Stochastic Differential Equations associateded with Infinite particle systems with long ranged interaction

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8th World Congress in Probability and Statistics (2012 July 13)

Introduction

Dyson's Brownian motion model [JMP 62] is a one parameter family of the systems of one dimensional Brownian motions with long ranged repulsive interaction, whose strength is represented by a parameter $\beta > 0$. It soves the stochastic differential equation

$$X_j(t) = x_j + B_j(t) + \frac{\beta}{2} \sum_{\substack{k:1 \leq k \leq n \\ k \neq j}} \int_0^t \frac{ds}{X_j(s) - X_k(s)}, \ 1 \leq j \leq n \qquad (1)$$

where $B_j(t), j = 1, 2, \dots, n$ are independent one dimensional Brownian motions.We consider the case that $\beta = 2$ and call the model in the special case Dyson model .

Introduction

The Dyson model is realized by the following three processes:
(i) The process of eigenvalues of Hermitian matrix valued diffusion process in the Gaussian unitary ensemble (GUE).
(ii) The system of one-dimensional Brownian motions conditioned never to collide with each other.
(iii) The harmonic transform of the absorbing Brownian motion in a Weyle chamber of type A_{n-1}:

$$\mathbb{W}_n = \Big\{ \mathbf{x} = (x_1, x_2, \cdots, x_n) : x_1 < x_2 < \cdots < x_n \Big\}.$$

with harmonic function given by the Vandermonde determinant:

$$h_n(\mathbf{x}) = \prod_{1 \le j < k \le n} (x_k - x_j) = \det_{1 \le j,k \le n} \left[x_k^{j-1} \right].$$

Introduction

$n \times n$ Hermitian matrix valued process $(n \in \mathbb{N})$

$$M(t) = \begin{pmatrix} M_{11}(t) & M_{12}(t) & \cdots & M_{1n}(t) \\ M_{21}(t) & M_{22}(t) & \cdots & M_{2n}(t) \\ & & \cdots & \\ M_1(t) & M_{n2}(t) & \cdots & M_{nn}(t) \end{pmatrix}, \quad M_{\ell k}(t) = M_{k\ell}(t)^{\dagger}.$$

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(GOE) $B^{\mathrm{R}}_{k\ell}(t)$, $1 \le k \le \ell \le n$: indep. BMs

$$M_{k\ell}(t) = \frac{1}{\sqrt{2}} B^{\mathrm{R}}_{k\ell}(t),\ 1 \le k < \ell \le n, \quad M_{kk}(t) = B^{\mathrm{R}}_{kk}(t),\ 1 \le k \le n,$$

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(GUE) $B^{\mathrm{R}}_{k\ell}(t)$, $B^{\mathrm{I}}_{k\ell}(t)$,$1 \le k \le \ell \le n$: indep. BMs

$$M_{k\ell}(t) = \frac{1}{\sqrt{2}} B^{\mathrm{R}}_{k\ell}(t) + \frac{\sqrt{-1}}{\sqrt{2}} B^{\mathrm{I}}_{k\ell}(t), \quad 1 \le k < \ell \le n,$$

$$M_{kk}(t) = B^{\mathrm{R}}_{kk}(t), \quad 1 \le k \le n,$$

Introduction

(GSE) $B^{\alpha}_{k\ell}(t)$, $\alpha = 0, 1, 2, 3$, $1 \le k \le \ell \le n$: indep. BMs

$$M^0_{k\ell}(t) = \frac{1}{\sqrt{2}} B^0_{k\ell}(t),\ 1 \le k < \ell \le n, \quad M^0_{kk}(t) = B^0_{kk}(t),\ 1 \le k \le n,$$

For $\alpha = 1, 2, 3$,

$$M^{\alpha}_{k\ell}(t) = \frac{\sqrt{-1}}{\sqrt{2}} B^{\alpha}_{k\ell}(t),\ 1 \le k < \ell \le n, \quad M^{\alpha}_{kk}(t) = 0,\ 1 \le k \le n,$$

$2n \times 2n$ self dual Hermitian matrix valued process

$$M(t) = M^0(t) \otimes I + \sum_{\alpha=1}^{3} M^{\alpha}(t) \otimes e_{\alpha}$$

Here

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Introduction

In this talk we discuss the following problems:

(i) Conditions that $(X_1(t), X_2(t), \dots, X_n(t))$ converges to some process, say $X(t)$, as $n \to \infty$.

