Diffusions associated with Gaussian analytic functions

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Workshop on "Probabilistic models with determinantal structure"

We construct unlabeled diffusion reversible to random point fields given by zero points of GAF.

Outline of talk:

- the standard planar GAF and diffusions
- A general theory for ISDEs

The standard planar GAF

The standard planar GAF is the random entire function with Gaussian coefficients:

$$f(z) = \sum_{k=0}^{\infty} \frac{\xi_k}{\sqrt{k!}} z^k$$

- $\{\xi_k\}$ is i.i.d. standard complex Gaussian.
- The zero points of f are regarded as configuration on $\mathbb{C}$ ($\mathbb{R}^2$).
- Let μ_{GAF} be its distribution. Rotation & translation invariant.

Problem 1. We discuss three problems:

- What is the natural μ_{GAF}-reversible diffusion $\mathsf{X} = \{\mathsf{X}_t\}$. Here

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \qquad \text{(unlabeled diffusion)}$$

- How to construct $\mathsf{X} = \{\mathsf{X}_t\}$?
- What is the SDE representation of $\mathbf{X}_t = (X_t^i)$?

Main theorem: Set Up

- Let S be the configuration space. Let $\mathsf{s} = \sum_i \delta_{s_i} \in \mathsf{S}$.
- Let $\mathbb{D}$ is the standard square field on S:

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_{i=1}^{\infty} \frac{\partial \tilde{f}}{\partial s_i} \cdot \frac{\partial \tilde{g}}{\partial s_i}$$

Here f is a local and smooth function on S, and $\tilde{f}(s_1,\ldots,)$ is a symmetric function such that $f(\mathsf{s}) = \tilde{f}(s_1,\ldots,)$.

- Let $\mathcal{D}_0$ be the set of local smooth functions. Let

$$\mathcal{E}^{\mu_{\mathsf{GAF}}}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g] d\mu_{\mathsf{GAF}}$$

on $L^2(\mathsf{S}, \mu_{\mathsf{GAF}})$ with domain

$$\mathcal{D}_0^{\mu_{\mathsf{GAF}}} = \{f \in L^2(\mu_{\mathsf{GAF}}); f \in \mathcal{D}_0,\ \mathcal{E}^{\mu_{\mathsf{GAF}}}(f,f) < \infty\}.$$

Thm 1. *$(\mathcal{E}^{\mu_{\mathsf{GAF}}}, \mathcal{D}_0^{\mu_{\mathsf{GAF}}})$ is closable on $L^2(\mu_{\mathsf{GAF}})$.*

- Proof of Thm 1 consists of “Ghosh’s quantitative bound of GAF” and “a generalization of [O. ’13]”.
- From Thm 1 we obtain L^2-Markovian semi-group.

Main theorem: GAF diffusion

Let $(\mathcal{E}^{\mu_{\mathsf{GAF}}}, \mathcal{D}^{\mu_{\mathsf{GAF}}})$ be the closure on $L^2(\mu_{\mathsf{GAF}})$.

Thm 2. (Construction of dynamics)

(1) μ_{GAF}*-reversible unlabeled diffusions* X

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$$

associated with $(\mathcal{E}^{\mu_{\mathsf{GAF}}}, \mathcal{D}^{\mu_{\mathsf{GAF}}})$ *on* $L^2(\mu_{\mathsf{GAF}})$ *exists.*

(2) $\mathbf{X} = (X_t^i)_{i \in \mathbb{N}}$ *is a* $\mathbb{C}^{\mathbb{N}}$*-valued diffusion.*

(3) *Each tagged particle* X_t^i *does not collide each other.*

- Thm 2 follows from a general theory in [O.'96,'04,'10,'13]" and the closability in Thm 1.
- We have not yet obtained the infinite-dimensional stochastic differential equation describing the labeled dynamics $\mathbf{X} = (X_t^i)$.

This is a problem to calculate the logarithmic derivative of μ_{GAF}.

A general theory for ISDEs

I have been developing a general theory for interacting Brownian motions in infinite dimentions, and like to apply to GAF. I would explain about this.

Outline:

- Examples: Sine, Airy, Bessel & Ginibre
- quasi-Gibbs measures and unlabeled diffusion

  and a generalization to GAF
- logarithmic derivative and SDE representation
- Calculation of logarithmic derivatives
- Examples: Ginibre and Airy RPFs

- We solve ISDEs of the form

$$dX_t^i = dB_t^i + b(X_t^i, \mathbf{X}_t^{\diamondsuit i})dt \quad (i \in \mathbb{N}) \tag{1}$$

Here $\mathbf{X}_t = (X_t^1, \dots,) \in (\mathbb{R}^2)^{\mathbb{N}}$-valued, and

$$\mathbf{X}_t^{\diamondsuit i} = (X_t^j)_{j \in \mathbb{N} \setminus \{i\}}.$$

The coefficient $b(x, \mathbf{y})$ is symmetric in $\mathbf{y} = (y_i)_{i \in \mathbb{N}}$ for each $x \in \mathbb{R}^2$.
$\mathbf{B}_t = (B_t^1, \dots,)$ is $(\mathbb{R}^2)^{\mathbb{N}}$-valued standard Brownian motion.
We will construct weak solution $(\mathbf{X}, \mathbf{B})$.
Our method can be applied to the case with $\sigma(X_t^i, \mathbf{X}_t^{\diamondsuit i})dB_t^i$.
For simplicity we talk about (1) only.

