+1} \).
Algebraic knot

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$K_f$ is a $(2n - 1)$-dim. closed manifold embedded in $S^{2n+1}_\varepsilon$. 
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2. $\partial X = (K_0 \times \{0\}) \cup (-K_1 \times \{1\})$.

$X$ is called a **cobordism** between $K_0$ and $K_1$. 
Cobordism vs Isotopy

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Cobordism vs Isotopy

\[ S^{m+2} \times \{0\} \xrightarrow{X} S^{m+2} \times \{1\} \]

Isotopic

\[ K_0 \times \{0\} \xrightarrow{\downarrow} X \xleftarrow{\uparrow} K_1 \times \{1\} \]

Cobordant
If two algebraic knots $K_f$ and $K_g$ are **cobordant**, then the topological types of $f$ and $g$ are mildly related.
Problem

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Today’s Topic: Problem 1.2 (2) for weighted homogeneous polynomials (in particular, Brieskorn–Pham type polynomials).
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- Weighted homogeneous polynomials
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- Criterion for isomorphism over $\mathbb{R}$
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Two-variable case

Case of $n = 1$ and the polynomials are irreducible at 0.
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**Theorem 2.1 (Lê, 1972)**

For algebraic knots $K_f$ and $K_g$ in $S^3_\varepsilon$, the following three are equivalent.

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1. \( K_f \) and \( K_g \) are isotopic.
2. \( K_f \) and \( K_g \) are cobordant.
3. Alexander polynomials coincide: \( \Delta_f(t) = \Delta_g(t) \).
Higher dimensions

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**du Bois–Michel, 1993**

Examples of two algebraic (spherical) knots that are cobordant, but are not isotopic.
Algebraic cobordism

Let $L_i : G_i \times G_i \to \mathbb{Z}, i = 0, 1$, be two bilinear forms defined on free $\mathbb{Z}$-modules of finite ranks.
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**Definition 2.2** Suppose $m = \text{rank } G$ is even. A direct summand $M \subset G$ is called a *metabolizer* if $\text{rank } M = m/2$ and $L$ vanishes on $M$. 
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$L_0$ is *algebraically cobordant* to $L_1$ if there exists a metabolizer satisfying additional properties about $S = L \pm L^T$. 
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**Theorem 2.3 (Blanlœil–Michel, 1997)** For \( n \geq 3 \),
two algebraic knots \( K_f \) and \( K_g \) are cobordant
\( \iff \) Seifert forms \( L_f \) and \( L_g \) are algebraically cobordant.
Witt equivalence

Remark 2.4 At present, there is no efficient criterion for algebraic cobordism. It is usually very difficult to determine whether given two forms are algebraically cobordant or not.
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Two forms $L_0$ and $L_1$ are *Witt equivalent over $\mathbb{R}$* if there exists a metabolizer over $\mathbb{R}$ for $L_0 \otimes \mathbb{R}$ and $L_1 \otimes \mathbb{R}$. 
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Lemma 2.5  If two algebraic knots $K_f$ and $K_g$ are cobordant, then their Seifert forms $L_f$ and $L_g$ are Witt equivalent over $\mathbb{R}$.
Weighted homogeneous polynomials

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§3. Proofs

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Weighted homogeneous polynomials

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\( f \) is \textit{non-degenerate} if it has an isolated critical point at 0.
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In the following, we will always assume $\forall$ weights $\geq 2$. 
Criterion for Witt equivalence over $\mathbb{R}$

Set

$$P_f(t) = \prod_{j=1}^{n+1} \frac{t - t^{1/w_j}}{t^{1/w_j} - 1}.$$ 

$P_f(t)$ is a polynomial in $t^{1/m}$ over $\mathbb{Z}$ for some integer $m > 0$. 
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$P_f(t)$ is a polynomial in $t^{1/m}$ over $\mathbb{Z}$ for some integer $m > 0$. Two non-degenerate weighted homogeneous polynomials $f$ and $g$ have the same weights if and only if $P_f(t) = P_g(t)$. 

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Two non-degenerate weighted homogeneous polynomials $f$ and $g$ have the same weights if and only if $P_f(t) = P_g(t)$.

