# Strong Markov property of determinantal processes associated with extended kernels

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# Introduction

The configuration space of unlabelled particles:

$$\mathfrak{M} = \Big\{ \xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot) : \sharp\{j \in \mathbb{I} : x_j \in K\} < \infty, \text{ for any } K \text{ compact} \Big\}$$

$\mathfrak{M}$ is a Polish space with the vague topology.
The configuration space of noncolliding systems:

$$\begin{aligned} \mathfrak{M}_0 &= \Big\{ \xi \in \mathfrak{M} : \xi(\{x\}) = 1, \text{ for any } x \in \operatorname{supp} \xi \Big\} \\ &= \Big\{ \{x_j\} : \sharp\{j : x_j \in K\} < \infty, \text{ for any } K \text{ compact} \Big\}. \end{aligned}$$

A configuration space $\mathcal{X}$ is relative compact, if

$$\sup_{\xi \in \mathcal{X}} \xi(K) < \infty, \quad \text{for any } K \subset \mathbb{R} \text{ compact}$$

# Introduction

The moment generating function of multitime distribution of the $\mathfrak{M}$-valued process $\Xi(t)$ is defined as

$$\Psi^{(t_1,\dots,t_M)}(f_1,\dots,f_M) = \Psi^{\mathbf{t}}(\mathbf{f}) = \mathbb{E}\left[\exp\left\{\sum_{m=1}^{M}\int_{\mathbb{R}} f_m(x)\Xi(t_m,dx)\right\}\right] \quad (1)$$

for $0 \le t_1 < t_2 < \cdots < t_M$, $\mathbf{f} = (f_1, f_2, \dots, f_M) \in \mathrm{C}_0(\mathbb{R})^M$.

$$\Psi^{\mathbf{t}}(\mathbf{f}) = \sum_{\substack{N_m \ge 0, \\ 1 \le m \le M}} \int_{\prod_{m=1}^{M} \mathbb{R}^{N_m}} \prod_{m=1}^{M} \left\{ \frac{1}{N_m!} d\mathbf{x}_{N_m}^{(m)} \prod_{i=1}^{N_m} \chi_m\left(x_i^{(m)}\right) \right\}$$
$$\times \rho_\xi\left(t_1, \mathbf{x}_{N_1}^{(1)}; \dots; t_M, \mathbf{x}_{N_M}^{(M)}\right),$$

with $\chi_m(\cdot) = e^{f_m(\cdot)} - 1$ and the multitime correlation functions $\rho_\xi(\dots)$.

## Introduction

A processs $\Xi(t)$ is said to be determinantal if the moment generating function (1) is given by the Fredholm determinant

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \underset{\substack{(s,t)\in\{t_1,t_2,\dots,t_M\}^2,\\ (x,y)\in\mathbb{R}^2}}{\mathrm{Det}} \Big[\delta_{st}\delta_x(y) + \mathbb{K}(s,x;t,y)\chi_t(y)\Big], \tag{2}$$

In other words, the multitime correlation functions are represented as

$$\rho\left(t_1, \mathbf{x}^{(1)}_{N_1}; \dots; t_M, \mathbf{x}^{(M)}_{N_M}\right) = \det_{\substack{1\le j\le N_m, 1\le k\le N_n\\ 1\le m,n\le M}} \left[\mathbb{K}(t_m, x^{(m)}_j; t_n, x^{(n)}_k)\right].$$

The function $\mathbb{K}$ is called the correlation kernel of the process $\Xi(t)$.

# Examples of determinantal processes (DP)

(1) Non-colliding Brownian motion (The Dyson model):

$$X_j(t) = x_j + B_j(t) + \sum_{\substack{k:1\le k\le N \\ k\neq j}} \int_0^t \frac{ds}{X_j(s) - X_k(s)}, \ 1 \le j \le N$$

where $B_j(t), j = 1, 2, \ldots, N$ are independent one dimensional BMs.