- Because of the symmetry of $b(x, \mathbf{y})$ in $\mathbf{y}$, we can rewrite

$$dX_t^i = dB_t^i + b(X_t^i, \mathsf{X}_t^{\diamondsuit i})dt \quad (i \in \mathbb{N}) \tag{2}$$

Here we regard $b(x, \cdot)$ as a function on the configuration space, and

$$\mathsf{X}_t^{\diamondsuit i} = \sum_{j \neq i} \delta_{X_t^j}$$

- Gibbsian examples for suitable α and d: $(i \in \mathbb{N})$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j\neq i}^{\infty} \{ \frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}} \} dt \qquad \text{(LJ 6-12)}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j\neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{\alpha}} dt. \qquad \text{(Riesz)}$$

- (LJ 6-12): $d = 3$ Lennard-Jones 6-12 potential
- (Riesz): $\alpha > d + 2$ Riesz potential (Gibbsian case)

- We recall the examples: $(i \in \mathbb{N})$ and $\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x)$.

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{1}{X_t^i - X_t^j} dt \tag{Sine}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt \tag{Airy}$$

$$dX_t^i = dB_t^i + \frac{a}{2X_t^i} dt + \frac{\beta}{2} \sum_{j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j} dt \tag{Bessel}$$

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{\substack{|X_t^i - X_t^j|<r \\ j\neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \tag{Ginibre}$$

Algebraic construction in 1D. Let $d = 1$ and $\beta = 2$.

- Sine, Airy, and Bessel can be constructed by space-time correlation functions. So there are two very different constructions for 1D system woth $\beta = 2$ arising from Random matrix theory.

Thm 3 (O.-Tanemura '14). *Let μ be Sine, Airy or Bessel RPFs. Stochastic dynamics constructed by stochastic analysis and the space-time correlation functions are equal.*

- The importance is the following. From algebraic construction we can obtain quantative infomation such as moment bounds of linear statistics. From analytic construction, we can obtain qualitative information such as semi-martingale property of tagged particles, non-collision property, non-explosion property, Itô formula, and so on.

- At present, such a algebraic construction is restricted $d = 1$, $\beta = 2$ and dynamics coming from Random matrix theory (logarithmic interactions).

Algebraic construction in 1D.

As an example, we explain Airy.

- Space-time correlation functions are given by the extended Airy kernel:

$$K_{\mathrm{Ai}}(s,x;t,y) = \begin{cases} \int_0^\infty du e^{-u(t-s)/2} \mathrm{Ai}(u+x)\mathrm{Ai}(u+y), & t \geq s \\ -\int_{-\infty}^0 du e^{-u(t-s)/2} \mathrm{Ai}(u+x)\mathrm{Ai}(u+y), & t < s \end{cases}.$$

The unlabeled process $\mathsf{Z}_t = \sum_{i=1}^\infty \delta_{Z_t^i}$ is given by its moment generating function ($\mathbf{f} = (f_1, \dots, f_M)$, $\mathbf{t} = (t_1, \dots, t_M)$, $t_i < t_{i+1}$)

$$\Psi^{\mathbf{t}}[\mathbf{f}] = E[\exp\{\sum_{m=1}^{M} \int_{\mathbb{R}} f_m(x) \mathsf{Z}_{t_m}(dx)\}]$$

defined as a Fredholm determinant

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \mathrm{Det}_{(s,t)\in I^2,\,(x,y)\in\mathbb{R}^2}[\delta_{st}\delta(x-y) + K_{\mathrm{Ai}}(s,x;t,y)\chi_t(y)].$$

Here $I = \{t_1, \dots, t_M\}$ and $\chi_{t_m}(y) = e^{f_m(y)} - 1$, Ginibre interacting Brownian motions in infinite-dimensions.

- We write Ginibre in non-consice form SDEs:

$$dX_t^1 = dB_t^1 + \lim_{r\to\infty} \sum_{j\neq 1,\,|X_t^1-X_t^j|<r}^{\infty} \frac{X_t^1 - X_t^j}{|X_t^1 - X_t^j|^2} dt$$

$$dX_t^2 = dB_t^2 + \lim_{r\to\infty} \sum_{j\neq 2,\,|X_t^2-X_t^j|<r}^{\infty} \frac{X_t^2 - X_t^j}{|X_t^2 - X_t^j|^2} dt$$

$$dX_t^3 = dB_t^3 + \lim_{r\to\infty} \sum_{j\neq 3,\,|X_t^3-X_t^j|<r}^{\infty} \frac{X_t^3 - X_t^j}{|X_t^3 - X_t^j|^2} dt$$

$$dX_t^4 = dB_t^4 + \lim_{r\to\infty} \sum_{j\neq 4,\,|X_t^4-X_t^j|<r}^{\infty} \frac{X_t^4 - X_t^j}{|X_t^4 - X_t^j|^2} dt$$

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Ginibre interacting Brownian motions in infinite-dimensions.

