Strong Markov property of determinatal processes

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Introduction

The configuration space of unlabelled particles:

$$\mathfrak{M} = \Big\{ \xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot) : \sharp\{j \in \mathbb{I} : x_j \in K\} < \infty, \text{ for any } K \text{ compact} \Big\}$$

$\mathfrak{M}$ is a Polish space with the vague topology.
The configuration space of noncolliding systems:

$$\begin{aligned} \mathfrak{M}_0 &= \Big\{ \xi \in \mathfrak{M} : \xi(\{x\}) = 1, \text{ for any } x \in \operatorname{supp} \xi \Big\} \\ &= \Big\{ \{x_j\} : \sharp\{j : x_j \in K\} < \infty, \text{ for any } K \text{ compact} \Big\}. \end{aligned}$$

A configuration space $\mathcal{X}$ is relative compact, if

$$\sup_{\xi \in \mathcal{X}} \xi(K) < \infty, \quad \text{for any } K \subset \mathbb{R} \text{ compact}$$

Introduction

The moment generating function of multitime distribution of the $\mathfrak{M}$-valued process $\Xi(t)$ is defined as

$$\Psi^{(t_1,\dots,t_M)}(f_1,\dots,f_M) = \Psi^{\mathbf{t}}(\mathbf{f}) = \mathbb{E}\left[\exp\left\{\sum_{m=1}^{M}\int_{\mathbb{R}} f_m(x)\Xi(t_m,dx)\right\}\right] \quad (1)$$

for $0 \le t_1 < t_2 < \cdots < t_M$, $\mathbf{f} = (f_1, f_2, \dots, f_M) \in \mathrm{C}_0(\mathbb{R})^M$.

$$\Psi^{\mathbf{t}}(\mathbf{f}) = \sum_{\substack{N_m \ge 0,\\ 1\le m\le M}} \int_{\prod_{m=1}^{M}\mathbb{R}^{N_m}} \prod_{m=1}^{M}\left\{\frac{1}{N_m!} d\mathbf{x}_{N_m}^{(m)} \prod_{i=1}^{N_m} \chi_m\left(x_i^{(m)}\right)\right\} \times \rho_\xi\left(t_1, \mathbf{x}_{N_1}^{(1)}; \dots; t_M, \mathbf{x}_{N_M}^{(M)}\right),$$

with $\chi_m(\cdot) = e^{f_m(\cdot)} - 1$ and the multitime correlation functions $\rho_\xi(\dots)$.

Introduction

A processs $\Xi(t)$ is said to be determinantal if the moment generating function (1) is given by the Fredholm determinant

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \operatorname*{Det}_{\substack{(s,t)\in\{t_1,t_2,\dots,t_M\}^2, \\ (x,y)\in\mathbb{R}^2}} \Big[\delta_{st}\delta_x(y) + \mathbb{K}(s,x;t,y)\chi_t(y)\Big], \tag{2}$$

In other words, the multitime correlation functions are represented as

$$\rho\left(t_1, \mathbf{x}^{(1)}_{N_1}; \dots ; t_M, \mathbf{x}^{(M)}_{N_M}\right) = \det_{\substack{1\le j\le N_m, 1\le k\le N_n \\ 1\le m,n\le M}} \left[\mathbb{K}(t_m, x_j^{(m)}; t_n, x_k^{(n)})\right].$$

The function $\mathbb{K}$ is called the correlation kernel of the process $\Xi(t)$.

Examples of determinatal processes

(1) Non-colliding Brownian motion (The Dyson model):

$$X_j(t) = x_j + B_j(t) + \sum_{\substack{k:1\le k\le N \\ k\neq j}} \int_0^t \frac{ds}{X_j(s) - X_k(s)}, \ 1 \le j \le N$$

where $B_j(t), j = 1, 2, \dots, N$ are independent one dimensional BMs.