(ii) Stochastic differential equation that the limit process $\mathbf{X}(t)$ solves.

(iii) Invariant distributions of the limit process $\mathbf{X}(t)$.

The configuration space of unlabelled particles:

$$\mathfrak{M} = \Big\{\xi : \xi \text{ is a nonnegative integer valued Radon measures in } \mathbb{R}\Big\}$$

$$= \Big\{\xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot) : \sharp\{j \in \mathbb{I} : x_j \in K\} < \infty, \text{ for any } K \text{ compact}\Big\}$$

$\mathfrak{M}$ is a Polish space with the vague topology.

Answer to (i)

For $L > 0, \alpha > 0$ and $\xi \in \mathfrak{M}$ we put

$$M(\xi, L) = \int_{[-L,L]\setminus\{0\}} \frac{\xi(dx)}{x}, \qquad M_\alpha(\xi, L) = \left(\int_{[-L,L]\setminus\{0\}} \frac{\xi(dx)}{|x|^\alpha} \right)^{1/\alpha}$$

and

$$M(\xi) = \lim_{L\to\infty} M(\xi, L), \qquad M_\alpha(\xi) = \lim_{L\to\infty} M_\alpha(\xi, L),$$

if the limits exist. We introduce the following two conditions:

(**C.1**) there exists $C_0 > 0$ such that $|M(\xi)| \le C_0$,

(**C.2**) there exist $\alpha \in (1, 2)$ and $C_1 > 0$ such that $M_\alpha(\xi) \le C_1$,
(ii) there exist $\beta > 0$ and $C_2 > 0$ such that

$$M_1(\tau_{-a^2}\xi^{\langle 2\rangle}) \le C_2(|a| \vee 1)^{-\beta} \quad \forall a \in \mathrm{supp}\xi.$$

Answer to (i)

Examples. Put

$$\eta^\kappa = \sum_{x\in\mathbb{Z}} \delta_{\mathrm{sgn}(x)|x|^\kappa}.$$

In case $\kappa > 1/2$, η^κ satusfies the conditions (**C**.**1**) and (**C**.**2**).

THEOREM (Katori-T. CMP '10, Katori-T. arXiv:math.PR 1008.2821)
Suppose that $\xi \in \mathfrak{M}_0 \equiv \{\xi \in \mathfrak{M} : \xi(x) \leq 1 \text{ for all } x \in \mathbb{R}\}$ satisfies the conditions (**C**.**1**) and (**C**.**2**). Then

$$(\Xi(t), \mathbb{P}_{\xi\cap[-L,L]}) \to (\Xi(t), \mathbb{P}_\xi), \quad L \to \infty$$

weakly on $\mathrm{C}([0,\infty) \to \mathfrak{M})$. In particular, the process $(\Xi(t), \mathbb{P}_\xi)$ has a modification which is almost-surely continuous on $[0,\infty)$ with $\Xi(0) = \xi$.

Answer to (ii)

The distribution of eigenvalues of GUE with size $n \times n$ are given by

$$m_2^n(d\mathbf{x}_n) = \frac{1}{Z} \prod_{i<j} |x_i - x_j|^2 \exp\left\{ -\frac{1}{2} \sum_{i=1}^n |x_i|^2 \right\} d\mathbf{x}_n,$$

on the configuration space $\mathbb{W}_{n-1}$.

(Bulk scaling limit) For the eigenvalues $\{\lambda_1^n, \dots, \lambda_n^n\}$

$$\{\sqrt{n}\lambda_1^n, \dots, \sqrt{n}\lambda_n^n\} \to \mu_{\sin,\beta}, \quad \text{weakly as } n \to \infty.$$

Answer to (ii)

For $\beta = 2$ (GUE) $\mu_{\sin,2}$ is the determinantal point process(DPP) , in which any spatial correlation function ρ_m is given by a determinant with the *sine kernel*

$$K_{\sin,2}(x,y) = K_{\sin}(x,y) \equiv \frac{\sin\{\pi(y-x)\}}{\pi(y-x)}, \quad x, y \in \mathbb{R}.$$

The moment generating function is given by a Fredholm determinant

$$\int_{\mathfrak{M}} \exp\left\{ \int_{\mathbb{R}} f(x)\xi(dx) \right\} \mu_{\sin,2}(d\xi) = \mathop{\mathrm{Det}}_{(x,y)\in\mathbb{R}^2} \Big[\delta_x(y) + K_{\sin}(x,y)\chi(y) \Big],$$

for $f \in \mathrm{C}_c(\mathbb{R})$, where $\chi(\cdot) = e^{f(\cdot)} - 1$.

