Infinite-dimensional stochastic differential equations arising from random matrix theory

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UK-Japan Stochastic Analysis School (JSPS Core-to-Core programme)

Total plan:

- 1'st Talk: Examples & strategy of the proof
- 2'nd Talk: Weak solutions
- 3'rd Talk: Strong solutions and pathwise uniqueness

Outline of 1'st talk:

- Dynamical soft edge scaling limit: Airy_β RPFs ($\beta = 1, 2, 4$)
- Dynamical bulk scaling limit: Sine RPFs and an SDE gap
- Ginbre and Bessel RPFs
- Histrical back ground of interacting Brownian motions
- Strategy: Outline of the proof.

Geometric soft edge/bulk scaling limits
of Gaussian ensembles

- GUE (Gaussian unitary ensemble): Gaussian random matrices:

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & \cdots & M_{1N} \\ M_{21} & M_{22} & \cdots & \cdots & M_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{N1} & M_{N2} & \cdots & \cdots & M_{NN} \end{pmatrix} \tag{1}$$

such that M are $N \times N$-hermitian matrices whose entries satisfying that

$$M_{ij} = \frac{G_{ij,1} + \sqrt{-1}G_{ij,2}}{\sqrt{2}} \quad (i < j), \quad M_{kk} = G_k \tag{2}$$

and that $G_{ij,1}$, $G_{ij,2}$, G_k are i.i.d. Gaussian random variables with mean 0 and variance 1.

- We define GOE and GSE similarly as real/quoternion symmetric Gaussian random variables.

Geometric scaling limit

- The distribution of eigen values of the G(O/U/S)E Random Matrices are given by ($\beta = 1, 2, 4$)

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^{N} |x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \tag{3}$$

This means the system can be regarded as particles interacting through logaritimic potential (2D Coulomb potentials).

- Wigner's theorem: The distribution of

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i/\sqrt{N}} \qquad \text{under } m_\beta^N$$

converges to the semi-circle law

$$\varsigma(x)dx = \frac{1}{2\pi}\sqrt{4 - x^2}dx \tag{4}$$

- This convergence corresponds to the law of large numbers.

Bulk/Soft edge scaling

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^{N}|x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \tag{3}$$

$$\varsigma(x)dx = \frac{1}{2\pi}\sqrt{4 - x^2}dx$$

- Bulk scaling: For $-2 < \theta < 2$ take $x_i = (s_i - \theta)/\sqrt{N}$ in (3):

$$\mu_{\mathrm{sin},\beta,\theta}^N(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N}|s_i - s_j|^\beta \prod_{k=1}^{N} e^{-\beta|s_k-\theta|^2/4N} d\mathbf{s}_N \tag{5}$$

- Soft edge scaling: Take $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$ in (3):

$$\mu_{\mathrm{Ai},\beta}^N(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N.$$

Soft edge scaling limit

Airy RPF: $\mu_{\mathrm{Ai},\beta}$ $(\beta = 1, 2, 4)$

- Take the scaling $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$ in

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^{N}|x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N$$

and set

$$\mu_{\mathrm{Ai},\beta}^N(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N.$$

Then $\mu_{\mathrm{Ai},\beta}^N$ converge to Airy RPF $\mu_{\mathrm{Ai},\beta}$:

$$\lim_{N\to\infty}\mu_{\mathrm{Ai},\beta}^N = \mu_{\mathrm{Ai},\beta}$$

Airy RPF – Soft edge scaling limit

- $\beta = 2 \Rightarrow \mu_{\mathsf{Ai},\beta}$ is a determinantal RPF given by (K_{Ai}, dx):

$$K_{\mathsf{Ai}}(x,y) = \frac{\mathsf{Ai}(x)\mathsf{Ai}'(y) - \mathsf{Ai}'(x)\mathsf{Ai}(y)}{x-y}$$

Here $\mathsf{Ai}(\cdot)$ is the Airy function.
The correlation function ρ^n_{Ai} is defined as

$$\rho^n_{\mathsf{Ai}}(\mathbf{x}) = \det[K_{\mathsf{Ai}}(x_i, x_j)]^n_{i,j=1}.$$

- If $\beta = 1, 4$, the correlation func of $\mu_{\mathsf{Ai},\beta}$ are given by similar formula of quaternion determinant.

- We discuss a dynamical counter part of this scaling limit.

Airy RPF – Soft edge scaling limit

- I give here minimal definition.
- Let $S=\mathbb{R}^d$, $[0,\infty)$, e.t.c.. S: configuration space over S

$$\mathsf{S}=\{\mathsf{s}=\sum_i \delta_{s_i}\,;\ s_i\in S,\ \mathsf{s}(|s|<r)<\infty\ (\forall r\in\mathbb{N})\}$$

- S is a Polish space with the vague topology.
- A prob meas. μ on S is called a random point field (RPF) on S.
- S is the set of unlabeled particles.
- $S^{\mathbb{N}}$ is the space of labeled particles.

- A symmetric function ρ^n is called the n-correlation function of μ w.r.t. Radon m. m if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(\mathbf{x}_n)\prod_{i=1}^{n} m(dx_i)=\int_{\mathsf{S}}\prod_{i=1}^{m}\frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!}d\mu$$

for any disjoint $A_i\in\mathcal{B}(S)$, $k_i\in\mathbb{N}$ s.t. $k_1+\ldots+k_m=n$.

Airy RPF – Soft edge scaling limit

- μ is called the determinantal RPF generated by (K, m) if its n-correlation functions ρ^n is given by

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{1 \le i,j \le n}$$

- It is known that (K, m) determines the RPF uniquely.
- The N-particle system of Airy RPF is a determinantal RPF whose kernel $K^N_{\mathsf{Ai}}(x, y)$ is given by orthogonal polynomials.