(2) Non-colliding Bessel process:

$$X_j(t) = x_j + B_j(t) + \frac{2\nu + 1}{2} \int_0^t \frac{ds}{X_j(s)} + \sum_{\substack{k:1\le k\le N \\ k\neq j}} \int_0^t \left\{ \frac{1}{X_j(s) - X_k(s)} + \frac{1}{X_j(s) + X_k(s)} \right\} ds, \ 1 \le j \le N$$

## Special cases

(1)' The Dyson model starting from $N$ points all at the origin is determinantal with the correlation kernel

$$\mathbb{K}_N(s,x;t,y) = \begin{cases} \dfrac{1}{\sqrt{2s}} \displaystyle\sum_{k=0}^{N-1} \left(\frac{t}{s}\right)^{k/2} \varphi_k\left(\frac{x}{\sqrt{2s}}\right) \varphi_k\left(\frac{y}{\sqrt{2t}}\right) & \text{if } s \le t, \\ \dfrac{-1}{\sqrt{2s}} \displaystyle\sum_{k=N}^{\infty} \left(\frac{t}{s}\right)^{k/2} \varphi_k\left(\frac{x}{\sqrt{2s}}\right) \varphi_k\left(\frac{y}{\sqrt{2t}}\right) & \text{if } s > t, \end{cases}$$

where $h_k = \sqrt{\pi} 2^k k!$ and

$$\varphi_k(x) = \frac{1}{\sqrt{h_k}} e^{-x^2/2} H_k(x),$$

with the Hermite polynomials $H_k, k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

(2)' The noncolliding squared Bessel process starting from $N$ points all at the origin is determinantal with the correlation kernel

$$\mathbb{K}_N^{(\nu)}(s,x;t,y) = \begin{cases} \dfrac{1}{2s}\displaystyle\sum_{k=0}^{N-1}\left(\frac{t}{s}\right)^k \varphi_k^\nu\left(\frac{x}{2s}\right)\varphi_k^\nu\left(\frac{y}{2t}\right), & \text{if } s \le t, \\ -\dfrac{1}{2s}\displaystyle\sum_{k=N}^{\infty}\left(\frac{t}{s}\right)^k \varphi_k^\nu\left(\frac{x}{2s}\right)\varphi_k^\nu\left(\frac{y}{2t}\right), & \text{if } s > t, \end{cases}$$

where

$$\varphi_k^\nu(x) = \sqrt{\Gamma(k+1)/\Gamma(\nu+k+1)}x^{\nu/2}L_k^\nu(x)e^{-x/2}$$

with the Laguerre polynomials $L_k^\nu(x), k \in \mathbb{N}_0$, $\nu > -1$.

## The *extended sine kernel*

(3) The bulk scaling limit:

$$\mathbb{K}_N\left(\frac{2N}{\pi^2}+s,x;\frac{2N}{\pi^2}+t,y\right)\to \mathbf{K}_{\sin}(s,x;t,y), \quad N\to\infty,$$

$$\mathbf{K}_{\sin}(s,x;t,y) = \begin{cases} \displaystyle\int_0^1 du\, e^{\pi^2 u^2 (t-s)/2}\cos\{\pi u(y-x)\} & \text{if } t>s \\ \frac{\sin\pi(x-y)}{\pi(x-y)} = K_{\sin}(x,y) & \text{if } t=s \\ \displaystyle -\int_1^\infty du\, e^{\pi^2 u^2 (t-s)/2}\cos\{\pi u(y-x)\} & \text{if } t<s. \end{cases}$$

$x,y\in\mathbb{R}$.

## The *extended Bessel kernel*

(4) The hard edge scaling limit:

$$\mathbb{K}_N^{(\nu)}(N+s,x;N+t,y) \to \mathbf{K}_{J_\nu}(s,x;t,y). \quad N \to \infty,$$

$$\mathbf{K}_{J_\nu}(s,x;t,y) = \begin{cases} \displaystyle\int_0^1 du\, e^{-2u(s-t)} J_\nu(2\sqrt{ux}) J_\nu(2\sqrt{uy}) & \text{if} \quad s<t \\ \displaystyle\frac{J_\nu(2\sqrt{x})\sqrt{y}J_\nu'(2\sqrt{y}) - \sqrt{x}J_\nu'(2\sqrt{x})J_\nu(2\sqrt{y})}{x-y} & \text{if} \quad t=s \\ \displaystyle -\int_1^\infty du\, e^{-2u(s-t)} J_\nu(2\sqrt{ux}) J_\nu(2\sqrt{uy}) & \text{if} \quad s>t, \end{cases}$$

$x,y \in \mathbb{R}_+ \equiv \{x \in \mathbb{R} : x \geq 0\}$.
Where $J_\nu(\cdot)$ is the Bessel function with index $\nu > -1$.