Answer to (ii)

THEOREM (Osada[PTRF: online first])
There exists the diffusion process whose reversible probability measure $\mu_{\sin,2}$ which solves the SDE

$$dX_j(t) = dB_j(t) + \sum_{\substack{k\in\mathbb{N} \\ k\neq j}} \frac{dt}{X_j(s) - X_k(s)}, \quad j \in \mathbb{N}, \qquad (2)$$

where $B_j(t), j \in \mathbb{N}$ are independent one dimensional Brownian motions.

Dirichlet spaces

A function f defined on the configuration space $\mathfrak{M}$ is local if $f(\xi) = f(\xi_K)$ for some compact set K.
A local function f is smooth if $f(\sum_{j=1}^n \delta_{x_j}) = \tilde{f}(x_1, x_2, \dots, x_n)$ with some smooth function $\tilde{f}$ on $\mathbb{R}^n$ with compact support. Put

$$\mathcal{D}_0 = \{f : f \text{ is local and smooth}\}.$$

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$$\mathcal{D}_0 = \{f : f \text{ is local and smooth}\}.$$

We put

$$\mathbb{D}[f,g](\xi) = \frac{1}{2} \sum_{j=1}^{\xi(K)} \frac{\partial \tilde{f}(\mathbf{x})}{\partial x_j} \frac{\partial \tilde{g}(\mathbf{x})}{\partial x_j}$$

and for a probability measure μ we introduce the bilinear form

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathfrak{M}} \mathbb{D}[f,g] d\mu, \quad f,g \in \mathcal{D}_0.$$

quasi Gibbs measure

Let Φ be a free potential, Ψ be an interaction potential. For a given sequence $\{b_r\}$ of $\mathbb{N}$ we introduce a Hamiltonian on $I_r = (-b_r, b_r)$:

$$H_r(\xi) = H_r^{\Phi,\Psi}(\xi) = \sum_{x_j \in I_r} \Phi(x_j) + \sum_{x_j, x_k \in I_r, j<k} \Psi(x_j, x_k)$$

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Definition A probability measure μ is said to be a (Φ, Ψ)-quasi Gibbs measure if there exists an increasing sequence $\{b_r\}$ of $\mathbb{N}$ and measures $\{\mu_{r,k}^m\}$ such that for each $r, m \in \mathbb{N}$ satisfying

$$\mu_{r,k}^m \leq \mu_{r,k+1}^m, \quad k \in \mathbb{N}, \quad \lim_{k\to\infty} \mu_{r,k}^m = \mu(\cdot \cap \{\xi(I_r) = m\}), \mathit{weekly}$$

and that for all $r, m, k \in \mathbb{N}$ and for $\mu_{r,k}^m$-a.s. $\xi \in \mathfrak{M}$

$$c^{-1} e^{-H_r(\xi)} \mathbf{1}_{\{\xi(I_r)=m\}} \Lambda(d\zeta) \leq \mu_{r,k}^m(\pi_{I_r} \in d\zeta | \xi_{I_r^c}) \leq c e^{-H_r(\xi)} \mathbf{1}_{\{\xi(I_r)=m\}} \Lambda(d\zeta)$$

Here Λ is the Poisson random measure with intensity measure dx.

quasi regular Dirichlet space

Theorem (Bulk) [Osada:to appear in AOP]

Let $\beta = 1, 2, 4$.

(1) The probability measure $\mu_{\sin,\beta}$ is a quasi Gibbs measure with $\Phi(x) = 0$ and $\Psi(x) = -\beta \log |x - y|$.

(2) The closure of $(\mathcal{E}^{\mu_{\sin,\beta}}, \mathcal{D}_0, L^2(\mathfrak{M}, \mu_{\sin,\beta}))$ is a quasi Dirichlet space, and there exists a $\mu_{\sin}$-reversible diffusion process $(\Xi^{\sin,\beta}(t), P)$ associated with the Diriclet space.