- Ginibre in non-consice form SDEs in the 2'nd representation:

$$dX_t^1 = dB_t^1 - X_t^1 dt + \lim_{r\to\infty} \sum_{j\neq 1, |X_t^j|<r}^{\infty} \frac{X_t^1 - X_t^j}{|X_t^1 - X_t^j|^2} dt$$

$$dX_t^2 = dB_t^2 - X_t^2 dt + \lim_{r\to\infty} \sum_{j\neq 2, |X_t^j|<r}^{\infty} \frac{X_t^2 - X_t^j}{|X_t^2 - X_t^j|^2} dt$$

$$dX_t^3 = dB_t^3 - X_t^3 dt + \lim_{r\to\infty} \sum_{j\neq 3, |X_t^j|<r}^{\infty} \frac{X_t^4 - X_t^j}{|X_t^4 - X_t^j|^2} dt$$

$$dX_t^4 = dB_t^4 - X_t^4 dt + \lim_{r\to\infty} \sum_{j\neq 4, |X_t^j|<r}^{\infty} \frac{X_t^4 - X_t^j}{|X_t^4 - X_t^j|^2} dt$$

$$\bullet \quad \bullet \quad \bullet$$

Here is a simulation:

Cofiguration spaces

Set up:

- $S = \mathbb{R}^d$: Space, where particles move,
- $S_r = \{|x| \le r\}$,
- $\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i},\ \mathsf{s}(S_r) < \infty (\forall r)\}$:

  Configuration space over S.

  Polish space with vague topology.

  The space of unlabeled particles.
- $S^{\mathbb{N}}$ is the space of labeled particles.
- $\mathsf{s} = \sum_i \delta_{s_i}$ denotes unlabeled particles.

  $\mathbf{s} = (s_i) \in S^{\mathbb{N}}$ denotes labeled particles.
- Since $S^{\mathbb{N}}$ is too large, we use S instead.

- $\mathsf{B}_t = \sum_{i=1}^{\infty} \delta_{B_t^i}$ is S-valued Brownian motion.
- $\mathbf{B}_t = (B_t^i)_{i \in \mathbb{N}}$ is $S^{\mathbb{N}}$-valued Brownian motion.

Canonical square field

For a fun f on S let $f(\mathsf{s}) =: \tilde{f}(s_1, \ldots)$, where $\tilde{f}$ is symmetric, $\mathsf{s} = \sum \delta_{s_i}$.
Let $\mathcal{D}_0$ be the set of bounded, local, smooth functions f on S.
i.e. f is $\sigma[\pi_r]$-measurable for some $r < \infty$, $\tilde{f}$ is smooth.
Let $\mathbb{D}$ be the canonical square field on S:

$$\mathbb{D}[f, g](\mathsf{s}) = \frac{1}{2} \sum_i \nabla_i \tilde{f} \cdot \nabla_i \tilde{g}.$$

Here $\nabla_i = (\frac{\partial}{\partial s_{i1}}, \ldots, \frac{\partial}{\partial s_{id}})$.

The rhs is independent of particular choice of label.

- For a RPF μ we set

$$\mathcal{E}^{\mu}(f, g) = \int_{\mathsf{S}} \mathbb{D}[f, g] \mu(d\mathsf{s}),$$

$$\mathcal{D}_0^{\mu} = \{f \in \mathcal{D}_0; \mathcal{E}^{\mu}(f, f) < \infty, f \in L^2(\mu)\}$$

- If we take $\mu = \Lambda$, Poisson RPF with Lebesgue intensiy, then the bilinear form associates Brownian motion $\mathsf{B}_t = \sum_i \delta_{B_t^i}$.

In this sense $\mathbb{D}$ is the canonical square field.

From RPF to unlabeled diffusion

Outline of the proof:

$$\mu \Rightarrow (\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}, L^2(\mu)) \Rightarrow \mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i\in\mathbb{N}} \Rightarrow \text{ISDE}$$

- The first arrow is automatic. For a given RPF μ, we can associated a positive bilinear form through the square field $\mathbb{D}$.

- If $(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}, L^2(\mu))$ is closable and its closue is quasi-regular, then by Dirichlet form theory an associated μ-reversible diffusion X_t exists.
- For this we introduce a notion of quasi-Gibbs measure.

If μ is quasi-Gibbs with upper semi-continuous potential Ψ, then the bilinear form id closable. In addition, μ satisies a marginal condition (local boundedness of correlation functions, say), then the form becomes quasi-regular. Hence by the general theory of Dirichlet form there exists the associated unlabeled diffusion X_t.