# The *extended Airy kernel*

(5) The soft-edge scaling limit: Let $f_N(u) = 2N^{2/3} + N^{1/3}u - u^2/4$.

$$\mathbb{K}_N\Big(N^{1/3}+s, f_N(s)+x; N^{1/3}+t, f_N(t)+y\Big) \to \mathbf{K}_{\mathrm{Ai}}(s,x;t,y), \quad N \to \infty.$$

$$\mathbf{K}_{\mathrm{Ai}}(s,x;t,y) \equiv \begin{cases} \displaystyle\int_0^\infty du\, e^{-u(t-s)/2}\mathrm{Ai}(u+x)\mathrm{Ai}(u+y) & \text{if } t>s, \\ \displaystyle\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} & \text{if } t=s, \\ \displaystyle -\int_{-\infty}^0 du\, e^{-u(t-s)/2}\mathrm{Ai}(u+x)\mathrm{Ai}(u+y) & \text{if } t<s, \end{cases}$$

$x, y \in \mathbb{R}$.
Where $\mathrm{Ai}(\cdot)$ is the Airy function.

## Related results

(O) [Spohn: 1987, Prähofer-Spohn: '02, Johanson: '02]
Infinite many particle systems $\Xi_t$ obtaine by the Bulk or the Soft-edge scaling limits of the processes. equilibrium systems

(1) [Katori-T: '09, '10, '11, '13]:
Infinite many particle systems in non-equilibrium

(2) [Katori-T: '11]
Markov property of the reversible Markov processes.

(3) [Oasda: '12, '13, '13, Osada-T: in preparation, Osada-Honda: preprint]
Constructing diffusion processes $\widehat{\Xi}_t^{DF}$ by Dirichlet form technique and deriving ISDEs related to them.

# Infinite dimensional SDEs for $\widehat{\Xi}_t^{DF} = \sum_{j\in\mathbb{N}} \delta_{X_j(t)}$

Bulk scaling limit : [Osada:PTRF12] $\beta = 1, 2$ and $4$

$$dX_j(t) = dB_j(t) + \frac{\beta}{2} \sum_{\substack{k\in\mathbb{N} \\ k\neq j}} \frac{dt}{X_j(t) - X_k(t)}, \quad j \in \mathbb{N}. \tag{3}$$

Hard edge scaling limit : [Honda-Osada: preprint] $\nu > -1$

$$\begin{aligned} dX_j(t) &= dB_j(t) + \frac{2\nu + 1}{2} \frac{1}{X_j(t)} dt \\ &\quad + \frac{\beta}{2} \sum_{\substack{k\in\mathbb{N} \\ k\neq j}} \left\{ \frac{1}{X_j(t) - X_k(t)} + \frac{1}{X_j(t) + X_k(t)} \right\} dt, \ j \in \mathbb{N}. \end{aligned} \tag{4}$$

# Infinite dimensional SDEs for $\widehat{\Xi}_t^{DF} = \sum_{j\in\mathbb{N}} \delta_{X_j(t)}$

Soft edge scaling limit : [Osada-T1: in preparation] $\beta = 1, 2$ and 4

$$dX_j(t) = dB_j(t) + \frac{\beta}{2} \lim_{L\to\infty} \left\{ \sum_{\substack{k\in\mathbb{N}, k\neq j \\ |X_k(t)|\leq L}} \frac{1}{X_j(t) - X_k(t)} - \int_{|y|\leq L} \frac{\widehat{\rho}(y)}{-y} dy \right\} dt, \ j \in \mathbb{N}, \quad (5)$$

where $\widehat{\rho}(x) = \dfrac{\sqrt{-x}\mathbf{1}(x<0)}{\pi}$.

**Questions**

1. The Dyson model constructed by Spohn(1987) solves ISDE (1)? The infinite particle system constructed by Prähofer-Spohn(2002), Johannson(2002) solves ISDE (2)?
2. $\Xi_t = \widehat{\Xi}_t^{DF}$?

