Desingularizing Special Generic Maps

Osamu Saeki
(Institute of Mathematics for Industry, Kyushu University)

Joint work with Masamichi Takase (Seikei University)

November 27, 2012
This is a **singular** plane curve.
This is a **singular** plane curve.

But, this might be the projected image of a **non-singular** space curve.
$M^n$: closed $n$-dim. $C^\infty$ manifold,

$f : M^n \rightarrow \mathbb{R}^p$ a generic $C^\infty$ map ($n \geq p$).
$M^n$: closed $n$-dim. $C^\infty$ manifold,

$f : M^n \rightarrow \mathbb{R}^p$ a generic $C^\infty$ map ($n \geq p$). ← always singular
$M^n$: closed $n$-dim. $C^\infty$ manifold,

$f : M^n \to \mathbb{R}^p$ a generic $C^\infty$ map ($n \geq p$). ← always singular

For $m > n \geq p$, $\pi : \mathbb{R}^m \to \mathbb{R}^p$ will denote the standard projection.
$M^n$: closed $n$-dim. $C^\infty$ manifold,
$f : M^n \to \mathbb{R}^p$ a generic $C^\infty$ map ($n \geq p$). ← always singular

For $m > n \geq p$, $\pi : \mathbb{R}^m \to \mathbb{R}^p$ will denote the standard projection.

Problem 1.1
$M^n$: closed $n$-dim. $C^\infty$ manifold,

$f : M^n \to \mathbb{R}^p$ a generic $C^\infty$ map ($n \geq p$). $\leftarrow$ always singular

For $m > n \geq p$, $\pi : \mathbb{R}^m \to \mathbb{R}^p$ will denote the standard projection.

Problem 1.1

$\eta$: immersion or embedding
**Problem 1.1**

\[ M^n: \text{closed } n\text{-dim. } C^\infty \text{ manifold,} \]
\[ f: M^n \to \mathbb{R}^p \text{ a generic } C^\infty \text{ map } (n \geq p). \leftarrow \text{ always singular} \]

For \( m > n \geq p \), \( \pi: \mathbb{R}^m \to \mathbb{R}^p \) will denote the standard projection.

\[ \eta: \text{immersion or embedding} \]

Yes, if \( m \gg n \).
$M^n$: closed $n$-dim. $C^\infty$ manifold,
$f : M^n \to \mathbb{R}^p$ a generic $C^\infty$ map ($n \geq p$). \hspace{5mm} \text{always singular}
For $m > n \geq p$, $\pi : \mathbb{R}^m \to \mathbb{R}^p$ will denote the standard projection.

**Problem 1.1**

\[ M^n \to \mathbb{R}^m \to \mathbb{R}^p \]

$\eta$: immersion or embedding

Yes, if $m >> n$.

In this talk, we consider the case $m = n + 1$. 
Theorem 1.2 (Haefliger, 1960) $f : M^2 \to \mathbb{R}^2$ generic immersion

\exists \text{immersion } \eta : M^2 \to \mathbb{R}^3 \text{ s.t. } f = \pi \circ \eta

\iff For every singular set component $S \ (\cong S^1)$ of $f$:
- if $S$ has an annulus nbhd, $S$ contains an even number of cusps,
- if $S$ has a Möbius band nbhd, $S$ contains an odd number of cusps.
Theorem 1.2 (Haefliger, 1960) \( f : M^2 \to \mathbb{R}^2 \) generic
\[ \exists \text{immersion} \; \eta : M^2 \to \mathbb{R}^3 \text{ s.t. } f = \pi \circ \eta \]

\( \iff \) For every singular set component \( S \) \( (\cong S^1) \) of \( f \):
if \( S \) has an annulus nbhd, \( S \) contains an even number of cusps,
if \( S \) has a Möbius band nbhd, \( S \) contains an odd number of cusps.