- The convergence of $\mu^N_{\mathsf{Ai},\beta}$ follows from that of correlation functions.
- This follows from that of kernels $K^N_{\mathsf{Ai}}(x, y)$.
- This follows from a calculation of orthogonal polynomials (special functions).

Airy RPF – Dynamical soft edge scaling limit

- We return to a dynamical soft edge scaling limit.
- From

$$\mu^N_{\mathrm{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N$$

we deduce the SDE of the N particle system:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\sum_{j=1,j\neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}}dt - \frac{\beta}{2}\{N^{1/3} + \frac{1}{2N^{1/3}}X_t^{N,i}\}dt$$

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- From

$$\mu^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N$$

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- Indeed, $X_t^{N,i}$ are associated with the Dirichlet form:

$$\mathcal{E}^{\mu^N_{\mathsf{Ai},\beta}}(f,g) = \int_{\mathbb{R}^N} \frac{1}{2}\sum_i^N \frac{\partial f}{\partial s_i}\frac{\partial g}{\partial s_i}\mu^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) \text{ on } L^2(\mathbb{R}^N, \mu^N_{\mathsf{Ai},\beta}).$$

Then, by integration by parts, the generator is

$$-L^N = \frac{1}{2}\Delta_N + \frac{\beta}{2}\sum_{i=1}^N [\sum_{j\neq i}^N \frac{1}{s_i - s_j} - \frac{\beta}{2}\{N^{1/3} + \frac{s_i}{2N^{1/3}}\}]\frac{\partial}{\partial s_i}$$

Airy RPF – Dynamical soft edge scaling limit

- The SDE of the N particle system:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2} \{ N^{1/3} + \frac{1}{2N^{1/3}} X_t^{N,i} \} dt$$

- The dynamics are also given by the space-time correlation functions.

- Problem: What SDE does the limit $\mathbf{X}_t = \lim_{N\to\infty} \mathbf{X}_t^N$ satisfy?

$$\text{Does } \lim_{N\to\infty} \{ \sum_{j=1, j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} - N^{1/3} \} \quad \text{converge ?}$$

How to solve the limit ISDE?

Airy RPF – Dynamical soft edge scaling limit

For a configuration $\mathsf{s} = \sum_i \delta_{s_i}$, let $\ell(\mathsf{s}) = (s_1, s_2, \ldots,) = \mathbf{s} \in \mathbb{R}^{\mathbb{N}}$ be a label such that $s_1 > s_2 > \cdots$, which is well defined for $\mu^{\ell}_{\mathsf{Ai},\beta}$-a.s..

Thm 1 (O.-Tanemura '14). [Existence of strong solutions]
Let $\beta = 1, 2, 4$. Define ISDE (6) of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ as

$$dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X^j_t|<r} \frac{1}{X^i_t - X^j_t}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt \qquad (6)$$

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x).$$

- *For $\mu^{\ell}_{\mathsf{Ai},\beta}$-a.s. $\mathbf{s}$, ISDE (6) has a strong solution with $\mathbf{X}_0 = \mathbf{s}$.*
- *The associated unlabeled dynamics $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X^i_t}$ is $\mu_{\mathsf{Ai},\beta}$-reversible.*
- *If $\beta = 2$ and $\mathbf{X}_0 \sim \mu^{\ell}_{\mathsf{Ai},2}$, then $X^1_t \sim F_2$. Here F_2 is the Tracy-Widom distribution and X^1_t is the Airy process $\mathcal{A}(t)$.*

Remarks:

- The key idea to derive the limit ISDE is to take the rescaled semi-circle law ς^N:

$$\varsigma^N(x) := N^{1/3}\varsigma(\frac{x}{N^{2/3}} + 2)$$
$$= \frac{1_{(-4N^{2/3},0)}}{\pi}\sqrt{-x(1+\frac{x}{4N^{2/3}})}$$

as the first approximation of the 1-correlation fun $\rho^{N,1}_{\mathsf{Ai},\beta}$.

- We expect that our method can be applied to other soft edge scaling.
- The SDE gives a kind of Girsanov formula.

Airy RPF – Dynamical soft edge scaling limit

Thm 2 (O.-Tanemura '14). [Pathwise uniqueness]

Let $\beta = 1, 2, 4$. *Then:*

- *Solutions of ISDE* (6) *of* $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ *starting at* s

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt \qquad (6)$$

satisfying abs cont cond (7) *are pathwise unique for* $\mu_{\mathrm{Ai},\beta}^{\ell}$*-a.s.* s.

$$\mu_{\mathrm{Ai},\beta,\mathrm{t}} \circ \mathsf{X}_t^{-1} \prec \mu_{\mathrm{Ai},\beta,\mathrm{t}} \quad \text{for } \mu_{\mathrm{Ai},\beta}\text{-a.s. t.} \qquad (7)$$

Here $\mu_{\mathrm{Ai},\beta,\mathrm{t}}$ *is a regular conditional probability w.r.t. to the tail* σ*-field* $\mathcal{T}$ *of the configuration space. Namely*

$$\mu_{\mathrm{Ai},\beta,\mathrm{t}} = \mu_{\mathrm{Ai},\beta}(\,\cdot\,|\mathcal{T})(\mathrm{t}), \qquad \mathcal{T} = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}]$$

where $\pi_A(\mathsf{s}) = \mathsf{s}(\cdot \cap A)$ *is a projection on configuration space, and* $S_r = \{|x| < r\}$. • • •

Airy RPF – Dynamical soft edge scaling limit

Thm 3 (O.-Tanemura '14). [Pathwise uniqueness]

Let $\beta = 1, 2, 4$. Then:

- *Solutions of ISDE* (6) *of* $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ *starting at* s