(2) Non-colliding Bessel process:

$$X_j(t) = x_j + B_j(t) + \frac{2\nu+1}{2} \int_0^t \frac{ds}{X_j(s)} + \sum_{\substack{k:1\le k\le N \\ k\neq j}} \int_0^t \left\{ \frac{1}{X_j(s) - X_k(s)} + \frac{1}{X_j(s) + X_k(s)} \right\} ds, \ 1 \le j \le N$$

Related results

(O) [Spohn: 1987, Prähofer-Spohn: JSP02, Johanson: CMP02]
Infinite many particle systems Ξ_t obtaine by the Bulk or the Soft-edge scaling limits of the processes. equilibrium systems

(1) [Katori-T: JSP09, CMP10, JSP11, ECP13]:
Infinite many particle systems in non-equilibrium

(2) [Katori-T: MPRF11]
Markov property of the reversible Markov processes.

(3) [Oasda: PTRF12, SPA13, AOP13, Osada-T: in preparation, Osada-Honda: preprint]
Constructing diffusion processes $\widehat{\Xi}_t^{DF}$ by Dirichlet form technique and deriving ISDEs related to them.

(4) [Osada-T: in preparation] The uniqueness of solutions of the ISDE.

Infinite dimensional SDEs for $\widehat{\Xi}_t^{DF} = \sum_{j\in\mathbb{N}} \delta_{X_j(t)}$

Bulk scaling limit : [Osada:PTRF12] $\beta = 1, 2$ and 4

$$dX_j(t) = dB_j(t) + \frac{\beta}{2} \sum_{\substack{k\in\mathbb{N} \\ k\neq j}} \frac{dt}{X_j(t) - X_k(t)}, \quad j \in \mathbb{N}. \tag{3}$$

Hard edge scaling limit : [Honda-Osada: preprint] $\nu > -1$

$$\begin{aligned} dX_j(t) &= dB_j(t) + \frac{2\nu+1}{2} \frac{1}{X_j(t)} dt \\ &\quad + \frac{\beta}{2} \sum_{\substack{k\in\mathbb{N} \\ k\neq j}} \left\{ \frac{1}{X_j(t) - X_k(t)} + \frac{1}{X_j(t) + X_k(t)} \right\} dt, \ j \in \mathbb{N}. \end{aligned} \tag{4}$$

Infinite dimensional SDEs for $\widehat{\Xi}_t^{DF} = \sum_{j\in\mathbb{N}} \delta_{X_j(t)}$

Soft edge scaling limit : [Osada-T: in preparation] $\beta = 1, 2$ and 4

$$
\begin{aligned}
dX_j(t) &= dB_j(t) \\
&+ \frac{\beta}{2} \lim_{L\to\infty} \left\{ \sum_{\substack{k\in\mathbb{N}, k\neq j \\ |X_k(t)|\leq L}} \frac{1}{X_j(t) - X_k(t)} - \int_{|y|\leq L} \frac{\widehat{\rho}(y)}{-y} dy \right\} dt, \ j \in \mathbb{N}, (5)
\end{aligned}
$$

where $\widehat{\rho}(x) = \frac{\sqrt{-x}\mathbf{1}(x<0)}{\pi}$.

Questions

1. The Dyson model constructed by Spohn(1987) solves ISDE (1)?
2. The infinite particle system constructed by Prähofer-Spohn(2002), Johannson(2002) solves ISDE (2)?

Main Theorem

In this talk we consider the case that $\beta = 2$, and $\nu = \frac{1}{2}, \frac{-1}{2}$.

Theorem

Let Ξ_t be the determinantal process obtained by (i) or (ii) or (iii):

(i) the bulk scaling limit of noncolliding BM

(ii) the soft-edge scaling limit of noncolliding BM

(iii) the hard-edge scalig limit of one or three dimensional Bessel process.