Theorem 1.3 (M. Yamamoto, 2007) \( f : M^2 \to \mathbb{R}^2 \) generic
There always exists an embedding \( \eta : M^2 \to \mathbb{R}^4 \) s.t. \( f = \pi \circ \eta \).
Theorem 1.2 (Haefliger, 1960) \( f : M^2 \to \mathbb{R}^2 \) generic
\[ \exists \text{immersion} \ \eta : M^2 \to \mathbb{R}^3 \ \text{s.t.} \ f = \pi \circ \eta \]
\[ \iff \text{For every singular set component } S \ (\cong S^1) \ \text{of } f : \]
\[ \text{if } S \ \text{has an annulus nbhd, } S \ \text{contains an even number of cusps,} \]
\[ \text{if } S \ \text{has a Möbius band nbhd, } S \ \text{contains an odd number of cusps.} \]

Theorem 1.3 (M. Yamamoto, 2007) \( f : M^2 \to \mathbb{R}^2 \) generic
There always exists an embedding \( \eta : M^2 \to \mathbb{R}^4 \) s.t. \( f = \pi \circ \eta. \)

Theorem 1.4 (Burlet–Haab, 1985) \( f : M^2 \to \mathbb{R} \) Morse
There always exists an immersion \( \eta : M^2 \to \mathbb{R}^3 \) s.t. \( f = \pi \circ \eta. \)
Theorem 1.5 (Saito, 1961) \( M^n: \text{orientable} \)
\[ f : M^n \to \mathbb{R}^n \text{ special generic map} \]
There always exists an immersion \( \eta : M^n \to \mathbb{R}^{n+1} \) s.t. \( f = \pi \circ \eta \).
Theorem 1.5 (Saito, 1961) \(M^n\): orientable
\(f : M^n \to \mathbb{R}^n\) special generic map
There always exists an immersion \(\eta : M^n \to \mathbb{R}^{n+1}\) s.t. \(f = \pi \circ \eta\).

Theorem 1.6 (Blank–Curley, 1985)
\(f : M^n \to \mathbb{R}^n\) generic,
\(\exists\) immersion \(\eta : M^n \to \mathbb{R}^{n+1}\) s.t. \(f = \pi \circ \eta\)
\(\iff \) \(\text{rk } df \geq n - 1, \text{ and } [\{\text{cusps}\}]^* + w_1(\nu) = 0 \text{ in } H^1(\{\text{folds}\}; \mathbb{Z}_2),\)
where \(\nu\) is the normal line bundle of \(\{\text{folds}\}\) in \(M^n\).
Today’s topic: Desingularization of special generic maps.
(Lifting special generic maps to immersions and embeddings in codimension 1.)
Today’s topic:
Desingularization of special generic maps.
(Lifting special generic maps to immersions and embeddings in codimension 1.)

Definition 1.7 A singularity of a $C^\infty$ map $M^n \to N^p$, $n \geq p$, that has the normal form

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{p-1}, x_p^2 + x_{p+1}^2 + \cdots + x_n^2)$$

is called a definite fold singularity.
Today’s topic: Desingularization of special generic maps. (Lifting special generic maps to immersions and embeddings in codimension 1.)

**Definition 1.7** A singularity of a $C^\infty$ map $M^n \to N^p$, $n \geq p$, that has the normal form

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{p-1}, x_p^2 + x_{p+1}^2 + \cdots + x_n^2)$$

is called a definite fold singularity.

**Definition 1.8** $f : M^n \to N^p$ is a special generic map (SGM, for short) if it has only definite fold singularities.
Examples

1. Desingularizing Singular Maps
2. Desingularizing Special Generic Functions
3. Desingularizing SGM’s into \( \mathbb{R}^2 \)
4. Further Results

Figure 1: Examples of special generic maps
§2. Desingularizing Special Generic Functions
Theorem 2.1 (Reeb, Smale, Cerf et al)

\( M^n: \) closed connected \( n \)-dim. \( C^\infty \) manifold

\( \exists \text{special generic function } M^n \to \mathbb{R} \)

\( \iff \)

(1) \( M^n \cong S^n \) (homeomorphic) \( (n \neq 4) \)

(2) \( M^n \cong S^n \) (diffeomorphic) \( (n = 4) \)
Theorem 2.1 (Reeb, Smale, Cerf et al)

\( M^n \): closed connected \( n \)-dim. \( C^\infty \) manifold

\( \exists \text{ special generic function } M^n \rightarrow \mathbb{R} \)

\( \iff \)

1. \( M^n \cong S^n \) (homeomorphic) \hspace{1cm} (\( n \neq 4 \))

2. \( M^n \cong S^n \) (diffeomorphic) \hspace{1cm} (\( n = 4 \))
In the following, $M^n$ will be closed and connected.
In the following, $M^n$ will be closed and connected.