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt \qquad (6)$$

satisfying abs cont cond (7) *are pathwise unique for* $\mu_{\mathrm{Ai},\beta}^{\ell}$*-a.s.* s*.*

$$\mu_{\mathrm{Ai},\beta,\mathrm{t}} \circ \mathsf{X}_t^{-1} \prec \mu_{\mathrm{Ai},\beta,\mathrm{t}} \qquad \textit{for } \mu_{\mathrm{Ai},\beta}\textit{-a.s. } \mathrm{t}. \qquad ((7))$$

Here $\mu_{\mathrm{Ai},\beta,\mathrm{t}}$ *is a regular conditional probability w.r.t. to the tail* σ*-field* $\mathcal{T}$ *of the configuration space.*

- *If* $\beta = 2$*, then* $\mathcal{T}$ *is* $\mu_{\mathrm{Ai},\beta}$*-trivial. Hence the uniqueness holds.*
- *The solutions in Thm 1 satisfy* (7)*. Hence tail preserving solutions exist uniquely.*
- *Weak solutions satisfying* (7) *are automatically unique strong solutions.*

Airy RPF – Dynamical soft edge scaling limit : algebraic construction

- If $\beta = 2$, then Johansson, Spohn, Katori-Tanemura, Corwin-Hammond & others show that there exist stochastic dynamics Z_t associated with $\mu_{\mathsf{Ai},2}$ given by the space-time correlation function.
- The dynamics is originally specified by the finite-dimensional distributions give by space-time-correlation functions. The space-time-correlation functions are defined as determinant of kernel (extended Airy kernel). Hence we call this approach algebraic.

- Continuity of sample path (Johansson).
- Strong Markov property of unlabeled infinte system, and calculation of the associated Dirichlet form. (Katori-Tanemura)
- Path level approach based on "Brownian-Gibbs property" (Corwin-Hammond).

Airy RPF – Dynamical soft edge scaling limit : algebraic construction

- Space-time correlation functions are given by the extended Airy kernel:

$$K_{\mathrm{Ai}}(s,x;t,y) = \begin{cases} \int_0^\infty du e^{-u(t-s)/2} \mathrm{Ai}(u+x)\mathrm{Ai}(u+y), & t \geq s \\ -\int_{-\infty}^0 du e^{-u(t-s)/2} \mathrm{Ai}(u+x)\mathrm{Ai}(u+y), & t < s \end{cases}.$$

The unlabeled process $\mathsf{Z}_t = \sum_{i=1}^\infty \delta_{Z_t^i}$ is given by its moment generating function ($\mathbf{f} = (f_1, \dots, f_M)$, $\mathbf{t} = (t_1, \dots, t_M)$, $t_i < t_{i+1}$)

$$\Psi^{\mathbf{t}}[\mathbf{f}] = E[\exp\{\sum_{m=1}^{M} \int_{\mathbb{R}} f_m(x) \mathsf{Z}_{t_m}(dx)\}]$$

defined as a Fredholm determinant

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \mathrm{Det}_{(s,t)\in I^2,\,(x,y)\in\mathbb{R}^2}[\delta_{st}\delta(x-y) + K_{\mathrm{Ai}}(s,x;t,y)\chi_t(y)].$$

Here $I = \{t_1, \dots, t_M\}$ and $\chi_{t_m}(y) = e^{f_m(y)} - 1$,

Thm 3 [O.-Tanemura, '14]
Let $\beta = 2$. Then these two dynamics are the same.

- This comes from the uniqueness of Dirichlet forms associated with these dynamics. To prove the uniqueness of Dirichlet forms, we use the uniqueness of weak solutions of the ISDE (6) .
- The first approach (ISDE) provides qualitative information, say, semimartingale property of each tagged particle, Hölder continuity of sample paths, non-collision property of tagged paticles, and so on.
- The second construction gives quantative information.
- By construction, if the total system start from the Airy$_2$ RPF $\mu_{\mathsf{Ai},2}$, then the distribution of the top particle X_t^1 equals $F_{2,edge}(x)$, the 2 Tracy-Widom distribution.
- If we label the particles decreasing order as $X_t^i > X_t^{i+1}$, then the top particle X_t^1 is the Airy process $\mathcal{A}(t)$ studied by Spohn.

Airy RPF – Dynamical soft edge scaling limit

Let $\mathbf{X}_t^N = (X_t^{N,i})_{i=1}^N$ be the N-particle system as before:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2} \{N^{1/3} + \frac{1}{2N^{1/3}} X_t^{N,i}\} dt$$

Set $\mathbf{X}^{N,m}$ be the first m-component.

$$\mathbf{X}^{N,m} = (X_t^{N,1}, \dots, X_t^{N,m})$$

Thm 4 [O.-Tanemura, O.-Kawamoto] (Finite-particle approximation)
Let $\beta = 1,2,4$. Then for each $0 \le \varphi \in L^1(\mu_{\mathrm{Ai},\beta}^{\ell})$ with $\int \varphi \mu_{\mathrm{Ai},\beta}^{\ell} = 1$, $\mathbf{X}^{N,m}$ with $\mathbf{X}_0^{N,m} \sim \varphi \mu_{\mathrm{Ai},\beta}^{\ell}$ converge to the first m-component $\mathbf{X}^m$ of the solution of the limit ISDE weakly in $C([0,\infty); \mathbb{R}^m)$.

- When $\beta = 2$, we have two proofs.