There exists the state spaces $\mathcal{S}$ associated with the process Ξ_t such that the process $(\mathbb{P}_\xi, \Xi_t)$, $\xi \in \mathcal{S}$ has the strong Markov property.

Answer the questions

Strong Markov property of Ξ_t

$\longrightarrow$ the quasi-regularity of the Dirichlet form associated with Ξ_t

$\longrightarrow$ Ξ_t solve the same ISDE as $\widehat{\Xi}_t^{DF}$(by the technique in (3))

$\longrightarrow$ the coincidence of the processes Ξ_t and $\widehat{\Xi}_t^{DF}$ (by (4))

Remark. The state space of $\widehat{\Xi}_t^{DF}$ is uniquely determined up to quasi everwhere. That is $\mathcal{S}$ is a version of the state space.

State Space

$\mathfrak{M}_{L_0}^{\kappa_1}(\zeta)$, $\zeta \in \mathfrak{M}$, $\kappa_1 > 0$, $L_0 \in \mathbb{N}$, is the set of configurations $\xi \in \mathfrak{M}_0$ satisfying

$$|\zeta([0,L] - \xi([0,L])| \le L^{\kappa_1}, \ |\zeta([-L,0] - \xi([-L,0])| \le L^{\kappa_1}, \quad L \ge L_0. \quad (6)$$

$\mathfrak{N}_{L_0}^{\kappa_2}$, $\kappa_2 > 0$, $L_0 \in \mathbb{N}$, is the set of configurations $\xi \in \mathfrak{M}_0$ satisfying

$$\xi\left([x - e^{-|x|^{\kappa_2}}, x + e^{-|x|^{\kappa_2}}] \setminus \{x\} \right) = 0, \ x \in \operatorname{supp} \xi, \ |x| > L_0. \qquad (7)$$

Put

$$\mathfrak{M}^{\kappa_1}(\zeta) = \bigcup_{L_0 \in \mathbb{N}} \mathfrak{M}_{L_0}^{\kappa_1}(\zeta), \quad \mathfrak{N}^{\kappa_2} = \bigcup_{L_0 \in \mathbb{N}} \mathfrak{N}_{L_0}^{\kappa_2}.$$

$$\mathcal{S}(\zeta) = \bigcup_{\kappa \in (0,1)} \left(\mathfrak{M}^{\kappa}(\zeta) \cap \mathfrak{N}^{\kappa} \right).$$

State space

We introduce the following configurations:

$$\zeta_\rho = \sum_{x:\sin(\rho x)=0} \delta_x, \quad \zeta_{\mathrm{Ai}} = \sum_{x:\mathrm{Ai}(x)=0} \delta_x, \quad \zeta^{(\nu)} = \sum_{x:J_\nu(x)=0} \delta_x.$$

Let

$\mu_{\sin}$: The determinatal point process with the sine kernel.

μ_{Ai} : The determinatal point process with the Airy kernel.

μ_{J_ν} : The determinatal point process with the Bessel kernel.

Let $\kappa_2 > \kappa_1 > 0$, It is readily checked that

$$\mu_{\sin}(\mathfrak{M}^{\kappa_1}(\zeta_\rho) \cap \mathfrak{N}^{\kappa_2}) = 1, \quad \kappa_1 > 0,$$

$$\mu_{\mathrm{Ai}}(\mathfrak{M}^{\kappa_1}(\zeta_{\mathrm{Ai}}) \cap \mathfrak{N}^{\kappa_2}) = 1 \quad \kappa_1 > 1/2,$$

$$\mu_{J_\nu}(\{\xi^{\langle 2\rangle} = \sum_{j\in\mathbb{N}} \delta_{x_j^2} : \xi = \sum_{j\in\mathbb{N}} \delta_{x_j} \in \mathfrak{M}^{\kappa_1}(\zeta^{(\nu)}) \cap \mathfrak{N}^{\kappa_2}\}) = 1 \quad \kappa_1 > 0.$$

Harmonic transformation (Brownian motions with a drift)

$\mathbf{B}(t) = (B_1(t), \dots, B_N(t))$: N-dim BM

Let g be a $\mathcal{F}_t$-adapted symmetric measurable function on $C([0, T], \mathbb{R}^N)$.