**Theorem 2.2** $n \geq 1$

$f : M^n \to \mathbb{R}$ special generic function

There always exists an immersion $\eta : M^n \to \mathbb{R}^{n+1}$ s.t. $f = \pi \circ \eta$. 

In the following, $M^n$ will be closed and connected.

**Theorem 2.2** $n \geq 1$

$f : M^n \to \mathbb{R}$ special generic function

There always exists an immersion $\eta : M^n \to \mathbb{R}^{n+1}$ s.t. $f = \pi \circ \eta$.

Two immersions are **regularly homotopic** if they are in the same connected component of the space $\{\text{immersions } M^k \to \mathbb{R}^\ell\}$. 
In the following, $M^n$ will be closed and connected.

**Theorem 2.2** \( n \geq 1 \)

\( f : M^n \rightarrow \mathbb{R} \) special generic function

There always exists an **immersion** \( \eta : M^n \rightarrow \mathbb{R}^{n+1} \) s.t.

\[
\pi = f = \pi \circ \eta.
\]

Two immersions are **regularly homotopic** if they are in the same connected component of the space \( \{\text{immersions } M^k \rightarrow \mathbb{R}^\ell\} \).

**Lemma 2.3 (Kaiser, 1988)**

Let \( i : S^{n-1} \rightarrow \mathbb{R}^n \) be the standard embedding.

For \( \forall \) diffeomorphism \( \varphi : S^{n-1} \rightarrow S^{n-1} \) preserving the orientation, the immersions \( i \) and \( i \circ \varphi \) are **regularly homotopic**.
Proof of Theorem 2.2

§1. Desingularizing Singular Maps §2. Desingularizing Special Generic Functions §3. Desingularizing SGM’s into $\mathbb{R}^2$ §4. Further Results

\begin{proof}
\end{proof}
Theorem 2.4  \( n \geq 2 \)
\[ f : M^n \to \mathbb{R} \text{ special generic function} \]
\[ \exists \text{embedding } \eta : M^n \to \mathbb{R}^{n+1} \text{ s.t. } f = \pi \circ \eta \]
\[ \iff M^n \cong S^n \text{ (diffeomorphic)} \]
Theorem 2.4 \[ n \geq 2 \]
\[ f : M^n \to \mathbb{R} \text{ special generic function} \]
\[ \exists \text{embedding } \eta : M^n \to \mathbb{R}^{n+1} \text{ s.t. } f = \pi \circ \eta \]
\[ \iff M^n \cong S^n \text{ (diffeomorphic)} \]

This implies that there exist special generic functions that can be desingularized by immersions, but not by embeddings.
Theorem 2.4 \( n \geq 2 \)
\[ f : M^n \to \mathbb{R} \text{ special generic function} \]
\[ \exists \text{embedding} \eta : M^n \to \mathbb{R}^{n+1} \text{ s.t. } f = \pi \circ \eta \]
\[ \iff M^n \cong S^n \text{ (diffeomorphic)} \]

This implies that there exist special generic functions that can be desingularized by immersions, but not by embeddings.

Proof of Theorem 2.4: For \( n \neq 5 \), \( \varphi \) is isotopic to the identity. For \( n = 5 \), \( i \circ \varphi \) is isotopic to \( i \). \( \square \)
Theorem 2.4 \( n \geq 2 \)
\( f : M^n \to \mathbb{R} \) special generic function
\( \exists \text{embedding} \eta : M^n \to \mathbb{R}^{n+1} \) s.t. \( f = \pi \circ \eta \)
\( \iff M^n \cong S^n \) (diffeomorphic)

This implies that there exist special generic functions that can be desingularized by immersions, but not by embeddings.

Proof of Theorem 2.4: For \( n \neq 5 \), \( \varphi \) is isotopic to the identity.
For \( n = 5 \), \( i \circ \varphi \) is isotopic to \( i \).

Remark 2.5 When \( n = 1 \), the existence problem of an embedding lift has recently been solved by Minoru Yamamoto.
§3. Desingularizing SGM’s into $\mathbb{R}^2$
Theorem 3.1 (Burlet–de Rham, 1974; Porto–Furuya, 1990; S, 1993)

Let $M^n$ be a closed, connected, orientable manifold $(n \geq 2)$.