Quasi-Gibbs measures:

- $\pi_r, \pi_r^c : \mathsf{S} \to \mathsf{S}$: projections

$$\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r), \qquad \pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$$

- For a RPF μ we set

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(S_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi))$$

- Let $\Psi : S \to \mathbb{R} \cup \{\infty\}$ (interaction).

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i - s_j)$$

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Ψ-Quasi-Gibbs meas.

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(S_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi)) \tag{QG}$$

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i - s_j) \tag{3}$$

Def: μ is Ψ-quasi-Gibbs measure if $\exists\ c_{r,\xi}^m$ s.t.

$${c_{r,\xi}^m}^{-1} e^{-\mathcal{H}_r} d\Lambda_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\Lambda_r^m$$

Here $\Lambda_r^m = \Lambda(\cdot | \mathsf{s}(S_r) = m)$ and Λ_r is the Poisson RPF with $1_{S_r} dx$.

- The above definition is a simplified version.
- Gibbs measures $\Rightarrow$ Quasi-Gibbs measures: If μ satisfies DLA eq.

$$\mu_{r,\xi}^m = c_{r,\xi}^m e^{-\mathcal{H}_r - \sum_{x_i \in S_r, \xi_j \in S_r^c} \Psi(x_i, \xi_j)} d\Lambda_r^m, \tag{DLA}$$

then μ is a canonical Gibbs m. (DLA) does not make sense for

$$\Psi(x, y) = -\log|x - y|$$

$$\mu \Rightarrow (\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}, L^2(\mu)) \Rightarrow \mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i\in\mathbb{N}} \Rightarrow \text{ISDE}$$

Unlabeled diffusions

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont

Thm 4 (O.'96 (CMP) (closability)).

(A1) $\Rightarrow$ $(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$ *is closable on* $L^2(\mu)$.

• Thm implies the existence of the associated L^2 Markovian semi-group.

Thm 1 (A1) $\Rightarrow$ $(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$ is closable on $L^2(\mu)$.

Proof. Outline of (1): • Let

$$\mathcal{E}^{\mu_{r,\xi}^m}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g]d\mu_{r,\xi}^m \quad \text{(reflecting BC)}.$$

Then $(\mathcal{E}^{\mu_{r,\xi}^m}, \mathcal{D}_0^{\mu_{r,\xi}^m})$ is closable on $L^2(\mu_{r,\xi}^m)$ by (A1).

• Recall the disintegration: $\mu(\cdot) = \sum_{m=1}^{\infty} \mu_{r,\xi}^m(\cdot)\mu(d\xi)$.

Then $(\widehat{\mathcal{E}}_r^{\mu}, \mathcal{D}_0^{\mu})$ are closable on $L^2(\mu)$. Here

$$\widehat{\mathcal{E}}_r^{\mu}(f,g) = \int_{\mathsf{S}} \sum_{m=1}^{\infty} \mathcal{E}^{\mu_{r,\xi}^m}(f,g)d\mu \quad \text{(reflecting BC)}.$$

• By the monotone convergence theorem of closable forms we see

$$\widehat{\mathcal{E}}^{\mu}(f,f) = \lim_{r\to\infty} \widehat{\mathcal{E}}_r^{\mu}(f,f), \quad \widehat{\mathcal{D}}_0 = \{f; \lim_{r\to\infty} \widehat{\mathcal{E}}_r^{\mu}(f,f) < \infty\}$$

is closable. Hence $(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$ is closable. $\square$

Application of quasi-Gibbs property to dynamics: existence of diffusions

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$
Here $\mathsf{S}_r^k = \{\mathsf{s}(S_r) = k\}$, σ_r^k is k-density fun on S_r^k.

Thm 5 (O.'96 (CMP) (existence of diffusions)).
Assume (A2). *Assume that* $(\mathcal{E}^\mu, \mathcal{D}_0^\mu)$ *is closable on* $L^2(\mu)$.
Then $\exists$ *diffusion* $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ *associated with the closure*

$$(\mathcal{E}^\mu, \mathcal{D}^\mu) \text{ of } (\mathcal{E}^\mu, \mathcal{D}_0^\mu) \text{ on } L^2(\mu).$$

Proof. This follows from a concrete construction of cut off function, which yields the quasi-regularity of Dirichlet forms. The general theory gives the diffusion. □

Remark 1. • In general, the closures of the limit Dirichlet forms

$$(\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}}) \quad \text{and} \quad (\mathcal{E}^\mu, \mathcal{D}^\mu)$$

are not equal. We will prove the coincidence of these by using the strong uniqueness of the solutions of the associated ISDEs.
• Lang's dynamics ('79) are given by the Dirichlet form $(\hat{\mathcal{E}}^\mu, \hat{\mathcal{D}})$.
O's ('96) dynamics are given by $(\mathcal{E}^\mu, \mathcal{D}^\mu)$. O.-Tanemura prove these are the same if tagged particles have no explosions.

Let $\Psi_2(x,y) = -\log|x-y|$ be the 2-dim Coulomb potential.