## Main Theorem

### Theorem

Let $\Xi_t$ be one of the determinantal processes associated with

(i) the extended sine kernel $\mathbf{K}_{\sin}(s, x; t, y)$,

(ii) the extended Airy kernel $\mathbf{K}_{\mathrm{Ai}}(s, x; t, y)$,

(iii) the extended Bessel kernel $\mathbf{K}^{(\nu)}(s, x; t, y)$ with index $\nu > -1$

There exists the state spaces $\mathcal{S}$ associated with the process $\Xi_t$ such that the process $(\mathbb{P}_\xi, \Xi_t)$, $\xi \in \mathcal{S}$ has the strong Markov property.

## Answer the questions

The Strong Markov property of the determinantal process$\Xi_t$

$\longrightarrow$ the quasi-regularity of the Dirichlet form associated with $\Xi_t$

(with the coincidence of two Dirichlet forms on the set of polynomials.)

$\longrightarrow$ $\Xi_t$ solve the same ISDE as $\widehat{\Xi}_t^{DF}$(by following the procedure in (3))

$\longrightarrow$ the coincidence of the processes $\Xi_t$ and $\widehat{\Xi}_t^{DF}$ by the uniqueness of solutions of the ISDE:

[Osada-T2: in preparation] Strong solutions of infinite-dimensional stochastic differential equations and tail $\sigma$-field.

## Configuration Space $\mathfrak{X}(\rho)$

For $\varepsilon, \kappa > 0$, $m_0, L_0 \in \mathbb{N}$, $\rho : \mathbb{R} \to [0, \infty)$, locally bounded, we denote by $\mathfrak{X}^{\varepsilon,\kappa}_{L_0,m_0}(\rho)$, the set of configurations $\xi \in \mathfrak{M}$ satisfying

$$\left| \int_{[0,L]} \rho(x)dx - \xi([0,L]) \right| \le L^{\varepsilon}, \quad \left| \int_{[-L,0]} \rho(x)dx - \xi([-L,0]) \right| \le L^{\varepsilon}, \tag{6}$$

for any $L \ge L_0$, and

$$\max_{k\in\mathbb{Z}} \xi\Big( [g^{\kappa}(k), g^{\kappa}(k+1)] \Big) \le m_0, \tag{7}$$

where $g^{\kappa}(x) = \mathrm{sign}(x)|x|^{\kappa}$, $x \in \mathbb{R}$.
The set $\mathfrak{X}^{\varepsilon,\kappa}_{L_0,m_0}(\rho)$ is relative compact with the vague topology.

## Determinatal point processes

Let $\mu_{\mathsf{sin}}$ be the DPP with the sine kernel with corr. functions $\rho_{\mathsf{sin}}$
$\mu_{\mathrm{Ai}}$ be the DPP with the Airy kernel with corr. functions $\rho_{\mathrm{Ai}}$
$\mu_{J_\nu}$ be the DPP with the Bessel kernel with corr. functions $\rho_{J_\nu}$

Put

$$\mathfrak{X}(\rho_{\mathsf{sin}}) = \bigcup_{L_0, m_0 \in \mathbb{N}} \bigcup_{\varepsilon \in (0,1), \kappa \in (1/2,1)} \mathfrak{X}^{\varepsilon,\kappa}_{L_0,m_0}(\rho_{\mathsf{sin}}).$$

$$\mathfrak{X}(\rho_{\mathrm{Ai}}) = \bigcup_{L_0, m_0 \in \mathbb{N}} \bigcup_{\varepsilon \in (0,1), \kappa \in (1/2,2/3)} \mathfrak{X}^{\varepsilon,\kappa}_{L_0,m_0}(\rho_{\mathrm{Ai}}).$$

$$\mathfrak{X}(\rho_{J_\nu}) = \bigcup_{L_0, m_0 \in \mathbb{N}} \bigcup_{\varepsilon \in (0,1), \kappa \in (1/2,2)} \mathfrak{X}^{\varepsilon,\kappa}_{L_0,m_0}(\rho_{J_\nu}) \cap \mathfrak{M}([0,\infty)).$$

It is readily checked that

$$\mu_{\mathsf{sin}}(\mathfrak{X}(\rho_{\mathsf{sin}})) = 1, \quad \mu_{J_\nu}(\mathfrak{X}(\rho_{\mathrm{Ai}})) = 1, \quad \mu_{J_\nu}(\mathfrak{X}(\rho_{J_\nu})) = 1.$$