Bulk scaling limit & an SDE gap

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Bulk scaling limit & an SDE gap

- Bulk scaling:
  For $-2 < \theta < 2$ take $x_i = (s_i - \theta)/\sqrt{N}$ in (3):

$$\mu^N_{\text{sin},\beta,\theta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N}|s_i - s_j|^\beta \prod_{k=1}^{N} e^{-\beta|s_k-\theta|^2/4N} d\mathbf{s}_N \tag{8}$$

As $N \to \infty$, $\mu^N_{\text{sin},\beta,\theta}$ converge to the sine$_\beta$ RPF such that

$$\lim_{N\to\infty} \mu^N_{\text{Sine},\beta,\theta} = \mu_{\text{Sine},\beta,\theta}.$$

The right-hand side is independent of θ up to constant scaling.
If $\beta = 2$, then $\mu_{\text{Sine},2,\theta}$ is determinantal with kernel

$$\mathsf{K}(x,y) = \sqrt{1 - (\frac{\theta}{2})^2}\,\frac{\sin(x-y)}{\pi(x-y)}$$

- We next consider the dynamical counter part of this scaling limit.

Bulk scaling (dynamical)

$$\mu^N_{\mathsf{sin},\beta,\theta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N}|s_i - s_j|^\beta \prod_{k=1}^{N} e^{-\beta|s_k-\theta|^2/4N} d\mathbf{s}_N \tag{8}$$

- The associated N particle system is given by the SDE:

$$dX^i_t = dB^i_t + \frac{\beta}{2}\sum_{j\neq i}^{N}\frac{1}{X^i_t - X^j_t}dt - \frac{\beta}{4N}X^i_t dt + \frac{\beta\theta}{4}dt \tag{9}$$

- Very loosely, the associated ∞ particle system is given by

$$dX^i_t = dB^i_t + \frac{\beta}{2}\sum_{j\neq i}^{\infty}\frac{1}{X^i_t - X^j_t}dt + \frac{\beta\theta}{4}dt \quad (i \in \mathbb{N}).$$

This is not the case for $\theta \neq 0$.

Sine RPF – Limit ISDE

For a configuration $\mathsf{s} = \sum_i \delta_{s_i}$, let $\ell(\mathsf{s}) = (s_1, s_2, \dots,) = \mathbf{s} \in \mathbb{R}^{\mathbb{N}}$ be a label which is defined for $\mu^{\ell}_{\mathsf{Sine},\beta}$-a.s..

Limit ISDE:

Thm 5 [O.-Tanemura '14, O.-Kawamoto '14] [Existence of strong solutions]
Let $\beta = 1, 2, 4$. Define ISDE (6) of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ as

$$dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r\to\infty} \{ \sum_{j\neq i,\ |X^j_t|<r} \frac{1}{X^i_t - X^j_t} \} dt \qquad (10)$$

- For $\mu^{\ell}_{\mathsf{Sine},\beta}$-a.s. $\mathbf{s}$, ISDE (10) has a strong solution with $\mathbf{X}_0 = \mathbf{s}$.
- The associated unlabeled dynamics $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X^i_t}$ is $\mu_{\mathsf{Sine},\beta}$-reversible.

Sine RPF - Dynamical bulk scaling limit

Thm 6 [O.-Tanemura '14] [Pathwise uniqueness]

Let $\beta = 1, 2, 4$. Then:

- Solutions of ISDE (10) of $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ starting at s

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \to \infty} \{ \sum_{j \neq i,\ |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \} dt \tag{10}$$

satisfying abs cont cond (11) are pathwise unique for $\mu_{\mathsf{Sine},\beta}^{\ell}$-a.s. s.

$$\mu_{\mathsf{Sine},\beta,\mathsf{t}} \circ \mathsf{X}_t^{-1} \prec \mu_{\mathsf{Sine},\beta,\mathsf{t}} \quad \text{for } \mu_{\mathsf{Sine},\beta}\text{-a.s. t.} \tag{11}$$

- If $\beta = 2$, then $\mathcal{T}$ is $\mu_{\mathsf{Sine},\beta}$-trivial. Hence the uniqueness holds.
- The solutions in Thm satisfy (11). Hence tail preserving solutions exist uniquely.
- Weak solutions satisfying (11) are automatically unique strong solutions.
- If $\beta = 2$, the solution equal to the stochastic dynamics given by space-time correlation functions (extended Sine kernels).

Sine RPF - Dynamical bulk scaling limit

Let $\mathbf{X}_t^N = (X_t^{N,i})_{i=1}^N$ be the N-particle system as before:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}}dt - \frac{\beta}{4N}X_t^{N,i}dt + \frac{\beta\theta}{4}dt \qquad (9)$$

Thm 7 [O.-Tanemura, O.-Kawamoto] (Finite-particle approxim)
Let $\beta = 1, 2, 4$. Then for each $0 \le \varphi \in L^1(\mu^{\ell}_{\mathrm{Ai},\beta})$ with $\int \varphi\mu^{\ell}_{\mathrm{Ai},\beta} = 1$, $\mathbf{X}^{N,m}$ with $\mathbf{X}_0^{N,m} \sim \varphi\mu^{\ell}_{\mathrm{Ai},\beta}$ converge to the first m-component $\mathbf{X}^m$ of the solution of the limit ISDE

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\{\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}\}dt \qquad (10)$$

weakly in $C([0,\infty);\mathbb{R}^m)$.

- The limit ISDE (10) is independent of θ.
- In this sense, an SDE gap occurs.

Bessel RPF & a hard edge scaling

- Bessel RPFs $\mu_{\text{bes},2}^{\alpha}$ $(-1 < \alpha < \infty)$ are probability measures on the configuration space S over $S = [0, \infty)$, whose n-point correlation functions ρ^n with respect to the Lebesgue measure are given by

$$\rho^n(x_1, \dots, x_n) = \det[\mathsf{K}(x_i, x_j)]_{1 \le i,j \le n}. \tag{12}$$

Here $\mathsf{K}(x, y)$ is called the Bessel kernel defined with the Bessel function J_α of order α such that for $x \neq y$

$$\mathsf{K}(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})\sqrt{y}J_\alpha(\sqrt{y})}{2(x - y)}. \tag{13}$$

We note that $0 \le \mathsf{K} \le \mathrm{Id}$ as an operator on $L^2(S, dx)$.