Thm 6 (O. AOP '13, O.-Honda '14, O.-Tanemura '14)**.**
(1) *Ginibre RPF is a* $2\Psi_2$*-quasi Gibbs measure.*
(2) Sine_β *RPF are* $\beta\Psi_2$*-quasi Gibbs m for* $\beta = 1,2,4$.
(3) Bessel_2^a *RPF is a* $2\Psi_2$*-quasi Gibbs m.*
(4) Airy_β *RPF are* $\beta\Psi_2$*-quasi Gibbs m for* $\beta = 1,2,4$.

• GAF is not quasi-Gibbsian. Indeed, Ghosh proved:
Thm 7 (Ghosh '12)**.** *Let* $\mu = \mu_{\mathsf{GAF}}$*. Then there exists constant* $c_{r,\xi}^m$ *such that*

$$\frac{1}{c_{r,\xi}^m} e^{-\mathcal{H}_r} d\Lambda_r^m[\mathsf{Ce}(\xi)] \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\Lambda_r^m[\mathsf{Ce}(\xi)]$$

for μ*-a.s.* ξ*. Here* Ce *is* π_r^c*-measurable, and*

$$\Lambda_r^m[M] = \Lambda_r^m(\cdot|\sum_{i=1}^m s_i = M).$$

Proof of the Main Th (Closability of GAF bilinear form).

Let $\gamma_m = \frac{1}{\sqrt{m}}(1,\dots,1)$ be the unit vector on the diagonal.

$$\frac{\partial}{\partial s_i} - \frac{1}{\sqrt{m}}\frac{\partial}{\partial \gamma_m}$$

Let $S_r = \{|s| < r\}$ and $\mathsf{S}_r^m = \{\mathsf{s}(S_r) = m\}$. Set $\mathsf{s} = \sum_i \delta_{s_i}$ and

$$\mathbb{D}_r^{\mathsf{GAF}}[f,g] = \frac{1}{2}\sum_{m=1}^{\infty} 1_{\mathsf{S}_r^m}(\mathsf{s}) \sum_{s_i \in S_r} \Big(\frac{\partial}{\partial s_i} - \frac{1}{\sqrt{m}}\frac{\partial}{\partial \gamma_m}\Big)\check{f} \cdot \Big(\frac{\partial}{\partial s_i} - \frac{1}{\sqrt{m}}\frac{\partial}{\partial \gamma_m}\Big)\check{g}.$$

Then for any $f \in \mathcal{D}_0$

$$\mathbb{D}_r^{\mathsf{GAF}}[f,f] \uparrow \mathbb{D}[f,f] \tag{1}$$

Let $\mathcal{E}_r^{\mathsf{GAF}}(f,g) = \int_{\mathsf{S}} \mathbb{D}_r^{\mathsf{GAF}}[f,g] d\mu^{\mathsf{GAF}}$. Then by (1)

$$\mathcal{E}_r^{\mathsf{GAF}}(f,f) \uparrow \mathcal{E}^{\mathsf{GAF}}(f,f) \tag{1}$$

By quantitative bound $(\mathcal{E}_r^{\mathsf{GAF}}, \mathcal{D}_0)$ is closable. Hence $(\mathcal{E}^{\mathsf{GAF}}, \mathcal{D}_0)$ is closable from (2). □

- We can generalize the notion of quasi-Gibbsian for general sub-manifolds. GAF is a special case and only an example.

$$\mu \Rightarrow (\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}, L^2(\mu)) \Rightarrow \mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i\in\mathbb{N}} \Rightarrow \text{ISDE}$$

Labeled dynamics

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ.
(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$
(A3) $\{X_t^i\}$ do not collide each other (non-collision)
(A4) each tagged particle X_t^i never explode (non-explosion)

By (A3) and (A4) the labeled dynamics

$$\mathbf{X}_t = (X_t^1, X_t^2, \ldots)$$

can be constructed from the unlabeled dynamics

$$\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}.$$

Indeed, the particles keep the initial label forever.

Sufficient condition of (A3) & (A4)

Let $\mathsf{S}_{s,i} = \mathsf{S}_s \cap \mathsf{S}_i$:

$$\mathsf{S}_s = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(\{x\}) = 0 \text{ for all } x \in S\}, \quad \mathsf{S}_i = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S) = \infty\}.$$

- (A3) is equaivalent to

$$\mathrm{Cap}^{\mu}(\mathsf{S}_{s,i}^c) = 0. \tag{4}$$

Let ρ^n be a n-correlation function of μ.

Lem 1. *Suppose μ is quasi-Gibbs with Ψ. Let ρ^2 be 2-correlation function of μ. Suppose one of the following holds. Then* (A3) *holds.*
(1) $d \geq 2$ *and ρ^2 are locally bounded.*
(2) $d = 1$ *and*

$$\rho^2(x,y) \leq Ch(|x-y|) \text{ locally near } \{x = y\}.$$

Here $h(t)$ such that

$$\int_{0+}^{1} \frac{1}{h(t)} dt = \infty.$$

Corollary 1. *Sine$_\beta$, Airy$_\beta$, Bessel$_\beta$ ($\beta \geq 1$), Ginibre RPFs satsfy* (A2).