## Entire function

For $\xi = \sum_{i=1}^{n} \delta_{u_i} \in \mathfrak{M}$, put

$$\Pi_0(\xi, w) = \prod_{x \in \mathrm{supp}\ \xi \cap \{0\}^c} \left(1 - \frac{w}{x}\right)^{\xi(x)}$$

For $\xi \in \mathfrak{X}(\rho_{\mathsf{sin}}), \mathfrak{X}(\rho_{J_\nu})$

$$\Phi_0(\xi, w) = \lim_{L \to \infty} \Pi_0(\xi \cap [-L, L], w), \quad w \in \mathbb{C},$$

exists and put $\Phi_0(\xi, z, w) = \Phi_0(\tau_{-z}\xi, w - z)$, $z, w \in \mathbb{C}$.
For $\xi \in \mathfrak{X}(\rho_{\mathrm{Ai}})$

$$\Phi_{\mathrm{Ai}}(\xi, w) = \lim_{L \to \infty} \exp\left\{ w \int_{-L}^{L} \widehat{\rho}(x) dx \right\} \Pi_0(\xi \cap [-L, L], w), \quad w \in \mathbb{C},$$

exists and put $\Phi_{\mathcal{A}}(\xi, u, z) = \Phi_{\mathrm{Ai}}(\tau_{-z}\xi, w - z)$, $\quad z, w \in \mathbb{C}$.

# Non-equilibrium system 1 [Katori-T:2010]

For $\xi \in \mathfrak{X}(\rho_{\sin})$, Dyson model $(\Xi^{Dyson}(t), \mathbb{P}_\xi)$ starting from $\xi$ is the DP with the correlation kernel $\mathbb{K}^\xi(s,x;t,y)$. In particular, when $\xi \in \mathfrak{X}(\rho_{\sin}) \cap \mathfrak{M}_0$, $\mathbb{K}^\xi$ is given by

$$\mathbb{K}^\xi(s,x;t,y) = \int_{\mathbb{R}} \xi(dx') \int_{\mathbb{R}} du\, p_{\sin}(s,x|x')\Phi_0(\xi,x',iu)p_{\sin}(-t,iu|y) \\ -\mathbf{1}(s>t)p_{\sin}(s-t,x|y),$$

where $p_{\sin}(t,x|y)$ is the generalized heat kernel:

$$\begin{aligned} p_{\sin}(t,x|y) &= \frac{1}{\sqrt{2\pi|t|}} \exp\left\{ -\frac{(x-y)^2}{2t} \right\} \mathbf{1}(t \neq 0) \\ &\quad +\delta(y-x)\mathbf{1}(t=0), \quad t \in \mathbb{R},\ x,y \in \mathbb{C}. \end{aligned}$$

$\Xi^{Dyson}(t)$ has $\mu_{\sin}$ as reversible measure, and the reversible process $(\Xi^{Dyson}(t), \mathbb{P}_{\mu_{\sin}})$ is determinantal with the extended sine kernel.

# Non-equilibrium system 2 [Katori-T:2011]

For $\xi \in \mathfrak{X}(\rho_{J_\nu})$ the non-colliding squared Bessel process $(\Xi^{(\nu)}(t), P_\xi)$ starting from $\xi$ is the DP with the correlation kernel $\mathbb{K}_\nu^\xi(s,x;t,y)$. In particular, when $\xi \in \mathfrak{X}(\rho_{J_\nu}) \cap \mathfrak{M}_0$, $\mathbb{K}_\nu^\xi(s,x;t,y)$ is given by

$$\mathbb{K}_\nu^\xi(s,x;t,y) = \int_0^\infty \xi(dx') \int_{-\infty}^0 du\, p^{(\nu)}(s,x|x')\Phi_0(\xi^N, x', u) p^{(\nu)}(-t,u|y)$$
$$-\mathbf{1}(s>t)p^{(\nu)}(s-t,x|y),$$

where for $t \in \mathbb{R}$ and $x, y \in \mathbb{C}$

$$p^{(\nu)}(t,y|x) = \frac{1}{2|t|}\left(\frac{y}{x}\right)^{\nu/2} \exp\left(-\frac{x+y}{2t}\right) I_\nu\left(\frac{\sqrt{xy}}{|t|}\right)\mathbf{1}(t\neq 0, x\neq 0)$$
$$+\frac{y^\nu}{(2|t|)^{\nu+1}\Gamma(\nu+1)}\exp\left(-\frac{y}{2t}\right)\mathbf{1}(t\neq 0, x=0) + \delta(y-x)\mathbf{1}(t=0).$$

$\Xi^{(\nu)}(t)$ has $\mu_{J_\nu}$ as reversible measure, and the reversible process $(\Xi^{(\nu)}(t), \mathbb{P}_{\mu_{J_\nu}})$ is the determinantal with the extended Bessel kernel.