(1) **Noncolliding Brownian motions with constant drift**

$$\mathbb{E}_{\mathbf{x}}[g(\mathbf{Y}(\cdot))] = E_{\mathbf{x}}\left[g(\mathbf{B}(\cdot)) \frac{h_N(\mathbf{B}(T))}{h_N(\mathbf{x})} \prod_{j=1}^{N} e^{D(B_j - x_j)(T) - \frac{D^2 T}{2}} \right],$$

where

$$h_N(\mathbf{x}) = \prod_{1 \le i < j \le N} (x_j - x_i) = \det_{1 \le i,j \le N} \left[x_i^{j-1} \right].$$

Harmonic transformation (Bessel processes $\nu = \pm\frac{1}{2}$)

Suppose that $g(x_1, x_2, \ldots, x_N) = g(|x_1|, |x_2|, \ldots, |x_N|)$.

(2) **Noncolliding 3-dimensional Bessel process (Type C)**

$$\mathbb{E}_{\mathbf{x}}[g(\mathbf{Y}(\cdot))] = E_{\mathbf{x}}\left[g(\mathbf{B}(\cdot))\frac{h_N^{(0)}(\mathbf{B}(T))}{h_N^{(0)}(\mathbf{x})}\prod_{j=1}^{N}\frac{B_j(T)}{x_j}\right],$$

where

$$h_N^{(0)}(\mathbf{x}) = \prod_{1\le i<j\le N}(x_j^2 - x_i^2) = \det_{1\le i,j\le N}\left[x_i^{2(j-1)}\right].$$

(3) **Noncolliding 1-dimensional Bessel process (Type D)**

$$\mathbb{E}_{\mathbf{x}}[g_1(\mathbf{Y}(\cdot))] = E_{\mathbf{x}}\left[g(\mathbf{B}(\cdot))\frac{h_N^{(0)}(\mathbf{B}(T))}{h_N^{(0)}(\mathbf{x})}\right].$$

Entire functions

Suppose that $\xi = \sum_{i=1}^{\xi(\mathbb{R})} \delta_{u_i} \in \mathfrak{M}_0$ with $\xi(\mathbb{R}) \in \mathbb{N}$.

$$\Phi_0(\xi, u, z) = \prod_{r \in \mathrm{supp}\ \xi - \{u\}} \left(1 - \frac{z-u}{r-u}\right), \quad z \in \mathbb{C}, u \in \mathbb{R}$$

(1)(Noncolliding Brownian motions with constant drift)

$$\Phi^D(\xi, u, z) = e^{D(z-u)} \Phi_0(\xi, u, z), \quad u \in \mathbb{R}.$$

(2)(Noncolliding 3-dimensional Bessel process (Type C))

$$\Phi^{(1/2)}(\xi, u, z) = \frac{z}{u} \Phi_0(\xi^{\langle 2 \rangle}, u^2, z^2), \quad u > 0.$$

(3)(Noncolliding 1-dimensional Bessel process (Type D))

$$\Phi^{(-1/2)}(\xi, u, z) = \Phi_0(\xi^{\langle 2 \rangle}, u^2, z^2), \quad u \geq 0.$$

Complex Brownian motion

$V_j(t), 1 \le j \le N$: indep. BMs with $V_j(0) = u_j \in \mathbb{R}$
$W_j(t), 1 \le j \le N$: indep. BMs with $W_j(0) = 0$,

$$Z_j(t) = V_j(t) + \sqrt{-1}W_j(t),\ 1 \le j \le N : \text{Complex BMs}$$

Put

$$\mathbf{Z}(t) = (Z_1(t), Z_2(t), \dots, Z_N(t)),\ \mathbf{V}(t) = (V_1(t), V_2(t), \dots, V_N(t)),$$