- By (A3) we represent one-labeled process $(X_t^1, \sum_{j=2}^{\infty} \delta_{X_t^j})$ by the Dirichlet space

$$(\mathcal{E}^{\mu^{[1]}}, \mathcal{D}^{\mu^{[1]}}, L^2(\mu^{[1]})).$$

Applying Takeda criteria based on Lyons-Zheng decomposition we deduce (A4) from $\exists T > 0$

$$\liminf_{r\to\infty} \{ \int_{|x|\le r+R} \rho^1(x)dx\} \{ \int_{\frac{r}{\sqrt{(r+R)T}}} \mathrm{g}(u)du \} = 0 \quad \text{for all } T. \tag{5}$$

Lem 2. (A4) *follows from* (5).

$$\mu \Rightarrow (\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}, L^2(\mu)) \Rightarrow \mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i\in\mathbb{N}} \Rightarrow \text{ISDE}$$

ISDE representation

Log derivative of μ: precise correspondence between RPFs & potentials

- Let μ_x be the (reduced) Palm m. of μ conditioned at x

$$\mu_x(\cdot) = \mu(\cdot - \delta_x | \mathsf{s}(x) \geq 1)$$

- Let μ^1 be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^1(A \times B) = \int_A \rho^1(x)\mu_x(B)dx$$

- $\mathsf{d}^\mu \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathsf{S}, \mu^1)$ is called the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^1 = -\int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}^\mu d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_0$$

  Here ∇_x is the nabla on $\mathbb{R}^d$.

- Very informally

$$\mathsf{d}^\mu = \nabla_x \log \mu^1$$

- A caluculation of log derivative of Gibbs measures are trivial.

  Indeed, it is immediate from DLR equation.

- This is not the case for RPFs appearing in RMT.

  We will give a sufficient condition later.

Log derivative

A very informal calculation shows:

- If $\mu^1(dxd\mathsf{s}) = m(x, s_1, \ldots)dx \prod_i ds_i$, then

$$
\begin{aligned}
& -\int \nabla_x f(x, s_1, \ldots)\mu^1(dxds_1 \cdots) \\
&= -\int \nabla_x f(x, s_1, \ldots)m(x, s_1, \ldots)dx \prod_i ds_i \\
&= \int f(x, s_1, \ldots)\nabla_x m(x, s_1, \ldots)dx \prod_i ds_i \\
&= \int f(x, s_1, \ldots)\frac{\nabla_x m(x, s_1, \ldots)}{m(x, s_1, \ldots)} m(x, s_1, \ldots)dx \prod_i ds_i .
\end{aligned}
$$

Hence

$$
\mathsf{d}^\mu = \frac{\nabla_x m(x, s_1, \ldots)}{m(x, s_1, \ldots)} = \nabla_x \log m(x, s_1, \ldots).
$$

General theorems on infinite-dim SDEs

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ.

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

(A3) $\{X_t^i\}$ do not collide each other

(A4) each tagged particle X_t^i never explode

(A5) The log derivative $\mathsf{d}^{\mu} \in L^1_{\text{loc}}(\mu^1)$ exists $\Rightarrow$(SDE representation)

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Thm 8. (O.12(PTRF)) (A1)–(A5) $\Rightarrow$ $\exists \mathsf{S}_0 \subset \mathsf{S}$ *such that* $\mu(\mathsf{S}_0) = 1$, *and that, for* $\forall \mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, *there exists a solution* $(\mathbf{X}, \mathbf{B})$ *satisfying*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j \neq i} \delta_{X_t^j})dt, \quad (X_0^i)_{i \in \mathbb{N}} = \mathbf{s}$$

$$\mathbf{X}_t \in \mathfrak{u}^{-1}(\mathsf{S}_0) \qquad \textit{for all } t$$

Here $\mathfrak{u} : S^{\mathbb{N}} \to \mathsf{S}$ such that $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$.

Corollary 2. *Suppose that there exists a RPF* μ *satisfying* (A1)–(A4) *and*

$$\nabla_x \log \mu^{[1]}(x, \mathsf{s}) = 2b(x, \mathsf{s}).$$

Then ISDE (1) *has a weak solution.*

Proof:

- $S^{\mathbb{N}}$ does not have good measures $\Rightarrow$ no Dirichlet forms on $S^{\mathbb{N}}$ $\Rightarrow$

Introduce a sequence of spaces with Campbel measures $\mu^{[M]}$:

$$S^M \times \mathsf{S}, \quad d\mu^{[M]} = \rho^M(\mathbf{x}_M)\mu_{\mathbf{x}_m}(d\mathsf{s})d\mathbf{x}_M$$

Here ρ^M is a M-correlation function of μ and $\mu_{\mathbf{x}_m}$ is the reduced Palm measure conditioned at $\mathbf{x}_M$.