Theorem 2.6  Let $f$ and $g$ be non-degenerate weighted homogeneous polynomials in $\mathbb{C}^{n+1}$. Then, their Seifert forms $L_f$ and $L_g$ are Witt equivalent over $\mathbb{R}$ iff

$$P_f(t) \equiv P_g(t) \mod t + 1.$$
Criterion for isomorphism over $\mathbb{R}$

The above theorem should be compared with the following.

**Remark 2.7**  The Seifert forms $L_f$ and $L_g$ associated with non-degenerate weighted homogeneous polynomials $f$ and $g$ are isomorphic over $\mathbb{R}$ iff

$$P_f(t) \equiv P_g(t) \mod t^2 - 1.$$
Brieskorn–Pham type polynomials

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§3. Proofs

Proposition 2.8 Let

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn–Pham type polynomials.
Brieskorn–Pham type polynomials

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be Brieskorn–Pham type polynomials. Then, their Seifert forms are Witt equivalent over $\mathbb{R}$ iff

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j}$$

holds for all odd integers $\ell$. 
Cobordism invariance of exponents

Theorem 2.9  Suppose that for each of the Brieskorn–Pham type polynomials

\[ f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}, \]

no exponent is a multiple of another one.
Cobordism invariance of exponents

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no exponent is a multiple of another one. Then, the knots $K_f$ and $K_g$ are cobordant iff

$$a_j = b_j, \quad j = 1, 2, \ldots, n + 1,$$

up to order.
Cobordism invariance of multiplicities

The smallest degree of a polynomial is called its **multiplicity**.
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**Zariski Conjecture**

The multiplicity is a topological invariant of a complex hypersurface singularity.
Cobordism invariance of multiplicities

The smallest degree of a polynomial is called its multiplicity.

Zariski Conjecture
The multiplicity is a topological invariant of a complex hypersurface singularity.

Proposition 2.10  Suppose that for each of the Brieskorn–Pham type polynomials

\[ f(z) = \sum_{j=1}^{n+1} z^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z^{b_j} \]

the exponents are pairwise distinct.
Cobordism invariance of multiplicities

The smallest degree of a polynomial is called its **multiplicity**.

**Zariski Conjecture**

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**Proposition 2.10** *Suppose that for each of the Brieskorn–Pham type polynomials*

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f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}
\]

*the exponents are pairwise distinct.*

If \( K_f \) and \( K_g \) are **cobordant**, then the **multiplicities** of \( f \) and \( g \) coincide.
Case of two or three variables

Proposition 2.11  Let $f$ and $g$ be weighted homogeneous polynomials of two variables with weights $(w_1, w_2)$ and $(w'_1, w'_2)$, respectively, with $w_j, w'_j \geq 2$. If their Seifert forms are Witt equivalent over $\mathbb{R}$, then $w_j = w'_j$, $j = 1, 2$, up to order.
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Proposition 2.12 Let $f(z) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ and $g(z) = z_1^{b_1} + z_2^{b_2} + z_3^{b_3}$ be Brieskorn–Pham type polynomials of three variables. If the Seifert forms $L_f$ and $L_g$ are Witt equivalent over $\mathbb{R}$, then $a_j = b_j$, $j = 1, 2, 3$, up to order.
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- Proof of Proposition 2.8
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- Proof of Theorem 2.9
- Open problem
- Cobordism and isotopy for Brieskorn–Pham type polynomials
Proof of Theorem 2.6

**Theorem 2.6** Let $f$ and $g$ be non-degenerate weighted homogeneous polynomials in $\mathbb{C}^{n+1}$. Then, their Seifert forms $L_f$ and $L_g$ are **Witt equivalent over** $\mathbb{R}$ iff

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Proof. For simplicity, we consider the case of $n$ even.
Proof of Theorem 2.6

**Theorem 2.6** Let \( f \) and \( g \) be non-degenerate weighted homogeneous polynomials in \( \mathbb{C}^{n+1} \). Then, their Seifert forms \( L_f \) and \( L_g \) are **Witt equivalent over** \( \mathbb{R} \) iff

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\]