Noting that

$$E^{\mathbf{W}}[h_N(\mathbf{Z}(T))] = h_N(\mathbf{V}(N)),$$

and for $\xi = \sum_{j=1}^N \delta_{x_j}$, $\mathbf{x} \in \mathbb{C}^N$,

$$\frac{h_N(\mathbf{z})}{h_N(\mathbf{x})} = \det_{1 \le j,k \le N} \Phi_0(\xi, x_j, z_k).$$

Theorem (Complex Brownian motion representations)

Let $0 < t < T < \infty$. For any $\mathcal{F}(t)$-measurable function F:

(1)(Noncolliding Brownian motions with constant drift)

$$\mathbb{E}_\xi \left[F(\Xi(\cdot)) \right] = \mathbf{E_u} \left[F\left(\sum_{i=1}^{\xi(\mathbb{R})} \delta_{V_i(\cdot)} \right) \det_{1\le i,j\le \xi(\mathbb{R})} \left[\Phi^D(\xi, u_i, Z_j(T)) \right] \right].$$

(2)(Noncolliding 3-dimensional Bessel process (Type C))

$$\mathbb{E}_\xi \left[F(\Xi(\cdot)) \right] = \mathbf{E_u} \left[F\left(\sum_{i=1}^{\xi(\mathbb{R})} \delta_{V_i(\cdot)} \right) \det_{1\le i,j\le \xi(\mathbb{R})} \left[\Phi^{(1/2)}(\xi, u_i, Z_j(T)) \right] \right].$$

(3)(Noncolliding 1-dimensional Bessel process (Type D))

$$\mathbb{E}_\xi \left[F(\Xi(\cdot)) \right] = \mathbf{E_u} \left[F\left(\sum_{i=1}^{\xi(\mathbb{R})} \delta_{|V_i(\cdot)|} \right) \det_{1\le i,j\le \xi(\mathbb{R})} \left[\Phi^{(-1/2)}(\xi, u_i, Z_j(T)) \right] \right].$$

Entire functions for configurations of infinitely many points

Lemma 1 If $\xi \in \mathfrak{M}^{\kappa_1}(\zeta_\rho) \cap \mathfrak{N}^{\kappa_2}$ with $0 < \kappa_1 < \kappa_2 < 1$,

$$\Phi_0(\xi, a, z) = \lim_{L\to\infty} \Phi_0(\xi \cap [-L, L], a, z), \quad \text{exists, and}$$

$$|\Phi_0(\xi, a, z)| \le C \exp\{c(|a|^\delta + |z|^\theta)\}, \quad a \in \mathrm{supp}\ \xi,\ z \in \mathbb{C},$$

for some $C = C(\kappa_1, \kappa_2, L_0) > 0, c = c(\kappa_1, \kappa_2, L_0) > 0$, $\delta \in (0, 1)$ and $\theta \in (1, 2)$.

THEOREM 1 (Katori-T. CMP10, ECP13)
Suppose that $\xi \in \mathcal{S}(\zeta_\rho)$. Then

$$(\Xi(t), \mathbb{P}_{\xi\cap[-L,L]}) \to (\Xi(t), \mathbb{P}_\xi), \quad L \to \infty$$

weakly on $\mathrm{C}([0, \infty) \to \mathfrak{M}_0)$. In particular, the process $(\Xi(t), \mathbb{P}_\xi)$ has a modification which is almost-surely continuous on $[0, \infty)$ with $\Xi(0) = \xi$.