# Non-equilibrium system 3 [Katori-T:2009]

For $\xi \in \mathfrak{X}(\rho_{Ai})$ Airy-Dyson model $(\Xi^{\mathrm{Ai}}(t), P_\xi)$ starting from $\xi$ is the DP with the correlation kernel $\mathbb{K}_{\mathrm{Ai}}^{\xi}(s,x;t,y)$ In particular, when $\xi \in \mathfrak{X}(\rho_{\mathrm{Ai}}) \cap \mathfrak{M}_0$, $\mathbb{K}_{\mathrm{Ai}}^{\xi}(s,x;t,y)$ is given by

$$\mathbb{K}_{\mathrm{Ai}}^{\xi}(s,x;t,y) = \int_{\mathbb{R}} \xi(dx') \int_{\mathbb{R}} du\, q(0,s,x-x')\Phi_{\mathrm{Ai}}(\xi,x',iu)q(t,0,iu-y) \\ -\mathbf{1}(s>t)q(t,s,x-y),$$

where $q(s,t,y-x), s,t \in \mathbb{R}, s \neq t, x,y \in \mathbb{C}$ is given by

$$q(s,t,y-x) = p_{\sin}\left(t-s, \left(y-\frac{t^2}{4}\right) - \left(x-\frac{s^2}{4}\right)\right) \\ = \frac{1}{\sqrt{2\pi|t-s|}} \exp\left[-\frac{(y-x)^2}{2(t-s)} + \frac{(t+s)(y-x)}{4} - \frac{(t-s)(t+s)^2}{32}\right].$$

$\Xi^{\mathrm{Ai}}(t)$ has $\mu_{\mathrm{Ai}}$ as reversible measure, and the reversible process $(\Xi^{\mathrm{Ai}}(t), \mathbb{P}_{\mu_{\mathrm{Ai}}})$ is the determinantal with the extended Airy kernel.

We equip $\mathfrak{X}(\rho_{\sin}), \mathfrak{X}(\rho_{\rm Ai}), \mathfrak{X}(\rho_{J_\nu})$ with the topologies obtained by the inductive limit. Suppose that
$(\mu, \rho, \Xi_t, \Phi, \mathbb{K}^\xi) = (\mu_{\sin}, \rho_{\sin}, \Xi^{Dyson}(t), \Phi_0, \mathbb{K}^\xi), (\mu_{J_\nu}, \rho_{J_\nu}, \Phi_0, \Xi^{(\nu)}(t), \mathbb{K}^\xi_\nu)$,
or $(\mu_{\rm Ai}, \rho_{\rm Ai}, \Xi^{\rm Ai}(t), \Phi_{\rm Ai}, \mathbb{K}^\xi_{\rm Ai})$.

**Lemma 1** Suppose $\xi_n, \xi \in \mathfrak{X}(\rho)$, and $u_n \in \text{supp } \xi_n, u \in \text{supp } \xi$.
If

$$\xi_n \to \xi, \ n \to \infty \quad \text{and} \quad u_n \to u, \ n \to \infty,$$

then

$$\Phi(\xi_n, u_n, z) \to \Phi(\xi, u, z), \quad n \to \infty,$$

$$\mathbb{K}^{\xi_n}(s, x; t, y) \to \mathbb{K}^\xi(s, x; t, y), \quad n \to \infty$$

and

$$(\Xi_t, \mathbb{P}_{\xi_n}) \to (\Xi_t, \mathbb{P}_\xi), \quad n \to \infty,$$

in the sence of finite dimensional distributions

We call a function $f$ is polynomial, if it is represented as

$$f(\xi) = Q_j\Big( \int_{\mathbb{R}} \varphi_1(x)\xi(dx), \dots, \int_{\mathbb{R}} \varphi_k(x)\xi(dx) \Big)$$

