Topology of
Definite Fold Singularities

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§1. Special Generic Maps
$M^m$: compact $C^\infty$ manifold without boundary
Morse function

§1. Special Generic Maps §2. 4-Dimensional Case §3. Broken Lefschetz Fibrations

$M^m$: compact $C^\infty$ manifold without boundary

**Definition 1.1** A **Morse function** $M^m \to \mathbb{R}$ is a $C^\infty$ function with each critical point being of the form

$$(x_1, x_2, \ldots, x_m) \mapsto \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_m^2 + c.$$

Number of negative signs is called the **index** of a critical point.
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\begin{aligned}
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$$\begin{cases} 
\text{local minimum} & \iff \text{index } 0 \\
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\end{cases}$$

They always appear if $M^m$ is compact.
Theorem 1.2 (Reeb, Smale, Cerf et al.)

$M^m$: compact $C^\infty$ manifold without boundary

$\exists$ Morse function $M^m \to \mathbb{R}$ with only critical points of index 0 or $m$

$\iff$

(1) $M^m \cong S^m$ (homeomorphic) ($m \neq 4$)

(2) $M^m \cong S^m$ (diffeomorphic) ($m = 4$)
Theorem 1.2 (Reeb, Smale, Cerf et al.)

\( M^m \): compact \( C^{\infty} \) manifold without boundary

\[ \exists \text{Morse function } M^m \rightarrow \mathbb{R} \text{ with only critical points of index 0 or } m \]

\[ \iff 
\begin{align*}
(1) & \quad M^m \approx S^m \text{ (homeomorphic)} \quad (m \neq 4) \\
(2) & \quad M^m \cong S^m \text{ (diffeomorphic)} \quad (m = 4)
\end{align*} \]

Remark 1.3

Generalized Poincaré conjecture is still open in dimension 4 in the \( C^\infty \) category.
Definition 1.4 A singularity of a $C^\infty$ map $M^m \to N^n$, $m \geq n$, that has the normal form

$$(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{n-1}, \pm x_n^2 \pm x_{n+1}^2 \pm \cdots \pm x_m^2)$$

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Definition 1.5  $f : M^m \to N^n$ is a **special generic map** (SGM, for short) if it has only **definite fold singularities**.

Example 1.6  A function $f : M^m \to \mathbb{R}$ is a SGM iff it is a Morse function with only critical points of index 0 or $m$. 
Examples of SGMs

§1. Special Generic Maps  §2. 4-Dimensional Case  §3. Broken Lefschetz Fibrations

Figure 1: Examples of special generic maps
Definition 1.7  $M^m$: compact

$$S(M^m) = \{ n \in \mathbb{Z} \mid 1 \leq n \leq m, \exists f : M^m \to \mathbb{R}^n \text{ SGM} \}$$
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$$S(M^m) = \{n \in \mathbb{Z} | 1 \leq n \leq m, \exists f : M^m \to \mathbb{R}^n \text{ SGM}\}$$

This is a diffeomorphism invariant of $M^m$.

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(1) \( S(S^m) = \{1, 2, \ldots, m\} \)

(2) \( S(S^a \times S^b) = \{a + 1, a + 2, \ldots, a + b\} \quad (a \leq b) \)
**Theorem 1.9 (S., 1993)**

$M^m$: compact $C^\infty$ manifold of dimension $m$

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SGMs can detect the standard differentiable structure on a sphere!
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Example 1.10

\( \Sigma^7 \): Milnor’s exotic 7-sphere
\( \{1, 2, 7\} \subset S(\Sigma^7) \subset \{1, 2, 3, 7\} \)
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\[ \exists V^{m+1} : \text{compact manifold with } \partial V^{m+1} = M_0^m \cup M_1^m, \]
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\begin{align*}
F|_{M_0} &= f_0 : M_0^m \to \mathbb{R}^n \times \{0\} \\
F|_{M_1} &= f_1 : M_1^m \to \mathbb{R}^n \times \{1\}
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Set of cobordism classes of all SGMs of $m$-dim. compact manifolds into $\mathbb{R}^n$ forms an **abelian group**, denoted by $\Gamma(m, n)$. 
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Groups of SGMs