Log derivative

Let $\mu_{\mathbf{x}}$ be the Palm measure conditioned at $\mathbf{x} = (x_1, \dots, x_k \in \mathbb{R}^k$

$$\mu_{\mathbf{x}} = \mu\left(\cdot - \sum_{j=1}^{k} \delta_{x_j} \middle| \xi(x_j) \geq 1 \text{ for } j = 1, 2, \dots, k \right).$$

Let μ^k be the Campbell measure of μ:

$$\mu^k(A \times B) = \int_A \mu_{\mathbf{x}}(B) \rho^k(\mathbf{x}) d\mathbf{x}, \quad A \in \mathcal{B}(\mathbb{R}^k), B \in \mathcal{B}(\mathfrak{M}).$$

We call $\mathbf{d}^\mu \in L^1_{loc}(\mathbb{R} \times \mathfrak{M}, \mu^1)$ the log derivative of μ if $\mathbf{d}^\mu$ satisfies

$$\int_{\mathbb{R} \times \mathfrak{M}} \mathbf{d}^\mu(x, \eta) f(x, \eta) d\mu^1(x, \eta) = -\int_{\mathbb{R} \times \mathfrak{M}} \nabla_x f(x, \eta) d\mu^1(x, \eta),$$

$$f \in C_c^\infty(\mathbb{R}) \otimes \mathcal{D}_0.$$

ISDE

Theorem [Osada, PTRF (on line first)]
Assume that there exists a log derivative $\mathbf{d}^{\mu}$ (and some conditions). There exists $\mathfrak{M}_0 \subset \mathfrak{M}$ such that $\mu(\mathfrak{M}_0) = 1$, and for any $\xi = \sum_{j \in \mathbb{N}} \delta_{x_j} \in \mathfrak{M}_0$, there exists $\mathbb{R}^{\mathbb{N}}$-valued continuous process $\mathbf{X}(t) = (X_j(t))_{j=1}^{\infty}$ satisfying $\mathbf{X}(0) = \mathbf{x} = (x_j)_{j=1}^{\infty}$ and

$$dX_j(t) = dB_j(t) + \frac{1}{2}\mathbf{d}^{\mu}\left(X_j(t), \sum_{k:k\neq j} \delta_{X_k(t)}\right)dt, \quad j \in \mathbb{N}.$$

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Lemma (Bulk) [Osada, PTRF online first] Let $\beta = 1, 2, 4$. For $x \in \mathbb{R}$ and $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ with $\eta(\{x\}) = 0$,

$$\mathbf{d}^{\mu_{\sin,\beta}}(x, \eta) = \beta \lim_{L\to\infty} \sum_{j:|x-y_j|\leq L} \frac{1}{x - y_j}$$

Key lemma

The key part in the proof of Theorem 2 is to determine the log derivative of μ.

Key lemma (tacnode) Let $\beta = 1, 2, 4$. For $x \in \mathbb{R}$ and $\eta = \sum_{j \in \mathbb{N}} \delta_{y_j}$ with $\eta(\{x\}) = 0$,

$$\mathbf{d}^{\mu_{\mathrm{Ai},\beta}}(x, \eta) = \beta \lim_{L \to \infty} \left\{ \sum_{j : |x - y_j| \le L} \frac{1}{x - y_j} \right\}.$$

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Lemma (Bulk) [Osada, PTRF online first] For $x \in \mathbb{R}$ and $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ with $\eta(\{x\}) = 0$,

$$\mathbf{d}^{\mu_{\sin,\beta}}(x, \eta) = 2 \lim_{L\to\infty} \left\{ \sum_{j:|x-y_j|\le L} \frac{1}{x - y_j} - \int_{|u|\le L} \frac{\rho}{-u} du \right\}.$$

Proof

To prove the key lemma, we use n particle sysytem:

$$m_\beta^n(d\mathbf{u}_n) = \frac{1}{Z}\prod_{i<j}|u_i - u_j|^\beta \exp\Big\{-\frac{\beta}{4}\sum_{i=1}^n |u_i|^2\Big\} d\mathbf{u}_n,$$

We put $u_j = 2\sqrt{n} + \frac{x_j}{n^{1/6}}$ and intrduce the measure defined by

$$\mu_{\mathcal{A},\beta}^n(d\mathbf{x}_n) = \frac{1}{Z}\prod_{i<j}|x_i - x_j|^\beta \exp\left\{-\frac{\beta}{4}\sum_{i=1}^n |2\sqrt{n} + n^{-1/6}x_i|^2\right\} d\mathbf{x}_n,$$