- By definition $\mu_{\text{bes},2}^{\alpha}$ are determinantal random point fields with Bessel kernels K

Bessel RPF: hard edge scaling

Thm 8 [O.-Honda, '14] Let $\alpha > 1$ and $\beta = 2$. Let $\mu^{\alpha}_{\text{bes},2}$ be the Bessel$^{\alpha}_{2}$ RPF. Then the associated ISDE is given by

$$dX_t^i = dB_t^i + \frac{\alpha}{2X_t^i}dt + \frac{\beta}{2}\sum_{\substack{|X_t^j|<r \\ j\neq i}} \frac{1}{X_t^i - X_t^j}dt.$$

These ISDEs have unique, strong solutions (as in the same meaning of the previous theorems).

Bessel RPF: hard edge scaling

- These random point fields arise as a scaling limit at the hard left edge of the distributions $\mu_{\text{bes},2}^{\alpha,N}$ of the spectrum of the Laguerre ensemble.
- The random point fields $\mu_{\text{bes},2}^{\alpha}$ represent the thermodynamic limit of the N-particle systems $\mu_{\text{bes},2}^{\alpha,N}$, whose labeled densities $\sigma_{\alpha}^{N}(\mathbf{x})d\mathbf{x}$ are given by

$$\sigma_{\alpha}^{N}(\mathbf{x}) = \frac{1}{\mathcal{Z}_{\alpha}^{N}} e^{-\sum_{i=1}^{N} x_i/4N} \prod_{j=1}^{N} x_j^{\alpha} \prod_{k<l}^{N} |x_k - x_l|^2. \tag{14}$$

Thm 9 [O. Kawamoto] The associated N-particle system $\mathbf{X}_t^N = (X_t^{N,1}, \ldots, X_t^{N,N})$ converge to the limit $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}}$ in the same sense as before.

- $\beta = 1, 4$ is in progress.

Universality in one dimension

In one dimension, Sine_β, Airy_β, and Bessel_β may be regarded to have universality because they often appear in bulk, soft edge, and hard edge scaling limit, respectively. If this is the case, we expext that so is the ISDEs we discussed:
The following ISDE is universal.

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ \sum_{j\neq i,\ |X_t^i - X_t^j|<r} \frac{1}{X_t^i - X_t^j} \} dt \qquad \text{(bulk)}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{(\max\{-x, 0\})^\alpha}{-x} dx \} dt \qquad \text{(soft edge)}$$

$$dX_t^i = dB_t^i + \frac{\alpha}{2X_t^i} dt + \frac{\beta}{2} \sum_{\substack{|X_t^j|<r \\ j\neq i}} \frac{1}{X_t^i - X_t^j} dt. \qquad \text{(hard edge)}$$

Ginibre RPF : Non-hermitian Gaussian random matrixes

- Ginibre RPF μ_{gin} is a determinantal RPF on $\mathbb{C}$ ($\mathbb{R}^2$) with (K, g).
- $\mathrm{g}(z) = (1/\pi)e^{-|z|^2}$ is a Gauss measure on $\mathbb{C}$.
- K is an exponential kernel

$$\mathsf{K}(x,y) = e^{x\bar{y}}.$$

- n-correlation function ρ^n of Gnibre RPF w.r.t. $\mathrm{g}^n dx^n$ is defined as

$$\rho^n(x_1,\dots,x_n) = \det[\mathsf{K}(x_i,x_j)]_{i,j=1}^n$$

- The N-particle system is given by

$$\mu^N(d\mathbf{x}_N) = \frac{1}{\mathcal{Z}}\prod_{i<j}^N |x_i - x_j|^2 \mathrm{g}^n(\mathbf{x}_N)d\mathbf{x}_N$$

- μ^N is a determinantal RPF with (K^N, dx) such that

$$\mathsf{K}^N(x,y) = \sum_{m=0}^{N-1}\frac{(x\bar{y})^m}{m!}$$

Ginibre RPF

Thm 11 [O.,'13, O.-Tanemura '14]
Let μ_{gin} be a Ginibre RPF. Then the associated ISDE is given by the following, and has a unique strong solution as in the same meaning of the previous theorems.

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{\substack{|X_t^i - X_t^j| < r \\ j \neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N}).$$

The solution also satisfy the following ISDEs for all $a \in \mathbb{C}$:

$$dX_t^i = dB_t^i - (X_t^i - a)dt + \lim_{r\to\infty} \sum_{\substack{|a - X_t^j| < r \\ j \neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N})$$

The associated N-particle system is given by

$$dX_t^{N,i} = dB_t^i - X_t^{N,i}dt + \sum_{j\neq i}^{N} \frac{X_t^{N,i} - X_t^{N,j}}{|X_t^{N,i} - X_t^{N,j}|^2}dt.$$

Thm 12 [O.-Kawamoto] The N-particle system

$$\mathbf{X}^N = (X_t^{N,1}, \dots, X_t^{N,N})$$

converge to the limit $\mathbf{X}$ in the same sense as before.

Simulation!!

A histrical background of IBMs

- Interacting Brownian motions in infinite-dimensions $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ are stochastic dynamics in $(\mathbb{R}^d)^{\mathbb{N}}$ given by ISDE

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j\in\mathbb{N},\, j\neq i} \nabla\Psi(X_t^i - X_t^j)dt \quad (i \in \mathbb{N})$$

Here Ψ is an interaction potential and β is inverse temperature.
This ISDE has been studied by Lang ('79), Fritz ('87), Tanemura ('96), and others.
They construct strong solutions.