There exists a special generic map $f : M^n \to \mathbb{R}^2$ if and only if $M^n$ is diffeomorphic to

$$
\Sigma^n \# \left( \#_{i=1}^{r} (\Sigma_{i}^{n-1} \times S^1) \right)
$$

for some homotopy spheres $\Sigma^n$ and $\Sigma_{i}^{n-1}$ (for $n \leq 6$, they are standard spheres).
Theorem 3.2  \( M^n: \) orientable, \( n \geq 2. \)
\( f : M^n \to \mathbb{R}^2 \) special generic map
There always exists an immersion \( \eta : M^n \to \mathbb{R}^{n+1} \) s.t. \( f = \pi \circ \eta. \)
Theorem 3.2  \( M^n: \text{orientable}, \, n \geq 2. \)
\[ f : M^n \to \mathbb{R}^2 \text{ special generic map} \]
There always exists an \textbf{immersion} \( \eta : M^n \to \mathbb{R}^{n+1} \) s.t. \( f = \pi \circ \eta. \)

Remark 3.3  The case \( n = 2 \) is a consequence of Haefliger’s result.
Definition 3.4 \( f : M^n \to \mathbb{R}^p \) \( C^\infty \) map \( (n > p) \)

For \( x, x' \in M^n \), define \( x \sim_f x' \) if

(i) \( f(x) = f(x')(= y) \), and

(ii) \( x \) and \( x' \) belong to the same connected component of \( f^{-1}(y) \).
Definition 3.4 \( f : M^n \to \mathbb{R}^p \) \( C^\infty \) map \( (n > p) \)

For \( x, x' \in M^n \), define \( x \sim_f x' \) if

(i) \( f(x) = f(x')(= y) \), and

(ii) \( x \) and \( x' \) belong to the same connected component of \( f^{-1}(y) \).

\( W_f = M^n / \sim_f \) quotient space, \( q_f : M^n \to W_f \) quotient map

\( \exists! \bar{f} : W_f \to \mathbb{R}^p \) that makes the diagram commutative:

\[
\begin{array}{ccc}
M^n & \xrightarrow{f} & \mathbb{R}^p \\
\downarrow{q_f} & & \uparrow{\bar{f}} \\
W_f & & \\
\end{array}
\]
**Definition 3.4**  \( f : M^n \to \mathbb{R}^p \) \( \ C^\infty \) map \( (n > p) \)

For \( x, x' \in M^n \), define \( x \sim_f x' \) if

(i) \( f(x) = f(x')(= y) \), and

(ii) \( x \) and \( x' \) belong to the same connected component of \( f^{-1}(y) \).

\( W_f = M^n / \sim_f \) quotient space, \( q_f : M^n \to W_f \) quotient map

\( \exists! \bar{f} : W_f \to \mathbb{R}^p \) that makes the diagram commutative:

\[
\begin{array}{ccc}
M^n & \xrightarrow{f} & \mathbb{R}^p \\
q_f \downarrow & & \bar{f} \\
W_f & & \\
\end{array}
\]

The above diagram is called the **Stein factorization** of \( f \).
Figure 2: Stein factorization of a SGM
Proposition 3.5 \( f : M^n \rightarrow \mathbb{R}^p \) special generic map \((n > p)\).
Proposition 3.5 \( f : M^n \to \mathbb{R}^p \) special generic map \((n > p)\).

1. The singular point set \( S(f) \) is a regular submanifold of \( M^n \) of dimension \( p - 1 \),

2. \( W_f \) has the structure of a smooth \( p \)-dim. manifold with boundary such that \( \tilde{f} : W_f \to \mathbb{R}^p \) is an immersion.

3. \( q_f|_{S(f)} : S(f) \to \partial W_f \) is a diffeomorphism.

4. \( q_f|_{M^n \setminus S(f)} : M^n \setminus S(f) \to \text{Int } W_f \) is a smooth \( S^{n-p} \)-bundle.
Let $f : M \to \mathbb{R}^2 \ (p = 2)$ be a SGM.
We want to construct an immersion lift $\eta : M^n \to \mathbb{R}^{n+1}$ of $f$. 
Let $f : M \rightarrow \mathbb{R}^2 \ (p = 2)$ be a SGM.

We want to construct an immersion lift $\eta : M^n \rightarrow \mathbb{R}^{n+1}$ of $f$.

