Elimination of Definite Fold and Broken Lefschetz Fibrations

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§1. Introduction

Singularities
Broken Lefschetz
Fibration
Definite Fold and Cusp
Base Diagrams for Folds
Base Diagrams for Cusps
Excellent map

§2. Elimination of Definite Fold

§3. Lekili’s Moves for BLF

§3. Isotopies
We will work in the **smooth category**.
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**Definition 1.1**

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3. A singularity that has the normal form
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(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^2 + x_3^2 - x_4^2)
\]
is called an **indefinite fold singularity** (or a **round singularity**).
Definition 1.2 (Auroux, Donaldson and Katzarkov 2005, etc.)
Let $f : M \to \Sigma$ be a smooth map.
(1) $f$ is a \textbf{broken Lefschetz fibration} (BLF, for short) if it has at most \underline{Lefschetz} and \underline{indefinite fold} singularities.
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Remark 1.3
(1) A usual Lefschetz fibration is a special case of a BLF.
(2) Regular fibers of a BLF (or ABLF) may not be connected. Even if they are connected, their genera may not be constant.

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In either case, $Z(f)$, the set of indefinite fold singularities of $f$, is a closed submanifold of $M$ of dimension 1.
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\[(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^3 - 3x_1 x_2 + x_3^2 \pm x_4^2)\]

is called a **cusp**.
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Figure 1: **Indefinite fold**

![Indefinite fold diagram](image)
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Figure 3: Indefinite cusp
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Base Diagrams for Cusps

Figure 3: Indefinite cusp

Figure 4: Definite cusp
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**Whitney (1955)** Every smooth map $M \rightarrow \Sigma$ is homotopic to a map with at most definite fold, indefinite fold, and cusp singularities. Such a map is called an excellent map.

**Levine (1965)** Every smooth map $M \rightarrow \Sigma$ is homotopic to an excellent map without a cusp if $\chi(M)$ is even, and with exactly one cusp if $\chi(M)$ is odd.
§2. Elimination of Definite Fold

Sketch of Proof
Modifying Excellent Maps
Definite to Indefinite
Existence of BLF

§3. Lekili’s Moves for BLF

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Theorem 2.1 (S. 2006) Every smooth map \( g : M \to S^2 \) is homotopic to an excellent map without definite fold singularities, and possibly with a cusp.
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In other words, we can eliminate definite fold singularities by homotopy.
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In the following, \( S(g) \) denotes the set of singular points, and \( S_0(g) \) denotes the set of definite fold singular points.

Step 1. Arrange \( S_0(g) \) so that it consists of a single “unknotted” component. Use Levine’s cusp elimination technique (S. 1995).

Step 2. Arrange \( g \) so that \( g |_{S_0(g)} \) is an embedding into \( S^2 \). Use Reidemeister-like moves on \( S^2 \) and their “lifts.” This is possible, since the target is the \( S^2 \)-sphere.

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- Sketch of Proof
- Modifying Excellent Maps
- Definite to Indefinite
- Existence of BLF

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§3. Isotopies

Figure 5: Birth

\[ \emptyset \]
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   Modifying Excellent Maps
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**Figure 5: Birth**

**Figure 6: Merge**
Step 3. Change the definite fold circle into an indefinite one (Williams 2010).
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Figure 7: Sinking and Unsinking (Lekili 2009)
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Remark 2.3 For the existence of BLF (or ABLF), several proofs have been known (Auroux–Donaldson–Katzarkov, Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).
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3. Lekili’s Moves for BLF
   - Birth and Merge
   - Flip and Wrinkle
   - Example
   - William’s Theorem
   - Williams’ Idea
   - Final Step

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Figure 8: Birth
Birth and Merge

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Birth and Merge

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Figure 8: Birth

Figure 9: Merge
Flip and Wrinkle

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Figure 10: Flip

Figure 11: Wrinkle
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Figure 10: Flip

Figure 11: Wrinkle
One can convert each achiral Lefschetz singularity to one circle of indefinite fold and three Lefschetz singularities (Lekili 2009).

![Diagram showing the transformation of an achiral Lefschetz singularity to a circle of indefinite fold and Lefschetz singularities.]

Figure 12: Removing an achiral Lefschetz singularity.
Theorem 3.1 (Williams 2010) If two BLFs $M \to \Sigma$ are homotopic, then one is obtained from the other by a finite sequence of Birth, Merge, Flip, Wrinkle, and Sink operations (and their inverses), together with “Isotopies”.

Remark 3.2 During the moves, indefinite cusps may appear. However, these cusps can be turned into Lefschetz singularities by “unsinking.”

Idea of Proof of Theorem 3.1 Each BLF can be homotoped to an excellent map without definite fold (by Wrinkle moves). By singularity theory, the two excellent maps can be connected by a generic 1-parameter family $f_t$ of smooth maps.
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§3. Isotopies

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- Every \( f_t : M \to \sum \) is an excellent map, except for a finite number of values of \( t \), say \( t_1, t_2, \ldots, t_k \).
- For each bifurcation value \( t_i \), the difference between \( f_{t_i \pm \varepsilon} \) is “well-understood”.

The generic homotopy \( \dot{F} : M \times [0; 1] \to \sum \) defined by

\[
F(t; t) = (f_t(x); t)
\]

has folds, cusps, and swallowtails.

Note. The BLFs \( f_0 \) and \( f_1 \) do not have definite folds, while for \( 0 < t < 1 \), \( f_t : M \to \sum \) may have definite folds.
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We need to \textbf{eliminate the definite folds} appearing in the generic homotopy \( F \).
Williams’ Idea

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**Note.** The BLFs \( f_0 \) and \( f_1 \) **do not have definite folds**, while for \( 0 < t < 1 \), \( f_t : M \rightarrow \Sigma \) may **have definite folds**.

We need to **eliminate the definite folds** appearing in the generic homotopy \( F \).

Williams’ idea: Remove the definite folds of the homotopy \( F \) by modifying it by “surgery” (not by homotopy).
Suppose that the generic homotopy $F$ has no definite folds. Then, Lekili has shown that his moves (together with isotopies) generate $F$, by essentially using singularity theory.
Bifurcations during Isotopies

“Isotopies” are generated by the following moves.
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\[ \text{Figure 13: Moves involving isotopies} \]
Y.K.S. Furuya, Sobre aplicações genéricas $M^4 \rightarrow \mathbb{R}^2$

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**Number of essential change types**
- II: 8 types
- III: 13 types
- C: 6 types
The integers indicate the genus of the corresponding fiber component.

Figure 14: A type III move
Example of Furuya’s Move (2)

Figure 15: A type C move
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Remark 4.2 Perutz (2007) defines Lagrangian matching invariants for BLFs. We do not know if they are invariant under Lekili’s moves (or under isotopies). It is conjectured that Lagrangian matching invariants equal the Seiberg–Witten invariants.
Problem 4.3 (Baykur)

*Find a sufficient sequence of moves that guarantees to stay within the class of fibrations without null-homologous fiber components.*
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How about the class of fibrations with connected fibers?

Note.

These guarantee that if we start with a near-symplectic BLF, then we can perform the moves within the subclass of near-symplectic BLFs.
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  Bifurcations during Isotopies
  Furuya’s Result
  Example of Furuya’s Move (1)
  Example of Furuya’s Move (2)
  Concluding Remarks
  Open Problem

Thank you!
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