- So far Ψ is taken to be $C_0^3(\mathbb{R}^d)$ or exponential decay at infinity.
- Itô scheme (Picard approximation) is used here.

Known results.

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j \in \mathbb{N},\, j \neq i} \nabla \Psi(X_t^i - X_t^j) dt \quad (i \in \mathbb{N})$$

- There many interesting potentials Ψ with polynomial decay or unbounded at infinity:
- These are excluded by the classical approach based on Itô scheme.
- In this talk, we present a new scheme applicable to polynomial decay or logatithmic potentials:

$$\Psi(x) = -\log |x|.$$

This appears in random matrix theory and vortex dynamics. If $d = 1$, $\beta = 2$, and Ψ is as above, then the ISDE is

$$dX_t^i = dB_t^i + \sum_{j \neq i} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{N}).$$

Known results.

- Itô scheme uses Lipschitz continuity of coefficients, which does not hold in infinite dimensions.
- We localize ISDE with increasing sets H_k and exit times τ_{H_k} such that coefficients are Lipschits continuous on each H_k and that

$$\lim_{k\to\infty} \tau_{H_k} = \infty.$$

- Since ISDEs like as

$$dX_t^i = dB_t^i + \sum_{j\neq i} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{N}).$$

are complicated, it is hard to find out such a sequence of subsets $\{H_k\}$. We give an algorithm to find out such sets by Dirichlet form theory and tail analysis. (In our theoem, exit times do not appear).

Out line of the proof.

Our approach consists of 6 steps:
By the first three steps we construct weak solutions.
By the next three steps we lift them to strong solutions and prove the pathwise uniqueness of ISDEs.

Idea to solve ISDE: $\mathsf{S} \Rightarrow C([0,\infty);\mathsf{S}) \Rightarrow C([0,\infty);S^{\mathbb{N}})$

(Step 1) • We start with a random point field μ (a probability measure on configuration space S).

• We construct μ-reversible unlabeled diffusions X by Dirichlet forms.

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}.$$

For this we introduce the map from RPF μ on S to bilinear forms :

$$\mu \mapsto \mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g]d\mu \quad \text{ on } L^2(\mathsf{S},\mu).$$

Here $\mathbb{D}$ is the standard square field on S:

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_{i=1}^{\infty} \frac{\partial \tilde{f}}{\partial s_i} \cdot \frac{\partial \tilde{g}}{\partial s_i}$$

Here f is a local and smooth function on S, and $\tilde{f}(s_1,\ldots,)$ is a symmetric function such that $f(\mathsf{s}) = \tilde{f}(s_1,\ldots,)$, where $\mathsf{s} = \sum_{i=1}^{\infty} \delta_{s_i}$.

• If μ is the Poisson RPF $= \Lambda$ with Lebesgue intensity, then the associated diffusion X_t is S-valued Brownian motion $\mathsf{B}_t = \sum_{i=1}^{\infty} \delta_{B_t^i}$, which is a reason we call $\mathbb{D}$ the standard square field.
Thus this Dirichlet space is a distorted Brownian motion on S although μ does not have a density with respect to Λ usually.

• We assume:

μ is a Ψ-quasi-Gibbs measure.

Roughly speaking, quasi-Gibbs means that μ has a local density conditioned out side. Gibbs measures are of course quasi-Gibbs, and there exist RPF that are quasi-Gibbs for logarithmic potential Ψ.

• Assume that μ is Ψ-quasi-Gibbs with upper semicontinuous Ψ, and that $\sum_{m=1}^{\infty} m\mu(\mathsf{S}_r^m) < \infty$ ($\mathsf{S}_r^m = \{\mathsf{s}; \mathsf{s}(S_r) = m\}$), and that m-density functions on S_r are in $L^2(S_r^m)$ for all $r, m \in \mathbb{N}$. Here $S_r = \{|s| < r\}$.

• With these assumption, the bilinear form is closable and its closure is a quasi-regular Dirichlet form.

• We thus have unlabeled diffusions.

Step 2

$$\mathsf{S} \Rightarrow C([0,\infty);\mathsf{S}) \Rightarrow C([0,\infty);S^{\mathbb{N}})$$

(Step 2) • Assuming non-collision and non-explosion of tagged particles, we can construct labeled dynamics.
Indeed, particles keep their initial label forever.
Hence we have the correspondence:

$$\mathbf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \;\Rightarrow\; \mathbf{X} = (X_t^1, X_t^2, \ldots).$$

• The difficulty to construct $S^{\mathbb{N}}$-valued diffusion $\mathbf{X}$, there is no good measure on $S^{\mathbb{N}}$. (Hence no associated Dirichlet forms).
Even if Brownian motions, the measure should be $dx^{\mathbb{N}}$!
Hence we introduce a countable sequence of spaces

$$S^k \times \mathsf{S} \quad (k \in \mathbb{N})$$

Step 2

$$\mathsf{S} \Rightarrow C([0,\infty);\mathsf{S}) \Rightarrow C([0,\infty);S^{\mathbb{N}})$$

$$S^k \times \mathsf{S} \quad (k \in \mathbb{N}) \quad \Leftrightarrow \quad S^{\mathbb{N}}$$

- Hence we consider M-Campbell measure $\mu^{[M]}$ of μ.

Introduce the countable family of Dirichlet forms:

$$(\mathcal{E}^{\mu^{[M]}}, L^2(S^M \times \mathsf{S}, \mu^{[M]})), \quad \mathbf{X}^{[M]} := (X^{M,1}, \ldots, X^{M,M}, \sum_{i=M+1}^{\infty} \delta_{X^{M,i}})$$

There is natural coupling associated diffusions. $\Rightarrow$
$X^{M,i}$ are independent of M. $\Rightarrow$
From this consistency we can construct the labeled diffusion on $S^{\mathbb{N}}$.