Enough to construct an immersion

$$\tilde{\eta} : M^n \ni W_f \times \mathbb{R}^{n-1} \left( \frac{\bar{f} \times \text{id}}{\Psi} \ni \mathbb{R}^2 \times \mathbb{R}^{n-1} \right)$$

of the form $\tilde{\eta} = (q_f, \ast)$. 
Let $f : M \rightarrow \mathbb{R}^2 \ (p = 2)$ be a SGM.
We want to construct an immersion lift $\eta : M^n \rightarrow \mathbb{R}^{n+1}$ of $f$.
Enough to construct an immersion

$$\tilde{\eta} : M^n \nleftrightarrow W_f \times \mathbb{R}^{n-1} \left( \tilde{f} \times \text{id} \nleftrightarrow \mathbb{R}^2 \times \mathbb{R}^{n-1} \right)$$

of the form $\tilde{\eta} = (q_f, \ast)$.
Easy to construct $\tilde{\eta}$ on a nbhd of $S(f)$, i.e. over a nbhd of $\partial W_f$. 
Let us consider a handlebody decomposition: \( W_f = h^0 \cup \left( \bigcup_{j=1}^{r} h_j^1 \right) \).
Let us consider a handlebody decomposition: 

\[ W_f = h^0 \cup \left( \bigcup_{j=1}^{r} h_j^1 \right). \]

Extend \( \tilde{\eta} \) over the 1-handles \( h_j^1 \) using lifts of special generic functions.
Let us consider a handlebody decomposition: \( W_f = h^0 \cup \left( \cup_{j=1}^r h_j^1 \right) \).

Extend \( \tilde{\eta} \) over the 1-handles \( h_j^1 \) using lifts of special generic functions. Let \( D \) be the 2-disk over which \( \tilde{\eta} \) has not been defined. By construction, over \( \partial D \), we have a family of embeddings \( \eta_t : S^{n-2} \to \mathbb{R}^{n-1}, t \in \partial D \).
We need to extend this family of embeddings to a family of immersions over the whole $D$. 
We need to extend this family of embeddings to a family of immersions over the whole $D$. This is possible if the following natural homomorphism is the zero map.

$$\pi_1 \text{Emb}(S^{n-2}, \mathbb{R}^{n-1}) \to \pi_1 \text{Imm}(S^{n-2}, \mathbb{R}^{n-1})$$
We need to extend this family of embeddings to a family of immersions over the whole $D$.

This is possible if the following natural homomorphism is the zero map.

$$
\pi_1 \text{Emb}(S^{n-2}, \mathbb{R}^{n-1}) \rightarrow \pi_1 \text{Imm}(S^{n-2}, \mathbb{R}^{n-1})
$$

By Lashof et al., we have the exact sequence, for $n \geq 6$,

$$
\pi_1 \text{Emb}(S^{n-2}, \mathbb{R}^{n-1}) \rightarrow \pi_1 \text{Imm}(S^{n-2}, \mathbb{R}^{n-1}) \rightarrow \pi_1 \text{Imm}^{\text{TOP}}(S^{n-2}, \mathbb{R}^{n-1}),
$$

where $\text{Imm}^{\text{TOP}}(S^{n-2}, \mathbb{R}^{n-1})$ denotes the space of locally flat topological immersions.
We need to extend this family of embeddings to a family of immersions over the whole $D$.

This is possible if the following natural homomorphism is the zero map.

$$\pi_1 \text{Emb}(S^{n-2}, \mathbb{R}^{n-1}) \to \pi_1 \text{Imm}(S^{n-2}, \mathbb{R}^{n-1})$$

By Lashof et al., we have the exact sequence, for $n \geq 6$,

$$\pi_1 \text{Emb}(S^{n-2}, \mathbb{R}^{n-1}) \to \pi_1 \text{Imm}(S^{n-2}, \mathbb{R}^{n-1}) \to \pi_1 \text{Imm}^{\text{TOP}}(S^{n-2}, \mathbb{R}^{n-1}),$$

where $\text{Imm}^{\text{TOP}}(S^{n-2}, \mathbb{R}^{n-1})$ denotes the space of locally flat topological immersions.

By Lees, Lashof, Burghelea, et al., the second map is injective.