The log derivative $\mathbf{d}^n$ of the measure $\mu_{\mathcal{A},\beta}^n$ is given by

$$\mathbf{d}^n(x,\eta) = \mathbf{d}^n\left(x, \sum_{j=1}^{n-1}\delta_{y_j}\right) = \beta\Big\{\sum_{j=1}^{n-1}\frac{1}{x - y_j} - n^{1/3} - \frac{n^{-1/3}}{2}x\Big\}.$$

Proof

Lemma 3 is derived from the fact that

$$\mathbf{d}^{\mu^{\mathrm{Ai}}}(x,\eta) = \lim_{n\to\infty} \mathbf{d}^{n}(x,\eta) = \beta \lim_{L\to\infty} \left\{ \sum_{|x-y_j|<L} \frac{1}{x-y_j} - \int_{|u|\le L} \frac{\widehat{\rho}(y)}{-y} du \right\} \tag{3}$$

Proof

Lemma 3 is derived from the fact that

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To check (3) we divide $\mathbf{d}^n/\beta$ into three parts:

$$\mathrm{g}_L^n(x,\eta) = \sum_{|x-y_j|<L} \frac{1}{x-y_j} - \int_{|x-u|<L} \frac{\rho^n_{\mathcal{A},\beta,x}(u)}{x-u} du,$$

$$w_L^n(x,\eta) = \sum_{|x-y_j|\ge L} \frac{1}{x-y_j} - \int_{|x-u|\ge L} \frac{\rho^n_{\mathcal{A},\beta,x}(u)}{x-u} du,$$

$$u^n(x) = \int_{\mathbb{R}} \frac{\rho^n_{\mathcal{A},\beta,x}(u)}{x-u} du - n^{1/3} - \frac{n^{-1/3}}{2} x.$$

Proof

The fact (3) is obtained if the following conditions hold:

$$\lim_{n\to\infty} \mathrm{g}_L^n(x,\eta) = \mathrm{g}_L(x,\eta), \quad \text{in } L^{\hat{p}}(\mu^1_{\mathrm{Ai},\beta}) \text{ for any } L>0, \tag{4}$$

$$\lim_{L\to\infty} \limsup_{n\to\infty} \int_{[-r,r]\times\mathfrak{M}} |w_L^n(x,\mathrm{y})|^{\hat{p}} d\mu_{\mathcal{A}}^{n,1}(dxd\eta) = 0, \tag{5}$$

$$\lim_{n\to\infty} u^n(x) = u(x), \quad \text{in } L^{\hat{p}}_{loc}(\mathbb{R}, dx)\ , \tag{6}$$

with

$$\mathrm{g}_L(x,\eta) = \sum_{|x-y_j|<L} \frac{1}{x-y_j} - \int_{|x-u|<L} \frac{\rho_{\mathrm{Ai},\beta,x}(u)}{x-u} du,$$

and

$$u(x) = \lim_{L\to\infty} \left\{ \int_{|u|\le L} \frac{\rho_{\mathrm{Ai},\beta,x}(u)}{x-u} du - \int_{|u|\le L} \frac{\widehat{\rho}(u)}{-u} du \right\} \in L^{\hat{p}}_{loc}(\mathbb{R}, dx).$$

Proof

In the case $\beta = 2$, $\mu_{\mathcal{A}}^n$ is the DPP with the correlation kernel

$$K_{\mathcal{A}}^n(x,y) = n^{1/3}\frac{\Psi_n(x)\Psi_{n-1}(y) - \Psi_{n-1}(x)\Psi_n(y)}{x-y}$$

where $\Psi_n(x) = n^{1/12}\varphi_n\Big(\sqrt{2n} + \frac{x}{\sqrt{2}n^{1/6}}\Big)$, and $\varphi_k(x)$ is the normalized orthogonal functions on $\mathbb{R}$ comprising the Hermite polynomials $H_k(x)$.

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$$\widehat{\rho}^n(u) = \frac{1}{\pi}\sqrt{-u\left(1 + \frac{u}{4n^{2/3}}\right)}, \quad -4n^{2/3} \le u \le 0$$

and the facts

$$\int_{\mathbb{R}} \frac{\widehat{\rho}^n(u)}{-u} du = n^{1/3}, \quad \lim_{n\to\infty} \widehat{\rho}^n(u) \to \widehat{\rho}(u).$$

Thanks

Thank you for your attention!