**Proof.** For simplicity, we consider the case of \( n \) even.

Let \( \Delta_f(t) \) be the **characteristic polynomial of the monodromy**

\[
h_* : H_n(\text{Int} F_f; \mathbb{C}) \to H_f(\text{Int} F_f; \mathbb{C}),
\]

where \( h : \text{Int} F_f \to \text{Int} F_f \) is the characteristic diffeomorphism of the Milnor fibration \( \varphi_f : S_{\varepsilon}^{2n+1} \setminus K_f \to S^1. \)
Proof of Theorem 2.6 (Continued)

We have

\[ H^n(F_f; \mathbb{C}) = \bigoplus \lambda H^n(F_f; \mathbb{C})_\lambda, \]

where \( \lambda \) runs over all the roots of \( \Delta_f(t) \), and \( H^n(F_f; \mathbb{C})_\lambda \) is the eigenspace of \( h_* \) corresponding to the eigenvalue \( \lambda \).
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The **intersection form** \( S_f = L_f + L^T_f \) of \( F_f \) on \( H^n(F_f; \mathbb{C}) \) decomposes as the orthogonal direct sum of \( (S_f)|_{H^n(F_f; \mathbb{C})_\lambda} \).
Proof of Theorem 2.6 (Continued)

We have

\[ H^n(F_f; C) = \bigoplus \lambda H^n(F_f; C)_\lambda, \]

where \( \lambda \) runs over all the roots of \( \Delta_f(t) \), and \( H^n(F_f; C)_\lambda \) is the eigenspace of \( h_* \) corresponding to the eigenvalue \( \lambda \).

The intersection form \( S_f = L_f + L_f^T \) of \( F_f \) on \( H^n(F_f; C) \) decomposes as the orthogonal direct sum of \( (S_f)|_{H^n(F_f; C)_\lambda} \).

Let \( \mu(f)_+^\lambda \) (resp. \( \mu(f)_-^\lambda \)) denote the number of positive (resp. negative) eigenvalues of \( (S_f)|_{H^n(F_f; C)_\lambda} \).
Proof of Theorem 2.6 (Continued)

We have

\[ H^n(F_f; \mathbb{C}) = \bigoplus_\lambda H^n(F_f; \mathbb{C})_\lambda, \]

where \( \lambda \) runs over all the roots of \( \Delta_f(t) \), and \( H^n(F_f; \mathbb{C})_\lambda \) is the eigenspace of \( h_* \) corresponding to the eigenvalue \( \lambda \).

The **intersection form** \( S_f = L_f + L^T_f \) of \( F_f \) on \( H^n(F_f; \mathbb{C}) \) decomposes as the orthogonal direct sum of \( (S_f|_{H^n(F_f; \mathbb{C})_\lambda}) \).

Let \( \mu(f)^+_\lambda \) (resp. \( \mu(f)^-_\lambda \)) denote the number of positive (resp. negative) eigenvalues of \( (S_f|_{H^n(F_f; \mathbb{C})_\lambda}) \).

The integer

\[ \sigma_\lambda(f) = \mu(f)^+_\lambda - \mu(f)^-_\lambda \]

is called the **equivariant signature** of \( f \) with respect to \( \lambda \).
Proof of Theorem 2.6 (Continued)

Lemma 3.1 (Steenbrink, 1977)

Set \( P_f(t) = \sum c_\alpha t^\alpha \). Then we have

\[
\sigma_\lambda(f) = \sum_{\lambda = \exp(-2\pi i \alpha)} c_\alpha - \sum_{\lambda = \exp(-2\pi i \alpha), \lfloor \alpha \rfloor: \text{odd}} c_\alpha
\]

for \( \lambda \neq 1 \), where \( i = \sqrt{-1} \), and \( \lfloor \alpha \rfloor \) is the largest integer not exceeding \( \alpha \).
Proof of Theorem 2.6 (Continued)

Lemma 3.1 (Steenbrink, 1977)

Set \( P_f(t) = \sum c_\alpha t^\alpha \). Then we have

\[
\sigma_\lambda(f) = \sum_{\lambda = \exp(-2\pi i \alpha), \lfloor \alpha \rfloor \text{: even}} c_\alpha - \sum_{\lambda = \exp(-2\pi i \alpha), \lfloor \alpha \rfloor \text{: odd}} c_\alpha
\]

for \( \lambda \neq 1 \), where \( i = \sqrt{-1} \), and \( \lfloor \alpha \rfloor \) is the largest integer not exceeding \( \alpha \).

