PROBLEMS

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Problem 1. Let $M$ and $N$ be smooth manifolds and $f : M \to N$ a smooth map. Define the notion of the “most natural map” (or the “simplest map”, or the “standard map”, or anything similar) among the generic smooth maps in the homotopy class of $f$, and study such maps (existence, uniqueness, their topological properties, etc.).

Problem 2 ([13]). Describe the Euler class $e$ of an oriented $S^1$-bundle in terms of the space $C^\infty(S^1, \mathbb{R}^2)$. Note that Kazarian [7] has obtained some results in terms of $C^\infty(S^1, \mathbb{R})$.

For example, for an oriented $S^1$-bundle $E$, if there exists a map $E \to \mathbb{R}^2$ that is an immersion of rotation number $\pm 1$ on each fiber, then the $S^1$-bundle is necessarily trivial, i.e. $e(E) = 0$.

Problem 3 ([13]). Characterize those Morse functions $S^1 \to \mathbb{R}$ which can be lifted to an embedding into $\mathbb{R}^2$.

Problem 4. Does there exist a special generic map $f : M^n \to \mathbb{R}^2$ of a closed orientable $n$-dimensional manifold $M^n$ into the plane that cannot be lifted to an immersion into $\mathbb{R}^{n+1}$?

Problem 5 ([8]). Let $G$ be an arbitrary finite graph without loops or isolated vertices.

(1) Is there an embedding $\eta : M^2 \to \mathbb{R}^3$ of a closed orientable surface such that the Reeb graph of the associated height function is homeomorphic to $G$?

(2) Is there an embedding $\eta : M^2 \to \mathbb{R}^3 \setminus \{0\}$ of a closed orientable surface such that the Reeb graph of the associated distance function from the origin is homeomorphic to $G$?

Problem 6. It is known that every graph 3-manifold $M^3$ admits a simple stable map into $\mathbb{R}^2$ and that the singular set of such a map is a graph link in $M^3$ [9]. Characterize those graph links which appear as the singular set of a simple stable map.

Problem 7. Let $M^3$ be a graph 3-manifold. Determine the smallest number of singular set components for simple stable maps $M^3 \to \mathbb{R}^2$.

Problem 8. Let $M^3$ be a closed 3-manifold. Let $\mathcal{W}(M^3)$ be the set of all compact polyhedrons that appear as the quotient space $W_f$ for a stable map $f : M^3 \to \mathbb{R}^2$. Does $\mathcal{W}(M^3_1) = \mathcal{W}(M^3_2)$ imply that $M^3_1$ is diffeomorphic to $M^3_2$?

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Problem 9 ([10]). For a smooth closed connected orientable 3-manifold $M$ and a positive integer $g$, are the following two conditions equivalent?

1. There exists a Morse function $f : M^3 \to \mathbb{R}$ such that the genus of every component of every regular fiber is at most $g$.
2. $M^3$ is diffeomorphic to the connected sum of finitely many closed orientable 3-manifolds of Heegaard genus at most $g$.

It is known that they are equivalent for $g = 1$.

Problem 10. Let $M^4$ be a closed oriented 4-dimensional manifold. For a $C^\infty$ stable map $f : M^4 \to \mathbb{R}^3$, it is known that the number of singular fibers of type III$^g$, counted with signs, coincides with the signature $\sigma(M^4)$ of $M^4$ (see [15]). Does there always exist a stable map $f : M^4 \to \mathbb{R}^3$ such that the number of singular fibers of type III$^g$ (counted without signs) coincides with $|\sigma(M^4)|$?

Problem 11. Let $M^4$ be a simply connected smooth closed 4-dimensional manifold. If $M^4$ admits a simple fold map into $\mathbb{R}^2$, then does it admit a special generic map into $\mathbb{R}^3$?

Problem 12. Let $G$ be a finitely presentable group. Does there exist a closed orientable 4-dimensional manifold $M^4$ and a simple stable map $M^4 \to \mathbb{R}^3$ such that $\pi_1(M) \cong G$? Or does there exist a closed orientable 4-dimensional manifold $M^4$ and a stable map $f : M^4 \to \mathbb{R}^2$ such that every component of every regular fiber is diffeomorphic to $S^2$? (See [14].)

Problem 13. Let $M^4_1$ and $M^4_2$ be smooth 4-dimensional manifolds that are homeomorphic. If there exist proper special generic maps $f_1 : M^4_1 \to \mathbb{R}^3$ and $f_2 : M^4_2 \to \mathbb{R}^3$, then are $M^4_1$ and $M^4_2$ diffeomorphic? (This would mean that the differentiable structure on a topological 4-manifold that allows the existence of a proper special generic map into $\mathbb{R}^3$ is unique.) See [12].

Problem 14. Does every closed non-orientable 4-dimensional manifold admit a fold map into $\mathbb{R}^3$?

Problem 15. It is known that closed manifolds whose tangent bundles satisfy certain conditions admit fold maps for which all the fold indices appear [1, 5]. Study the existence of fold maps with restricted set of fold indices. The extremal case corresponds to that of special generic maps.

Problem 16 (Gay–Kirby [6]). Let $M^n$ be a closed connected $n$-dimensional manifold $(n \geq 3)$. It is known that every smooth map $M^n \to S^2$ is homotopic to an excellent map (i.e. a smooth map with only folds and cusps as its singularities) without definite folds [11]. If $M^n$ is 1-connected, is every smooth map $M^n \to S^2$ homotopic to an excellent map without folds of index 0, 1, $n - 2$, $n - 1$?

Problem 17. Characterize those surface links that appear as the singular set of a stable map $S^4 \to \mathbb{R}^3$.

Problem 18. Let $C$ be a plane projective curve in $\mathbb{C}P^2$. Study the condition for $C$ to be topologically equivalent to a plane projective curve defined by a polynomial of real coefficients.

Problem 19. Let

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$


be Brieskorn–Pham type polynomials. It is known that if their associated algebraic knots are cobordant then their Seifert forms are Witt equivalent over $\mathbb{R}$. Furthermore, their Seifert forms are Witt equivalent over $\mathbb{R}$ if and only if
\[
\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j}
\]
holds for all odd integer $\ell$ (see [3]). Does it imply that $a_i = b_j$ up to renumbering the indices?

**Problem 20.** Let $f(z)$ be a Brieskorn–Pham type polynomial as above. Describe the condition on the exponents $a_j$ such that $H_{n-1}(K_f; \mathbb{Z})$ is torsion free, where $K_f = \overline{f^{-1}(0) \cap S^{2n+1}}$ is the $(2n - 1)$-dimensional closed manifold called the **link** of $f$. The condition for the vanishing of $H_{n-1}(K_f; \mathbb{Z})$ has been described in [4].

**Problem 21.** ([2]) Is the multiplicity of a complex holomorphic function germ at an isolated singular point a cobordism invariant of the associated algebraic knot? This is known to be true for the case of algebraic 1-knots.

**References**


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