$\Rightarrow$ DONE!
We need to extend this family of embeddings to a family of immersions over the whole $D$.
This is possible if the following natural homomorphism is the zero map.

$$\pi_1 \text{Emb}(S^{n-2}, \mathbb{R}^{n-1}) \rightarrow \pi_1 \text{Imm}(S^{n-2}, \mathbb{R}^{n-1})$$

By Lashof et al., we have the exact sequence, for $n \geq 6$,

$$\pi_1 \text{Emb}(S^{n-2}, \mathbb{R}^{n-1}) \rightarrow \pi_1 \text{Imm}(S^{n-2}, \mathbb{R}^{n-1}) \rightarrow \pi_1 \text{Imm}^{\text{TOP}}(S^{n-2}, \mathbb{R}^{n-1}),$$

where $\text{Imm}^{\text{TOP}}(S^{n-2}, \mathbb{R}^{n-1})$ denotes the space of locally flat topological immersions.
By Lees, Lashof, Burghelea, et al., the second map is injective.
$\Rightarrow$ DONE!
For $n = 3, 4, 5$, we use some arguments on $\text{Diff}(S^{n-2})$. 

$\square$
Non-orientable case

§1. Desingularizing Singular Maps  §2. Desingularizing Special Generic Functions  §3. Desingularizing SGM’s into $\mathbb{R}^2$  §4. Further Results

Theorem 3.6 $M^n$: non-orientable, $n \geq 2$.

$f : M^n \rightarrow \mathbb{R}^2$ special generic map

$\exists \text{immersion } \eta : M^n \rightarrow \mathbb{R}^{n+1}$ s.t. $f = \pi \circ \eta$

$\Leftrightarrow$

$n = 2, 4$ or $8$, and the tubular neighborhood of $S(f)$ in $M$ is orientable.
Non-orientable case

§1. Desingularizing Singular Maps §2. Desingularizing Special Generic Functions §3. Desingularizing SGM’s into $\mathbb{R}^2$ §4. Further Results

Theorem 3.6 $M^n$: non-orientable, $n \geq 2$.

$f : M^n \to \mathbb{R}^2$ special generic map

$\exists$ immersion $\eta : M^n \to \mathbb{R}^{n+1}$ s.t. $f = \pi \circ \eta$

$\iff$

$n = 2, 4$ or $8$, and the tubular neighborhood of $S(f)$ in $M$ is orientable.

Turning the sphere $S^{n-2} \subset \mathbb{R}^{n-1}$ inside out (sphere eversion) is possible if and only if $n = 2, 4, 8$. 
Theorem 3.7 \( f : M^n \to \mathbb{R}^2 \) special generic map, \( n \geq 3 \)

∃ embedding \( \eta : M^n \to \mathbb{R}^{n+1} \) s.t. \( f = \pi \circ \eta \)

\( \iff \) \( M \cong S^n \) or \( \#^k(S^1 \times S^{n-1}) \) (diffeomorphic).

Proof of (\( \leftarrow \rightarrow \))\( : \) The universal cover of \( \#^k(S^1 \times S^{n-1}) \) embeds in \( S^n \).
(Use the Schottky group argument. The free group of rank \( k \) can act on \( S^n \) as a Schottky group with totally disconnected limit set.)
Theorem 3.7 \( f : M^n \to \mathbb{R}^2 \) special generic map, \( n \geq 3 \)

\[ \exists \text{embedding } \eta : M^n \to \mathbb{R}^{n+1} \text{ s.t. } f = \pi \circ \eta \]

\[ \iff M \cong S^n \text{ or } \#^k(S^1 \times S^{n-1}) \text{ (diffeomorphic).} \]

Proof of (\( \iff \)):

The universal cover of \( \#^k(S^1 \times S^{n-1}) \) embeds in \( S^n \).

(Use the Schottky group argument. The free group of rank \( k \) can act on \( S^n \) as a Schottky group with totally disconnected limit set.)

Therefore, every homotopy \((n - 1)\)-sphere embedded in \( \#^k(S^1 \times S^{n-1}) \) is standard.
Theorem 3.7 \( f : M^n \to \mathbb{R}^2 \) special generic map, \( n \geq 3 \)

\[ \exists \text{embedding } \eta : M^n \to \mathbb{R}^{n+1} \text{ s.t. } f = \pi \circ \eta \]

\[ \iff M \cong S^n \text{ or } \sharp^k(S^1 \times S^{n-1}) \text{ (diffeomorphic)}. \]

Proof of (\( \iff \)): The universal cover of \( \sharp^k(S^1 \times S^{n-1}) \) embeds in \( S^n \).
(Use the Schottky group argument. The free group of rank \( k \) can act on \( S^n \) as a Schottky group with totally disconnected limit set.)