- We use unlabeled diffusion X_t to couple with these $\mathbf{X}^{[M]}$.

(Step 3) Calculate the logarithmic derivative d^{μ}. ISDE becomes

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i} \delta_{X_t^j})dt$$

In the case of Ginibre, Sine$_\beta$(Dyson), Bessel, and Gibbs measures:

$$\beta \nabla \Phi(x) + \beta \lim_{r\to\infty} \sum_{j\neq i,\, |x-s_j|<r} \nabla \Psi(x - s_j)$$

Then we have the ISDE (weak solution):

$$dX_t^i = dB_t^i - \frac{\beta}{2}\nabla \Phi(X_t^i) - \frac{\beta}{2} \lim_{r\to\infty} \sum_{j\neq i,\, |X_t^i - X_t^j|<r} \nabla \Psi(X_t^i - X_t^j)dt$$

To calculate the logarithmic derivative we use finite particle approximation. In particular, orthogonal polynomials.
The shape of Airy RPF is different.

(Step 4) Introduce:

The infinite system of finite-dimensional SDEs with consistency (IFC):

Let $(\mathbf{X}, \mathbf{B})$ be a weak solution.

We regard $\mathbf{X}$ as a part of coefficients of SDEs.

For each M consider SDE of $\mathbf{Y}^M = (Y^{M,1}, \dots, Y^{M,M})$:

$$dY_t^{M,i} = dB_t^i - \frac{\beta}{2}\nabla\Phi(Y_t^{M,i}) - \frac{\beta}{2}\sum_{j=1, j\neq i}^{M} \nabla\Psi(Y_t^{M,i} - Y_t^{M,j})dt - \frac{\beta}{2}\sum_{j=M+1}^{\infty} \nabla\Psi(Y_t^{M,i} - X_t^j)dt.$$

These (time inhomogeneous, finite-dimensional) SDEs have unique strong solution (under suitable assumptions). Hence

$$\mathbf{Y}^M = \mathbf{X}^M := (X^1, \dots, X^M)$$

- We solve infinite-many finite-dimensional SDEs with consistency in stead of solving a single ISDE.

(Step 5)

- Let $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ be the tail σ-field of labeled path space w.r.t. label.

$$\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}}) = \bigcap_{M=1}^{\infty} \sigma[X^M, \dots,].$$

- $\mathbf{Y}^M$ is a functional of $(\mathbf{B}, (X^{M+1}, \dots,))$.

$\Rightarrow$ If $\lim_{M\to\infty} \mathbf{Y}^M$ exists, then $\sigma[\mathbf{B}] \vee \mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$-measurable.

$\Rightarrow$ Since $\lim_{M\to\infty} \mathbf{Y}^M = \mathbf{X}$, $\mathbf{X}$ is $\sigma[\mathbf{B}] \vee \mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$-measurable.

$\Rightarrow$ If $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ is trivial, then $\mathbf{X}$ is a strong solution.

- Since we see in the (Step 5) that

$$\mathbf{Y}^M = \mathbf{X}^M := (X^1, \dots, X^M),$$

$\mathbf{Y}^M$ satisy these.

(Step 6) • We say unlabeled diffusion satisfies the absolutely continuity condition (ACC) if

$$P_\mu(\mathsf{X}_t \in \cdot) \prec \mu \quad \text{for all } t.$$

• If ACC is satisfied and if μ is tail trivial, then $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is trivial.

• Tail triviality of RPF $\Rightarrow$ tail tiriviality of labeled path space.

• $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is the tail σ-field of the labeled path space w.r.t. the label.

• We regard $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ as a *boundary condition of ISDE*.
So if its trivial and unique, then the solution of ISDE is unique.

• Our pathwise uniqueness does not exclude the posiibility of the existence of a *tail moving* or *shock* solution. It is related to the uniqueness of Dirichlet forms (domain choice).

• We have not yet solved the non-equilibrium problem. We have not yet fully utilize the property of this method, and expect that with this we can solve the non-equilibrium problem at the lebel of Fritz (1987).

Tail triviality of μ is not a real restriction. Indeed,

Prop 1. *Determinantal RPFs (in continuous spaces) are tail trivial. In particular, Ginibre RPF is tail trivial.*

This result is a generalization of Shirai-Talahashi, and Russel Lyons for discrete spaces.

Note that RPFs appearing in random matrix theory are determinantal random point fields if $\beta = 2$. So our results provide the uniqueness for these.

Even if μ is not tail trivial, we can still apply our results to quasi-Gibbs measures because of the following result.

Prop 2. *Quasi-Gibbs measures μ have decomposition w.r.t. their tail σ-fields $\mathcal{T}(\mathsf{S})$ such that each components are tail trivial: For μ-a.s.* s

$$\mu(\mathsf{A}|\mathcal{T}(\mathsf{S}))(\mathsf{s}) = 1_{\mathsf{A}}(\mathsf{s}) \qquad \textit{for all } \mathsf{A} \in \mathcal{T}(\mathsf{S}).$$

This is an analogy of the result of Georgii on Gibbs measures on discrete spaces.