Remark 3.2  The equivariant signature for \( \lambda = 1 \) is always equal to zero.
Proof of Theorem 2.6 (Continued)

Seifert forms $L_f$ and $L_g$ are Witt equivalent over $\mathbb{R}$.

$$\Rightarrow \quad \sigma_{\lambda}(f) = \sigma_{\lambda}(g) \quad \text{for all } \lambda.$$
Proof of Theorem 2.6 (Continued)

Seifert forms $L_f$ and $L_g$ are Witt equivalent over $\mathbb{R}$.

$$\sigma_\lambda(f) = \sigma_\lambda(g)$$

for all $\lambda$.

Set

$$P_f(t) = P_f^0(t) + P_f^1(t), \quad \text{where}$$

$$P_f^0(t) = \sum_{[\alpha] \equiv 0 \pmod{2}} c_\alpha t^\alpha,$$

$$P_f^1(t) = \sum_{[\alpha] \equiv 1 \pmod{2}} c_\alpha t^\alpha.$$

We define $P_g^0(t)$ and $P_g^1(t)$ similarly.
Proof of Theorem 2.6 (Continued)

Seifert forms $L_f$ and $L_g$ are Witt equivalent over $\mathbb{R}$.

$$\Rightarrow \quad \sigma_\lambda(f) = \sigma_\lambda(g) \quad \text{for all } \lambda.$$  

Set

$$P_f(t) = P^0_f(t) + P^1_f(t), \quad \text{where}$$

$$P^0_f(t) = \sum_{[\alpha] \equiv 0 \pmod{2}} c_\alpha t^\alpha,$$

$$P^1_f(t) = \sum_{[\alpha] \equiv 1 \pmod{2}} c_\alpha t^\alpha.$$

We define $P^0_g(t)$ and $P^1_g(t)$ similarly.

Since the equivariant signatures of $f$ and $g$ coincide, we have

$$tP^0_f(t) - P_f(t) \equiv tP^0_g(t) - P_g(t) \pmod{t^2 - 1},$$

$$tP^1_f(t) - P^0_f(t) \equiv tP^1_g(t) - P^0_g(t) \pmod{t^2 - 1}.$$
Adding up these two congruences we have

\[(t - 1)P_f(t) \equiv (t - 1)P_g(t) \mod t^2 - 1, \quad (1)\]
Proof of Theorem 2.6 (Continued)

Adding up these two congruences we have

\[(t - 1)P_f(t) \equiv (t - 1)P_g(t) \mod t^2 - 1,\]  \hspace{1cm} (1)

which implies

\[P_f(t) \equiv P_g(t) \mod t + 1.\]  \hspace{1cm} (2)
Proof of Theorem 2.6 (Continued)

Adding up these two congruences we have

\[(t - 1)P_f(t) \equiv (t - 1)P_g(t) \mod t^2 - 1,\]  

(1)

which implies

\[P_f(t) \equiv P_g(t) \mod t + 1.\]  

(2)

Conversely, suppose that (2) holds.

\[\implies (1) \text{ holds.}\]

\[\implies f \text{ and } g \text{ have the same equivariant signatures.}\]
Proof of Theorem 2.6 (Continued)

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which implies

\[P_f(t) \equiv P_g(t) \mod t + 1. \quad (2)\]

Conversely, suppose that (2) holds.

\[\implies (1) \text{ holds.} \]

\[\implies f \text{ and } g \text{ have the same equivariant signatures.} \]

Then, we can prove that they are Witt equivalent over \(\mathbb{R}\).

This completes the proof.
Proposition 2.8  Let

\[ f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j} \]

be Brieskorn–Pham type polynomials. Then, their Seifert forms are Witt equivalent over \( \mathbb{R} \) iff

\[ \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j} \]

holds for all odd integers \( \ell \).
Proof of Proposition 2.8 (Continued)

Proof.