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§1. Special Generic Maps  §2. 4-Dimensional Case  §3. Broken Lefschetz Fibrations

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**Theorem 1.12 (S., 2002)** $m \geq 6$

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Special generic maps $\uparrow$ closely related !
Differentiable structures
§2. 4-Dimensional Case
Theorem 2.1 (Sakuma-S., 1990’s)

\[ \exists (M^4_1, M^4_2): \text{pair of compact } C^\infty \text{ 4-manifolds such that} \]
\[ M^4_1 \approx M^4_2 \quad (\text{homeomorphic}), \]
\[ \exists f_1 : M^4_1 \to \mathbb{R}^3 \text{ SGM}, \]
\[ \nexists f_2 : M^4_2 \to \mathbb{R}^3 \text{ SGM}. \]

In fact, there are infinitely many such pairs.
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\( \exists (M_1^4, M_2^4): \) pair of compact \( C^\infty \) 4-manifolds such that

\( M_1^4 \cong M_2^4 \) (homeomorphic),

\( \exists f_1 : M_1^4 \rightarrow \mathbb{R}^3 \) SGM,

\( \forall f_2 : M_2^4 \rightarrow \mathbb{R}^3 \) SGM.

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\[ M_1^4 \not\cong M_2^4 \] non-diffeomorphic
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SGMs can detect distinct differentiable structures on a given topological 4-manifold.
Compact 1-connected 4-manifolds

§1. Special Generic Maps  §2. 4-Dimensional Case  §3. Broken Lefschetz Fibrations

Theorem 2.2 (S. (1993) + 3-dim. Poincaré Conj.)

\( M^4 \): compact simply connected \( C^\infty \) 4-manifold

\[ \exists f : M^4 \to \mathbb{R}^3 \quad \text{special generic map} \]

\[ \iff M^4 \cong \#^k (S^2 \times S^2) \text{ or } \#^k (S^2 \tilde{\times} S^2) \] (diffeomorphic)
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Remark 2.4 Smooth structures on $\#^k(S^2 \times S^2)$ are not unique. In fact, there are \textit{infinitely many} such structures if $k$ is a sufficiently big odd integer (Jongil Park, 2002).
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SGMs can detect the standard differentiable structure.
Remarks

§1. Special Generic Maps §2. 4-Dimensional Case §3. Broken Lefschetz Fibrations

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SGMs can detect the standard differentiable structure.

Remark 2.5 $M_1^4, M_2^4$: compact orientable $C^\infty$ 4-manifolds
If $M_1^4 \simeq M_2^4$ (homeomorphic), then
\[\exists f_1 : M_1^4 \to \mathbb{R}^3 \text{ smooth map with only fold singularities (} = \text{fold map})\]
\[\iff \exists f_2 : M_2^4 \to \mathbb{R}^3 \text{ fold map}\]
Remarks

§1. Special Generic Maps  §2. 4-Dimensional Case  §3. Broken Lefschetz Fibrations

Remark 2.4 Smooth structures on $\#^k (S^2 \times S^2)$ are not unique. In fact, there are infinitely many such structures if $k$ is a sufficiently big odd integer (Jongil Park, 2002).

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If $M^4_1 \approx M^4_2$ (homeomorphic), then
$\exists f_1 : M^4_1 \rightarrow \mathbb{R}^3$ smooth map with only fold singularities (= fold map)
$\iff \exists f_2 : M^4_2 \rightarrow \mathbb{R}^3$ fold map

Fold maps cannot detect distinct differentiable structures.
Theorem 2.6 (S., 2010)

\(M^4\): open simply connected \(C^\infty\) 4-manifold of “finite type”

\(\exists f : M^4 \to N^3\) proper special generic map

for some 3-manifold \(N^3\) with \(S(f) \neq \emptyset\)

\(\iff\) \(M^4\) is diffeomorphic to the connected sum

of a finite number of copies of the following manifolds:

- \(R^4(= S^4 \setminus \{\text{point}\})\),
- \(\text{Int} \left( \bigvee^k (S^2 \times D^2) \right) = S^4 \setminus (\bigvee^k S^1)\),
- \(S^2 \times S^2\),
- \(S^2 \tilde{\times} S^2\),
- \(R^2\)-bundle over \(S^2\)
Corollary 2.7