Let $\mathbb{D}^{[M]}$ be the natural square field of $S^M \times \mathsf{S}$. Let

$$\mathcal{E}^{[M]}(f,g) = \int_{S^M \times \mathsf{S}} \mathbb{D}^{[M]}[f,g]d\mu^{[M]},$$

$$L^2(\mu^{[M]}), \quad C_0^\infty(S^M) \otimes \mathcal{D}_\circ.$$

Lem 3. *These bilinear forms are closable, and their closures are quasi-regular Dirichlet forms. Hence associated diffusion* $(\mathbf{X}_t^M, \mathsf{X}_t^{M*})$ *exists:*

$$(\mathbf{X}_t^M, \mathsf{X}_t^{M*}) = (X_t^{M,1}, \ldots, X_t^{M,M}, \sum_{i=M+1}^{\infty} \delta_{X_t^{M,i}})$$

Coupling of Dirichlet forms:

- Let fix a label ℓ. Let

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$$

be the unlabeld diffusion associated with the original unlabeled Dirichlet form

$$(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}, L^2(\mu)).$$

Thm 9. *Associated diffusions have consistency*

$$(X_t^{M,1}, \dots, X_t^{M,M}, X_t^{M,M+1}, \dots) = (X_t^1, \dots, X_t^M, X_t^{M+1}, \dots) \quad \textit{in law}$$

or equivalently

$$(\mathbf{X}_t^M, \mathsf{X}_t^{M*}) = (X_t^1, \dots, X_t^M, \sum_{i=M+1}^{\infty} \delta_{X_t^i}) \quad \textit{in law}$$

From this coupling and Fukushima decomposition (Itô formula) we prove that (X_t^i) satisfies the ISDE. We use the M-labeled process $(\mathbf{X}_t^M, \mathsf{X}_t^{M*})$, to apply Itô formula to coordinate functions $x_1, \dots, x_M$.

Coupling of Dirichlet forms:

- The key point here is that, instead of large space

$$S^{\mathbb{N}}$$

we use a system of countably infinite *good* infinite dimensional sapce

$$S^1 \times \mathsf{S},\ S^2 \times \mathsf{S},\ S^3 \times \mathsf{S},\ S^4 \times \mathsf{S},\ S^5 \times \mathsf{S},\ S^6 \times \mathsf{S},\ S^7 \times \mathsf{S},\ \cdots$$

- By the diffusion X on the original unlabeled space

$$\mathsf{S},$$

we construct a coupling of diffusions $(\mathbf{X}^M, \mathsf{X}^{M*})$ on these inifinite many spaces $S^M \times \mathsf{S}$.

- From this coupling, we have the ISDE representation. Indeed, we can apply Itoô formula to each coordinate functions $f(\mathbf{x}) = x_k$. We use $\mathcal{E}^{[M]}(f, g)$ for $1 \le k \le M$.

Log derivative of μ: precise correspondence between RPFs & potentials

- The log derivative gives the precise correspondence between RPFs μ and potentials (Φ, Ψ).

- We next give examples of logarithmic derivatives

$\mathsf{d}^{\mu} = \nabla_x \log \mu^1$

Thm 10 (O. PTRF 12).

(1) *Let* μ_{gin} *be the Ginibre RPF. Then*

$$\mathsf{d}^{\mu_{\mathsf{gin}}}(x, \mathsf{s}) = \lim_{r\to\infty} 2 \sum_{|x-s_i|<r} \frac{x - s_i}{|x - s_i|^2}$$

$$\mathsf{d}^{\mu_{\mathsf{gin}}}(x, \mathsf{s}) = -2x + \lim_{r\to\infty} 2 \sum_{|s_i|<r} \frac{x - s_i}{|x - s_i|^2}$$

(2) *Let* $\mu_{\mathsf{sin},\beta}$ *be the* Sine$_\beta$ *RPF. Suppose* $\beta = 1, 2, 4$. *Then*

$$\mathsf{d}^{\mu_{\mathsf{sin},\beta}}(x, \mathsf{s}) = \lim_{r\to\infty} \beta \sum_{|x-s_i|<r} \frac{1}{x - s_i}$$

Thm 11 (O.-Honda). *Let* $\mu_{\mathsf{bes},2}^{a}$ *be the* Bessel$_2^a$ *RPF. Then*

$$\mathsf{d}^{\mu_{\mathsf{bes},2}^{a}}(x, \mathsf{s}) = \frac{a}{x} + 2 \sum_{|x-s_i|<r} \frac{1}{x - s_i}$$

Thm 12 (O.-Tanemura). [*Airy RPFs:* $\mu_{\mathsf{Ai},\beta}$]
Let $\beta = 1, 2, 4$. *Then the log derivative* $\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}$ *is*

$$\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}(x, \mathsf{s}) = \beta \lim_{r\to\infty} \{ (\sum_{|x-s_i|<r} \frac{1}{x - s_i}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \}$$

Here

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0)}(x)$$

- The significant problem is:

  To solve what is the log derivative $\mathsf{d}^{\mathsf{GAF}}$?

  To obtain the representation of $\mathsf{d}^{\mathsf{GAF}}$.