CBM representation for infinte particle systems

In case F is a polynomial, CBMR holds for $\xi \in \mathcal{S}(\zeta_\rho)$. For example

(1) For any measurable function f on $\mathbb{R}$ with compact support

$$\mathbb{E}_\xi \left[\int_{\mathbb{R}} f(u) \Xi_t(du) \right] = \int_{\mathbb{R}} \xi(du) \mathbf{E}_u \left[f(V_1(t)) \Phi_0(\xi, u, Z_1(T) \right]$$

(2) For any measurable symmetric function f on $\mathbb{R}^2$ with compact support

$$\mathbb{E}_\xi \left[\int_{u_1<u_2} \phi(u_1, u_2) \Xi_t \otimes \Xi_t(du_1 du_2) \right]$$

$$= \int_{u_1<u_2} \xi(du) \mathbf{E}_{(u_1,u_2)} \left[f(V_1(t), V_2(t)) \det_{1\le i,j\le 2} \Phi_0(\xi, u_i, Z_j(T)) \right]$$

State space

Lemma 2 Let $\zeta = \zeta_\rho, \zeta_{\mathrm{Ai}}$ or $\zeta^{(\nu)}$, $\nu = \pm\frac{1}{2}$, $1/2 < \kappa_1 < \kappa_2 < 1$. Then for any $\kappa_1' > \kappa_1$ and $\kappa_2' > \kappa_2$

$$\mathbb{P}_\xi \left(\Xi_t \in \mathfrak{M}^{\kappa_1'}(\zeta) \cap \mathfrak{N}^{\kappa_2'}, \quad t > 0 \right) = 1.$$

To prove this lemma we apply CBMR to functions F_1 and F_2:

$$F_1\left(\sum_j \delta_{X_j(\cdot)} \right) = \sum_j \sup_{t \in [0,T]} |f(X_j(t)) - f(X_j(0))|,$$

$$F_2\left(\sum_j \delta_{X_j(\cdot)} \right) = \sum_{j,k} \sup_{t \in [0,T]} f(|X_j(t) - X_k(t)|),$$

and the following identity:

Another key identity

For a symmetric function $F(v_1, v_2)$

$$\mathbf{E}_{(u_1,u_2)}\left[f(V_1, V_2)\det_{1\le i,j\le 2}\Phi_0(\xi, u_i, Z_j)\right]$$
$$= \mathbf{E}_{(u_1,u_2)}\left[f(V_1, V_2)\frac{h_2(V_1(T), V_2(T))}{h_2(u_1, u_2)}\right.$$
$$\left.\times \prod_{j=1,2}\Phi_0(\xi\cap\{u_1, u_2\}^c, u_j, Z_j(T))\right]. \qquad (8)$$

Then, $(Y_1(t), Y_2(t))$ can be treated as the non-colliding Brownian motion of 2-particles.

Soft-edge case

We introduce the entire function defined by

$$\Phi_1(\xi, 0, z) = \exp\Big\{ - \int_{\{0\}^c} \frac{z}{x} \xi(dx) \Big\} \Phi_0(\xi, 0, z).$$

Then $\mathrm{Ai}(z) = \exp\{d_1 z\}\Phi_1(\zeta_{\mathrm{Ai}}, 0, z)$. The soft-edge scaling limit case is associated with the entire function for ξ with $\xi(\mathbb{R}) = N$ defined by

$$\Phi_{\mathrm{Ai}}(\xi, 0, z) = \exp\Big\{ d_1 z + \int_{\{0\}^c} \frac{z}{x} (\xi - \zeta_{\mathrm{Ai}}^N)(dx) \Big\} \Phi_1(\zeta_{\mathrm{Ai}}, 0, z),$$

$$\Phi_{\mathrm{Ai}}(\xi, u, z) = \Phi_{\mathrm{Ai}}(\tau_{-u}\xi, 0, z - u).$$

where $\zeta_{\mathrm{Ai}}^N = \sum_{j=1}^N \delta_{a_j}$, with the decreasing sequence $\{a_j\}$ of zeros of Ai.