\( M^4 : C^\infty \) 4-manifold with \( M^4 \cong \mathbb{R}^4 \) (homeomorphic)

\( \exists f : M^4 \to \mathbb{R}^p \) proper SGM with \( S(f) \neq \emptyset \) for \( 1 \leq \exists p \leq 3 \)

\( \iff M^4 \cong \mathbb{R}^4 \) (diffeomorphic)
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Remark 2.8

It is known that \( \mathbb{R}^n, n \neq 4, \) has a unique differentiable structure (Munkres, Stallings, \( \sim 60'\text{s} \)).
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\( M^4 : C^\infty 4\text{-manifold with } M^4 \approx \mathbb{R}^4 \) (homeomorphic)

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Remark 2.8

It is known that \( \mathbb{R}^n, n \neq 4 \), has a unique differentiable structure (Munkres, Stallings, \( \sim 60\text{’s} \)).

However, \( \mathbb{R}^4 \) admits uncountably many differentiable structures (Donaldson, Freedman, Taubes, \( \sim 80\text{’s} \)).
**Remark 2.9** Every 4-manifold as in Theorem 2.6 admits infinitely many (or uncountably many) distinct differentiable structures.
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Theorem 2.6 implies that among them there is exactly one structure that allows the existence of a proper SGM into a 3-manifold.
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Conjecture 2.10

$M^4$: topological $4$-manifold

$\longrightarrow$ There exists at most one differentiable structure on $M^4$ that allows the existence of a proper SGM into $\mathbb{R}^3$. 
§3. Broken Lefschetz Fibrations
$M, \Sigma$: compact connected oriented $C^\infty$ manifolds
\dim M = 4, \dim \Sigma = 2
\( M, \Sigma \): compact connected oriented \( C^\infty \) manifolds
\( \dim M = 4, \dim \Sigma = 2 \)

**Definition 3.1**
A singularity of a smooth map \( M \to \Sigma \) that has the normal form

\[
(z, w) \mapsto zw
\]

w.r.t. complex coordinates compatible with the orientations, is called a **Lefschetz singularity**.
$M$, $\Sigma$: compact connected oriented $C^\infty$ manifolds
dim $M = 4$, dim $\Sigma = 2$

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A singularity of a smooth map $M \to \Sigma$ that has the normal form

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**Definition 3.2 (Auroux–Donaldson–Katzarkov 2005, etc.)**
Let $f : M \to \Sigma$ be a $C^\infty$ map. $f$ is a **broken Lefschetz fibration** (**BLF**, for short) if it has at most **Lefschetz** and **indefinite** fold singularities.
Fibers of a BLF

§1. Special Generic Maps  §2. 4-Dimensional Case  §3. Broken Lefschetz Fibrations

Image of indefinite fold singularities

Image of a Lefschetz critical point
Remark 3.3
Regular fibers of a BLF may not be connected. Even if they are connected, their genera may not be constant.
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Figure 2: Regular fibers near indefinite fold
Remark 3.3
Regular fibers of a BLF may not be connected.
Even if they are connected, their genera may not be constant.

For a BLF \( f : M^4 \to \Sigma^2 \), we denote by \( S_I(f) (\subset M^4) \) the **oriented** 1-dimensional submanifold of \( M^4 \) consisting of the indefinite fold singularities.
A usual Lefschetz fibration is a special case of a BLF.
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**Donaldson, Gompf, 1990’s**

**Lefschetz fibrations** ↔ **symplectic structures** (up to blow-up)
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\textbf{Donaldson, Gompf, 1990's}

\textbf{Lefschetz fibrations} $\iff$ \textbf{symplectic structures} (up to blow-up)

\textbf{Auroux–Donaldson–Katzarkov, 2005}

\textbf{broken Lefschetz fibrations} $\iff$ \textbf{near-symplectic structures}

$(S_I(f) \iff 1\text{-dim. sing. locus})$
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broken Lefschetz fibrations $\iff$ near-symplectic structures
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Symplectic structure: $\omega \in \Omega^2(M^4)$, $d\omega = 0$, non-degenerate ($\omega^2 > 0$)
A usual Lefschetz fibration is a special case of a BLF.