Calculation of logarithmic derivative

- Assume that n-point cor funs $\{\rho^{N,n}\}$ satisfy for each $r, n \in \mathbb{N}$

$$\lim_{N\to\infty} \rho^{N,n}(\mathbf{x}) = \rho^n(\mathbf{x}) \quad \text{uniformly on } S_r^n, \tag{6}$$

$$\sup_{N\in\mathbb{N}} \sup_{\mathbf{x}\in S_r^n} \rho^{N,n}(\mathbf{x}) \le C_1^{-n} n^{C_2 n}, \quad 0 < C < \infty, 0 < C_2 < 1, \ . \tag{7}$$

Calculation of logarithmic derivative

- We assume that μ^N have log derivative d^N such that

$$\mathsf{d}^N(x,\mathsf{y}) = u^N(x) + \mathsf{g}_s^N(x,\mathsf{y}) + w_s^N(x,\mathsf{y}) \tag{8}$$

Here g, g^N, v, $v^N : S^2 \to \mathbb{R}^d$ and $w : S \to \mathbb{R}^d$ and set ($\mathsf{y} = \sum_i \delta_{y_i}$)

$$\mathsf{g}_s(x,\mathsf{y}) = \int_{|x-y|<s} v(x,y)dy + \sum_{|x-y_i|<s} g(x,y_i),$$

$$\mathsf{g}_s^N(x,\mathsf{y}) = \int_{|x-y|<s} v^N(x,y)dy + \sum_{|x-y_i|<s} g^N(x,y_i),$$

$$w_s^N(x,\mathsf{y}) = \int_{s\le|x-y|} v^N(x,y)dy + \sum_{s\le|x-y_i|} g^N(x,y_i) \in L^{\hat{p}}_{\mathrm{loc}}(\mu^1).$$

Calculation of logarithmic derivative

- Let $1 < p < \hat{p} < \infty$. Assume that

$$\limsup_{N\to\infty} \int_{S_r \times \mathsf{S}} |\mathsf{d}^N - u^N|^{\hat{p}} d\mu^{N,1} < \infty \quad \text{for all } r \in \mathbb{N} \tag{9}$$

$$\lim_{N\to\infty} u^N = u \quad \text{in } L^{\hat{p}}_{\text{loc}}(S, dx) \tag{10}$$

$$\lim_{N\to\infty} \mathsf{g}_s^N = \mathsf{g}_s \quad \text{in } L^{\hat{p}}_{\text{loc}}(\mu^1) \quad \text{for all } s, \tag{11}$$

$$\lim_{s\to\infty} \limsup_{N\to\infty} \int_{S_r \times \mathsf{S}} |w_s^N(x,\mathsf{y}) - w(x)|^{\hat{p}} d\mu^{N,1} = 0. \tag{12}$$

Recall that

$$\mathsf{g}_s(x,\mathsf{y}) = \int_{|x-y|<s} v(x,y)dy + \sum_{|x-y_i|<s} g(x,y_i)$$

Thm 13. *Assume (6)–(12). Then* d^μ *exists in* $L^p_{\text{loc}}(\mu^1)$ *given by*

$$\mathsf{d}^\mu(x,\mathsf{y}) = u(x) + \lim_{s\to\infty} \mathsf{g}_s(x,\mathsf{y}) + w(x). \tag{13}$$

Calculation of logarithmic derivative

Recall that

$$\mathsf{g}_s(x,\mathsf{y}) = \int_{|x-y|<s} v(x,y)dy + \sum_{|x-y_i|<s} g(x,y_i)$$

Thm 13 The log derivative d^{μ} exists in $L^p_{\text{loc}}(\mu^1)$ and is given by

$$\mathsf{d}^{\mu}(x,\mathsf{y}) = u(x) + \lim_{s\to\infty} \mathsf{g}_s(x,\mathsf{y}) + w(x). \tag{14}$$

Example 1. In the case of Ginibre RPF, we take

$$u^N(x) = u(x) = -2x, \quad w(x) = 2x,$$
$$v^N(x,y) = v(x,y) = 0,$$
$$g^N(x,y) = g(x,y) = \frac{2(x-y)}{|x-y|^2}.$$

Example 2. In the case of Airy RPF, we take

$$u^N(x) = \beta\{\int_{\mathbb{R}} \frac{\rho^{N,1}_{\beta,x}(y)}{x-y}dy\} - N^{1/3} - \frac{N^{-1/3}}{2}x$$

$$u(x) = \beta \lim_{s\to\infty}\{\int_{|s|<s} \frac{\rho^{1}_{\beta,x}(y)}{x-y}dy - \int_{|y|<s} \frac{\varrho(y)}{-y}dy\}$$

$$w(x) = 0$$

$$v^N(x,y) = -\beta\frac{\rho^{N,1}_{\beta,x}(y)}{x-y}$$

$$v(x,y) = -\beta\frac{\rho^{1}_{\beta,x}(y)}{x-y}$$

$$g^N(x,y) = g(x,y) = \frac{\beta}{x-y}.$$