Therefore, every homotopy \( (n - 1) \)-sphere embedded in \( \sharp^k(S^1 \times S^{n-1}) \) is standard.

Then, one can construct an embedding lift using Theorem 2.4, with the help of a result of Schultz about “inertia group” of manifolds.
Theorem 3.7 \( f : M^n \to \mathbb{R}^2 \) special generic map, \( n \geq 3 \)

\[ \exists \text{embedding} \quad \eta : M^n \to \mathbb{R}^{n+1} \quad \text{s.t.} \quad f = \pi \circ \eta \]

\[ \iff M \cong S^n \text{ or } \#^k(S^1 \times S^{n-1}) \text{ (diffeomorphic)}. \]

Proof of (\( \Leftarrow \)):

The universal cover of \( \#^k(S^1 \times S^{n-1}) \) embeds in \( S^n \).

(Use the Schottky group argument. The free group of rank \( k \) can act on \( S^n \) as a Schottky group with totally disconnected limit set.)

Therefore, every homotopy \((n - 1)\)-sphere embedded in \( \#^k(S^1 \times S^{n-1}) \) is standard.

Then, one can construct an embedding lift using Theorem 2.4, with the help of a result of Schultz about “inertia group” of manifolds.

(\( \Rightarrow \)):

Standard argument.
§4. Further Results
**Theorem 4.1** \( M^n \): orientable, \( (n, p) = (5, 3), (6, 3), (6, 4) \) or \( (7, 4) \)

\( f : M^n \to \mathbb{R}^p \) special generic map

\( \exists \text{immersion} \ \eta : M^n \to \mathbb{R}^{n+1} \) s.t. \( f = \pi \circ \eta \)

\( \iff M^n \) is spin, i.e. \( w_2(M^n) = 0 \).
Theorem 4.1 \( M^n \): orientable, \((n, p) = (5, 3), (6, 3), (6, 4) \) or \((7, 4)\)

\( f : M^n \rightarrow \mathbb{R}^p \) special generic map

\( \exists \text{immersion} \ \eta : M^n \rightarrow \mathbb{R}^{n+1} \) s.t. \( f = \pi \circ \eta \)

\( \iff M^n \) is spin, i.e. \( w_2(M^n) = 0 \).

Key to the proof:

The Stein factorization induces a smooth \( S^{n-p} \)-bundle

\[ M^n \setminus S(f) \rightarrow \text{Int} \ W_f. \]
Theorem 4.1  \( M^n: \) orientable, \((n, p) = (5, 3), (6, 3), (6, 4) \) or \((7, 4)\)

\[ f : M^n \to \mathbb{R}^p \] special generic map

\[ \exists \text{immersion } \eta : M^n \to \mathbb{R}^{n+1} \text{ s.t. } f = \pi \circ \eta \]

\[ \iff M^n \text{ is spin, i.e. } w_2(M^n) = 0. \]

Key to the proof:
The Stein factorization induces a smooth \( S^{n-p} \)-bundle

\[ M^n \setminus S(f) \to \text{Int } W_f. \]

If \( w_2(M^n) = 0 \), then we can show that this is a trivial bundle.
Codimension $-1$ case

$f: M^n \rightarrow \mathbb{R}^p$ special generic map $(n > p)$

Orient $\mathbb{R}^p$. Then the quotient space $W_f$ has the induced orientation. Then $\partial W_f \cong S(f)$ also have the induced orientations.
$f : M^n \to \mathbb{R}^p$ special generic map ($n > p$)

Orient $\mathbb{R}^p$. Then the quotient space $W_f$ has the induced orientation.

Then $\partial W_f \cong S(f)$ also have the induced orientations.

**Theorem 4.2** $M^n$: orientable, $f : M^n \to \mathbb{R}^{n-1}$ special generic

∃**immersion** $\eta : M^n \to \mathbb{R}^{n+1}$ s.t. $f = \pi \circ \eta$

$\iff [S(f)] = 0 \text{ in } H_{n-2}(M^n; \mathbb{Z})$.
Codimension \(-1\) case

\[ f : M^n \to \mathbb{R}^p \] special generic map \((n > p)\)

Orient \(\mathbb{R}^p\). Then the quotient space \(W_f\) has the induced orientation. Then \(\partial W_f \cong S(f)\) also have the induced orientations.