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Symplectic structure: $\omega \in \Omega^2(M^4)$, $d\omega = 0$, non-degenerate ($\omega^2 > 0$)

Kähler $\implies$ symplectic $\implies$ almost complex
A usual Lefschetz fibration is a special case of a BLF.

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Symplectic structure: $\omega \in \Omega^2(M^4)$, $d\omega = 0$, non-degenerate ($\omega^2 > 0$)

Kähler $\implies$ symplectic $\implies$ almost complex

\[\Downarrow\]

Gauge theoretic invariants can be defined.
Remark 3.4
Not every 4-manifold admits a symplectic structure.
(e.g. $\#^n \mathbb{C}P^2$, $n \geq 2$, etc.)
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On the other hand, it is known that every closed oriented 4-manifold $M^4$ with $b_2^+(M^4) > 0$ admits a near-symplectic structure.
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(e.g. $\#^n CP^2$, $n \geq 2$, etc.)

On the other hand, it is known that every closed oriented 4-manifold $M^4$ with $b_2^+(M^4) > 0$ admits a near-symplectic structure.

In fact, there are a variety of such structures on a given 4-manifold $M^4$. 
Definition 3.5  A singularity that has the normal form

\[(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^3 - 3x_1 x_2 + x_3^2 \pm x_4^2)\]

is called a cusp.
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is called a **cusp**.

![Indefinite cusp](image)

**Figure 3: Indefinite cusp**
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is called a cusp.

Figure 3: Indefinite cusp  Figure 4: Definite cusp
Facts.

Whitney (1955)

Every smooth map $M \to \Sigma$ is homotopic to a map with at most definite fold, indefinite fold, and cusp singularities.
Facts.

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Every smooth map $M \rightarrow \Sigma$ is homotopic to a map with at most definite fold, indefinite fold, and cusp singularities. Such a map is called an excellent map.
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Whitney (1955)
Every smooth map $M \rightarrow \Sigma$ is homotopic to a map with at most definite fold, indefinite fold, and cusp singularities. Such a map is called an **excellent map**.

Levine (1965)
Every smooth map $M \rightarrow \Sigma$ is homotopic to an excellent map without a cusp if $\chi(M)$ is even, and with exactly one cusp if $\chi(M)$ is odd.
Theorem 3.6 (S., 2006)

Every smooth map \( g : M \rightarrow S^2 \) is homotopic to an excellent map without definite fold singularities, and possibly with a cusp.
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Every smooth map $g : M \to S^2$ is homotopic to an excellent map without definite fold singularities, and possibly with a cusp.

In other words, we can eliminate definite fold singularities by homotopy.
Corollary 3.7 (Baykur, 2008)

Every closed oriented 4-manifold admits a BLF over $S^2$. 

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Figure 5: Sinking and Unsinking (Lekili 2009)
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**Theorem 3.8** \( g : M^4 \rightarrow S^2 \) a \( C^\infty \) map  
\( L \subset M^4 \): a non-empty closed oriented 1-dim. submanifold  
\( \exists f : M^4 \rightarrow S^2 \) BLF homotopic to \( g \) s.t. \( S_1(f) = L \)  
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Using similar techniques in the context of near-symplectic structures (Perutz, 2006; Lekili, 2009), we can prove the following.

**Theorem 3.9** \( M^4 : \) closed oriented 4-manifold with \( b_2^+(M^4) > 0 \)

\( L \subset M^4 : \) a non-empty closed oriented 1-dim. submanifold

\( \exists \textbf{near-symplectic structure } \omega \) whose zero locus coincides with \( L \)

\( \iff [L] = 0 \text{ in } H_1(M^4; \mathbb{Z}) \)
Remark 3.10 For the existence of a BLF, several proofs have been known (Auroux–Donaldson–Katzarkov, Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).
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Elimination of definite fold for generic homotopy is possible.
Lekili’s moves

§1. Special Generic Maps  §2. 4-Dimensional Case  §3. Broken Lefschetz Fibrations

Figure 6: Lekili’s moves
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Singularities of $C^\infty$ maps are closely related to differentiable structures of manifolds!
Thank you!