$P_f(t)$ and $P_g(t)$ are polynomials in $s = t^{1/m}$ for some $m$. Put $Q_f(s) = P_f(t)$ and $Q_g(s) = P_g(t)$.

Then, $P_f(t) \equiv P_g(t) \mod t + 1$ holds

$\iff Q_f(\xi) = Q_g(\xi)$ for all $\xi$ with $\xi^m = -1$. 
Proof of Proposition 2.8 (Continued)

Proof.

$P_f(t)$ and $P_g(t)$ are polynomials in $s = t^{1/m}$ for some $m$. Put $Q_f(s) = P_f(t)$ and $Q_g(s) = P_g(t)$.

Then, $P_f(t) \equiv P_g(t) \mod t + 1$ holds

$\iff Q_f(\xi) = Q_g(\xi)$ for all $\xi$ with $\xi^m = -1$.

Note that $\xi$ is of the form

$$\exp\left(\pi\sqrt{-1}\ell/m\right)$$

with $\ell$ odd and that

$$\frac{-1 - \exp\left(\pi\sqrt{-1}\ell/a_j\right)}{\exp\left(\pi\sqrt{-1}\ell/a_j\right) - 1} = \sqrt{-1} \cot \frac{\pi \ell}{2a_j}.$$

Then, we immediately get Proposition 2.8.
Proof of Theorem 2.9

Theorem 2.9  Suppose that for each of the Brieskorn–Pham type polynomials

\[ f(z) = \sum_{j=1}^{n+1} z^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z^{b_j}, \]

no exponent is a multiple of another one.

Then, the knots \( K_f \) and \( K_g \) are \textbf{cobordant} iff

\[ a_j = b_j, \quad j = 1, 2, \ldots, n + 1, \]

up to order.
Proof of Theorem 2.9

Theorem 2.9  Suppose that for each of the Brieskorn–Pham type polynomials

\[ f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}, \]

no exponent is a multiple of another one. Then, the knots \( K_f \) and \( K_g \) are cobordant iff

\[ a_j = b_j, \quad j = 1, 2, \ldots, n+1, \]

up to order.

This is a consequence of the “Fox–Milnor type relation” for the Alexander polynomials of cobordant algebraic knots.
Open problem

Problem 3.3  Are the exponents cobordism invariants for Brieskorn–Pham type polynomials?
Open problem

Problem 3.3  *Are the exponents cobordism invariants for Brieskorn–Pham type polynomials?*

Proposition 2.8 reduces the above problem to a number theoretical problem involving cotangents.
Open problem

**Problem 3.3** Are the exponents cobordism invariants for Brieskorn–Pham type polynomials?

Proposition 2.8 reduces the above problem to a number theoretical problem involving cotangents.

\[
\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j} \quad \forall \text{odd integers } \ell
\]

\[\implies a_j = b_j \quad \text{up to order?}\]
Remark 3.4  Theorem 2.9 implies that two algebraic knots $K_f$ and $K_g$ associated with certain Brieskorn–Pham type polynomials are isotopic if and only of they are cobordant.
**Cobordism and isotopy for Brieskorn–Pham type polynomials**

**Remark 3.4** Theorem 2.9 implies that two algebraic knots $K_f$ and $K_g$ associated with certain Brieskorn–Pham type polynomials are **isotopic** if and only if they are **cobordant**.

According to **Yoshinaga–Suzuki**, two algebraic knots associated with Brieskorn–Pham type polynomials in general are isotopic if and only if they have the same set of exponents.
Remark 3.4  Theorem 2.9 implies that two algebraic knots $K_f$ and $K_g$ associated with certain Brieskorn–Pham type polynomials are isotopic if and only if they are cobordant.

According to Yoshinaga–Suzuki, two algebraic knots associated with Brieskorn–Pham type polynomials in general are isotopic if and only if they have the same set of exponents.

In fact, they showed that the characteristic polynomials coincide if and only if the Brieskorn–Pham type polynomials have the same set of exponents.
§1. Introduction

§2. Results

§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and isotopy for Brieskorn–Pham type polynomials

Thank you!