**Theorem 4.2**

\(M^n: \) orientable, \(f : M^n \to \mathbb{R}^{n-1}\) special generic immersion \(\eta : M^n \to \mathbb{R}^{n+1}\) s.t. \(f = \pi \circ \eta\)

\[ \iff [S(f)] = 0 \text{ in } H_{n-2}(M^n; \mathbb{Z}) . \]

Key to the proof:
The Stein factorization induces a smooth \(S^1\)-bundle

\[ M^n \setminus S(f) \to \text{Int } W_f . \]
$f : M^n \to \mathbb{R}^p$ special generic map ($n > p$)
Orient $\mathbb{R}^p$. Then the quotient space $W_f$ has the induced orientation. Then $\partial W_f \cong S(f)$ also have the induced orientations.

**Theorem 4.2** $M^n$: orientable, $f : M^n \to \mathbb{R}^{n-1}$ special generic

∃ immersion $\eta : M^n \to \mathbb{R}^{n+1}$ s.t. $f = \pi \circ \eta$

$\iff [S(f)] = 0$ in $H_{n-2}(M^n; \mathbb{Z})$.

Key to the proof:
The Stein factorization induces a smooth $S^1$-bundle

$$M^n \setminus S(f) \to \text{Int } W_f.$$ 

If $[S(f)] = 0$, then we can show that this is a trivial bundle.
Special generic function $M^n \to \mathbb{R}$ can always be desingularized by an **immersion** $M^n \to \mathbb{R}^{n+1}$. It can be desingularized by an **embedding** iff $M^n \cong S^n$ (diffeo.).

Special generic map $f : M^n \to \mathbb{R}^2$ can always be desingularized by an **immersion** $M^n \to \mathbb{R}^{n+1}$ if $M^n$ is orientable. It can be desingularized by an **embedding** iff $M^n \cong S^n$ or $\#^k(S^1 \times S^{n-1})$ (diffeomorphic).

When $M^n$ is non-orientable, $f$ can be desingularized by an **immersion** iff $n = 2, 4, 8$ and $S(f)$ has an orientable nbhd.

Special generic map $f : M^n \to \mathbb{R}^3$ with $M^n$ orientable can be desingularized by an **immersion** $M^n \to \mathbb{R}^{n+1}$ iff $M^n$ is spin for $n = 5$ and 6.

Special generic map $f : M^n \to \mathbb{R}^{n-1}$ with $M^n$ orientable can be desingularized by an **immersion** $M^n \to \mathbb{R}^{n+1}$ iff $[S(f)] = 0$ in $H_{n-2}(M^n; \mathbb{Z})$. 
Muito obrigado!
Theorem 4.3 \( M^n: \) orientable, \( f : M^n \to \mathbb{R}^p \) special generic map
\((n, p) = (2, 1), (3, 2), (4, 3), (5, 3), (6, 3), (6, 4) \) or \((7, 4)\)
\( \implies \exists \) regular homotopy of immersions \( \eta_t : M^n \to \mathbb{R}^{n+1}, t \in [0, 1], \)
with \( f = \pi \circ \eta_0 \) s.t. \( f_t = \pi \circ \eta_t \) is a special generic map, \( t \in [0, 1], \)
and \( \eta_1 \) is an embedding.
Theorem 4.3 \( M^n \): orientable, \( f : M^n \to \mathbb{R}^p \) special generic map
\((n, p) = (2, 1), (3, 2), (4, 3), (5, 3), (6, 3), (6, 4) \) or \((7, 4)\)
\( \implies \exists \) regular homotopy of \textbf{immersions} \( \eta_t : M^n \to \mathbb{R}^{n+1}, t \in [0, 1], \)
with \( f = \pi \circ \eta_0 \) s.t. \( f_t = \pi \circ \eta_t \) is a special generic map, \( t \in [0, 1], \)
and \( \eta_1 \) is an \textbf{embedding}.

Theorem 4.4 \( M^4 \): orientable, \( \exists f : M^4 \to \mathbb{R}^3 \) special generic map
\( M^4 \) can be embedded into \( \mathbb{R}^5 \)
\( \iff M^4 \) is spin, i.e. \( w_2(M^4) = 0 \).