with some polynomial function on $\mathbb{R}^k$, and $\varphi_j \in \mathbb{C}_c^\infty(\mathbb{R})$, $j = 1, 2, \dots k$. For a polynomial function $f$ we put

$$T_t f(\xi) = \mathbb{E}_\xi[f(\Xi_t)],$$

**Proposition 2 [Katori-T: 2011]** The process $(\Xi(t), P_\mu)$ is a reversible Markov process, i.e.

$$T_{s+t} f(\xi) = T_s T_t f(\xi), \quad \mu - \text{a.s. } \xi.$$

From Lemma 1,

$$\xi_n \to \xi, \quad \text{in } \mathfrak{X}(\rho) \quad \Rightarrow \quad T_t f(\xi_n) \to T_t f(\xi).$$

The main theorem is derived from the following lemma.

**Key Lemma** There exists $\mathcal{S} \subset \mathfrak{X}(\rho)$ such that $\mu(\mathcal{S}) = 1$ and

$$P_\xi(\Xi(\cdot) \text{ is continuous in } \mathfrak{X}(\rho)) = 1, \quad \xi \in \mathcal{S}$$

**Remark** A function $\Xi(\cdot)$ is continuous in $\mathfrak{X}(\rho)$ if $\Xi(\cdot)$ is continous with the vague topology, and for any $T > 0$ there exist $L_0, m_0 \in \mathbb{N}$ and $\varepsilon \in (0,1), \kappa \in (1/2, \kappa(\rho))$ such that

$$\Xi(t) \in \mathfrak{X}^{\varepsilon,\kappa}_{L_0,m_0}(\rho), \quad t \in [0, T],$$

where $\kappa(\rho_{\sin}) = 1$, $\kappa(\rho_{\mathrm{Ai}}) = 2/3$, and $\kappa(\rho_{J_\nu}) = 2$.

# Outline of Proof of Key Lemma

1) $\Xi_t$ has vaguely continuous path : For any polynomial function $f$,

$$\mathbb{E}_\mu[|f(\Xi_t) - f(\Xi_s)|^\beta] \leq C|t-s|^\alpha, \quad 0 \leq s < t \leq T < \infty$$

for some $\alpha > 1, \beta > 0$ and $C > 0$.
2) There exist $\varepsilon \in (0,1), \kappa \in (1/2, \kappa(\rho))$ and $m \in \mathbb{N}$ such that

$$\sum_{L\in\mathbb{N}} \mathbb{P}_\mu \left( \left| \int_{[0,\pm L]} \rho(x)dx - \Xi_t([0,\pm L]) \right| > L^\varepsilon,\ t \in [0,T] \right) < \infty$$

and

$$\sum_{k\in\mathbb{Z}} \mathbb{P}_\mu \left( \Xi_t([g^\kappa(k), g^\kappa(k+1)]) \leq m,\ t \in [0,T] \right) < \infty$$

# Reversible processes

1. Bulk scaling limit:

$$dX_j^N(t) = dB_j(t) - \frac{\rho}{2N} X_j^N(t)dt + \sum_{\substack{1 \le k \le N \\ k \ne j}} \frac{1}{X_j^N(t) - X_k^N(t)} dt,$$

with the initial distribution whose density

$$\frac{1}{c_N} h_N(\mathbf{x})^2 \exp\{-\frac{\rho |\mathbf{x}|^2}{2N}\}.$$

Then

$$\sum_{j=1}^N \delta_{X_j^N(\cdot)} \to \Xi^{Dyson}(\cdot), \quad N \to \infty$$

weakly in the sense of finite dimensional distributions.

2. Hard-edge scaling limit:

$$dX_j^N(t) = dB_j(t) + \Big( -\frac{X_j^N(t)}{2N} + \frac{\nu + 1/2}{X_j^N(t)} \Big) dt + \sum_{\substack{1 \le k \le N \\ k \neq j}} \frac{2X_j^N(t)}{X_j^N(t)^2 - X_k^N(t)^2} dt,$$

with the initial distribution whose density

$$\frac{1}{c_N} h_N^{(2\nu+1)}(\mathbf{x})^2 \exp\{-\frac{|\mathbf{x}|^2}{2N}\}, q$$

with $h_N^{\alpha}(\mathbf{x}) = \prod_{1 \le i < j \le N} (x_j^2 - x_i^2) \prod_{j=1}^N x_j^{\alpha}$. Then

$$\sum_{j=1}^N \delta_{X_j^N(\cdot)^2} \to \Xi^{(\nu)}(\cdot), \quad N \to \infty$$

weakly in the sense of finite dimensional distributions.