Hard-edge case

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \Phi_0(\zeta_{J_\nu}^{\langle 2 \rangle}, z^2)$$

The hard-edge scaling limit case is associated with the entire functions defined by

$$\Phi^{(1/2)}(\xi, u, z) = \frac{z}{u} \Phi_0(\xi^{\langle 2 \rangle}, u^2, z^2),$$

$$\Phi^{(-1/2)}(\xi, u, z) = \Phi_0(\xi^{\langle 2 \rangle}, u^2, z^2).$$

Markov property

Lemma 3 Let $(\zeta, \Phi) = (\zeta_\rho, \Phi_0), (\zeta_{\rm Ai}, \Phi_{\rm Ai})$ or $(\zeta^{(\nu)}, \Phi^{(\nu)})$, $\nu = \pm\frac{1}{2}$, and let $1/2 < \kappa_1 < \kappa_2 < 1$.

(1) Suppose that $\xi \in \mathfrak{M}_{L_0}^{\kappa_1}(\zeta) \cap \mathfrak{N}_{L_0}^{\kappa_2}$, with $L_0 \in \mathbb{N}$. Then

$$\Phi(\xi \cap [-L, L], u, z) \to \Phi(\xi, u, z), \quad L \to \infty.$$

(2) Suppose $\xi_n, \xi \in \mathfrak{M}_{L_0}^{\kappa_1}(\zeta) \cap \mathfrak{N}_{L_0}^{\kappa_2}(\zeta)$, and $u_n \in \mathrm{supp}\ \xi_n, u \in \mathrm{supp}\ \xi$. If

$$\xi_n \to \xi, \quad \text{vaguely} \quad \text{and} \quad u_n \to u, \quad n \to \infty,$$

then

$$\Phi(\xi_n, u_n, z) \to \Phi(\xi, u, z), \quad n \to \infty$$

$(\mathbb{P}_{\xi\cap[-L,L]}, \Xi_t)$ is a Markov process, that is,

$$\mathbb{E}_{\xi\cap[-L,L]}\Big[f_1(\Xi_s)f_2(\Xi_t)\Big] = \mathbb{E}_{\xi\cap[-L,L]}\Big[f_1(\Xi_s)\mathbb{E}_{\Xi_s}[f_2(\Xi_{t-s})]\Big]$$

for $0 < s < t$ and polynomials f_1, f_2:

$$f_j(\xi) = Q_j\Big(\int_{\mathbb{R}} \varphi_1(x)\xi(dx), \dots, \int_{\mathbb{R}} \varphi_k(x)\xi(dx)\Big)$$

By taking $L \to \infty$ from Lemmas 1 and 3

$$\mathbb{E}_{\xi}\Big[f_1(\Xi_s)f_2(\Xi_t)\Big] = \mathbb{E}_{\xi}\Big[f_1(\Xi_s)\mathbb{E}_{\Xi_s}[f_2(\Xi_{t-s})]\Big]$$

Strong Markov property

For a polynomial function f we put

$$T_t f(\xi) = \mathbb{E}_\xi[f(\Xi_t)],$$

the semigroup associated with the Markov process $(\mathbb{P}_\xi, \Xi_t)$.
Suppose that $\xi_n, \xi \in \mathfrak{M}^{\varepsilon}_{L_0}(\xi_{\mathbb{Z}/\rho}) \cap \mathfrak{N}^{\kappa}_{L_0}$ for some $L_0 \in \mathbb{N}$.
From Lemmas 1 and 2,

$$\xi_n \to \xi, \quad \text{vaguely} \quad \Rightarrow \quad T_t f(\xi_n) \to T_t f(\xi).$$

We equip $\mathcal{S}^\delta$ with the topology by the inductive limit. Then, for $\xi_n, \xi \in \mathcal{S}^\delta$

$$\xi_n \to \xi, \qquad \Rightarrow \qquad T_t f(\xi_n) \to T_t f(\xi).$$

We have the Feller property.

Thanks

Thank you for your attention!