3. Soft-edge scaling limit:

$$dX_j^N(t) = dB_j(t) - \Big(N^{1/3} + \frac{X_j^N(t)}{N^{1/3}}\Big)dt + \sum_{\substack{1\le k\le N \\ k\neq j}} \frac{1}{X_j^N(t) - X_k^N(t)} dt,$$

with the initial distribution whose density

$$\frac{1}{c_N} h_N(\mathbf{x})^2 \exp\Big\{ -\frac{1}{2}\sum_{j=1}^N |2\sqrt{N} + n^{-1/6} x_j|^2 \Big\}$$

Then

$$\sum_{j=1}^N \delta_{X_j^N(\cdot)} \to \Xi^{\mathrm{Ai}}(\cdot) \quad N \to \infty$$

weakly in the sense of finite dimensional distributions.

## Lyons-Zheng decomposition

$\mathbf{Y}(t) = (Y_1(t), Y_2(t), \dots, Y_N(t))$ : reversible Markov

$$Y_j(t) = Y_j(0) + B_j(t) + \int_0^t b_j(\mathbf{Y}(s))ds, \quad 1 \le j \le N, \ t \in [0, T].$$

$\widehat{\mathbf{Y}}(t) = \mathbf{Y}(T-t)$ satisfies

$$\widehat{Y}_j(t) = \widehat{Y}_j(0) + \widehat{B}_j(t) + \int_0^t b_j(\widehat{\mathbf{Y}}(s))ds, \quad 1 \le j \le N, \ t \in [0, T],$$

with Brownian motions $\widehat{B}_j(t)$. Then

$$Y_j(t) - Y_j(0) = \widehat{Y}(T-t) - \widehat{Y}(T) = \widehat{B}_j(T-t) - \widehat{B}_j(T) - \int_0^t b_j(\mathbf{Y}(s))ds$$

$$Y_j(t) - Y_j(0) = \frac{1}{2}\left(B_j(t) + \widehat{B}_j(T-t) - \widehat{B}_j(T)\right).$$

1) $\Xi_t$ has vaguely continuous path : For any polynomial function $f$,

$$\mathbb{E}_\mu[|f(\Xi_t) - f(\Xi_s)|^\beta] \le C|t-s|^\alpha, \quad 0 \le s < t \le T < \infty$$

for some $\alpha > 1, \beta > 0$ and $C > 0$.
2) There exist $\varepsilon \in (0,1), \kappa \in (1/2, \kappa(\rho))$ and $m \in \mathbb{N}$ such that

$$\sum_{L \in \mathbb{N}} \mathbb{P}_\mu \left( \left| \int_{[0,\pm L]} \rho(x)dx - \Xi_t([0,\pm L]) \right| > L^\varepsilon, \ t \in [0,T] \right) < \infty$$

and

$$\sum_{k \in \mathbb{Z}} \mathbb{P}_\mu \left( \Xi_t([g^\kappa(k), g^\kappa(k+1)]) \le m, \ t \in [0,T] \right) < \infty$$

(i) There exist $\varepsilon \in (0,1)$ such that

$$\sum_{L \in \mathbb{N}} \mu \left( \xi : \left| \int_{[0,\pm L]} \rho(x)dx - \xi([0,\pm L]) \right| > L^{\varepsilon} \right) < \infty$$

(ii) For any $\kappa \in (1/2, \kappa(\rho))$ and any $\ell \in \mathbb{N}$ we can take $m \in \mathbb{N}$ such that

$$\sum_{k \in \mathbb{Z}} |k|^{\ell} \mu \left( \xi : \xi([g^{\kappa}(k), g^{\kappa}(k+1)]) \le m \right) < \infty$$

and so

$$\sum_{k \in \mathbb{Z}} \mu \left( \Xi_t([g^{\kappa}(k), g^{\kappa}(k+1)]) \le m, \ t = \frac{jT}{|k|^{\ell}}, \ j = 0, 1, \dots, |k|^{\ell} \right) < \infty.$$

## Thanks

# Thank you for your attention!