END

Weak solutions

2014/9/2/Tue Warwick

UK-Japan Stochastic Analysis School (JSPS Core-to-Core programme)

Workshop dates: 2014-09-01 – 2014-09-05

Outline:

- quasi-Gibbs measures and unlabeled diffusion
- logarithmic derivative and SDE representation
- A sufficient condition for quasi-Gibbs property
- Calculation of logarithmic derivatives
- Examples: Ginibre and Airy RPFs

- We solve ISDEs of the form

$$dX_t^i = dB_t^i + b(X_t^i, \mathbf{X}_t^{\diamondsuit i})dt \quad (i \in \mathbb{N}) \tag{1}$$

Here $\mathbf{X}_t = (X_t^1, \ldots,) \in (\mathbb{R}^2)^{\mathbb{N}}$-valued, and

$$\mathbf{X}_t^{\diamondsuit i} = (X_t^j)_{j \in \mathbb{N} \setminus \{i\}}.$$

The coefficient $b(x, \mathbf{y})$ is symmetric in $\mathbf{y} = (y_i)_{i \in \mathbb{N}}$ for each $x \in \mathbb{R}^2$.
$\mathbf{B}_t = (B_t^1, \ldots,)$ is $(\mathbb{R}^2)^{\mathbb{N}}$-valued standard Brownian motion.
We will construct weak solution $(\mathbf{X}, \mathbf{B})$.
Our method can be applied to the case with $\sigma(X_t^i, \mathbf{X}_t^{\diamondsuit i})dB_t^i$.
For simplicity we talk about (1) only.

- Because of the symmetry of $b(x, \mathbf{y})$ in $\mathbf{y}$, we can rewrite

$$dX_t^i = dB_t^i + b(X_t^i, \mathsf{X}_t^{\diamondsuit i})dt \quad (i \in \mathbb{N}) \tag{2}$$

Here we regard $b(x, \cdot)$ as a function on the configuration space, and

$$\mathsf{X}_t^{\diamondsuit i} = \sum_{j \neq i} \delta_{X_t^j}$$

- We recall the examples: $(i \in \mathbb{N})$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{1}{X_t^i - X_t^j} dt \qquad \text{(Sine)}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt \qquad \text{(Airy)}$$

$$dX_t^i = dB_t^i + \frac{a}{2X_t^i} dt + \frac{\beta}{2} \sum_{j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j} dt \qquad \text{(Bessel)}$$

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{\substack{|X_t^i - X_t^j|<r \\ j\neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \qquad \text{(Ginibre)}$$

- Gibbsian examples for suitable α and d: $(i \in \mathbb{N})$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^{\infty} \{ \frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}} \} dt \qquad \text{(LJ 6-12)}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j \neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{\alpha}} dt. \qquad \text{(Riesz)}$$

Cofiguration spaces

Set up:

- $S=\mathbb{R}^d$: Space, where particles move,
- $S_r=\{|x|\le r\}$,
- $\mathsf{S}=\{\mathsf{s}=\sum_i \delta_{s_i},\ \mathsf{s}(S_r)<\infty(\forall r)\}$:

  Configuration space over S.

  Polish space with vague topology.

  The space of unlabeled particles.
- $S^{\mathbb{N}}$ is the space of labeled particles.
- $\mathsf{s}=\sum_i \delta_{s_i}$ denotes unlabeled particles.

  $\mathbf{s}=(s_i)\in S^{\mathbb{N}}$ denotes labeled particles.
- Since $S^{\mathbb{N}}$ is too large, we use S instead.

- $\mathsf{B}_t=\sum_{i=1}^{\infty}\delta_{B_t^i}$ is S-valued Brownian motion.
- $\mathbf{B}_t=(B_t^i)_{i\in\mathbb{N}}$ is $S^{\mathbb{N}}$-valued Brownian motion.

Canonical square field

For a fun f on S let $f(\mathsf{s}) =: \tilde{f}(s_1, \ldots)$, where $\tilde{f}$ is symmetric, $\mathsf{s} = \sum \delta_{s_i}$.
Let $\mathcal{D}_0$ be the set of bounded, local, smooth functions f on S.
i.e. f is $\sigma[\pi_r]$-measurable for some $r < \infty$, $\tilde{f}$ is smooth.
Let $\mathbb{D}$ be the canonical square field on S:

$$\mathbb{D}[f, g](\mathsf{s}) = \frac{1}{2} \sum_i \nabla_i \tilde{f} \cdot \nabla_i \tilde{g}.$$

Here $\nabla_i = (\frac{\partial}{\partial s_{i1}}, \ldots, \frac{\partial}{\partial s_{id}})$.

The rhs is independent of particular choice of label.

- For a RPF μ we set

$$\mathcal{E}^\mu(f, g) = \int_{\mathsf{S}} \mathbb{D}[f, g] \mu(d\mathsf{s}),$$

$$\mathcal{D}_0^\mu = \{f \in \mathcal{D}_0; \mathcal{E}^\mu(f, f) < \infty, f \in L^2(\mu)\}$$

- If we take $\mu = \Lambda$, Poisson RPF with Lebesgue intensiy, then the bilinear form associates Brownian motion $\mathsf{B}_t = \sum_i \delta_{B_t^i}$.

In this sense $\mathbb{D}$ is the canonical square field.

From RPF to unlabeled diffusion

Outline of the proof:

$$\mu \Rightarrow (\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}, L^2(\mu)) \Rightarrow \mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i\in\mathbb{N}} \Rightarrow \text{ISDE}$$

- The first arrow is automatic. For a given RPF μ, we can associated a positive bilinear form through the square field $\mathbb{D}$.

- If $(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}, L^2(\mu))$ is closable and its closue is quasi-regular, then by Dirichlet form theory an associated μ-reversible diffusion X_t exists.
- For this we introduce a notion of quasi-Gibbs measure.

If μ is quasi-Gibbs with upper semi-continuous potential Ψ, then the bilinear form id closable. In addition, μ satisies a marginal condition (local boundedness of correlation functions, say), then the form becomes quasi-regular. Hence by the general theory of Dirichlet form there exists the associated unlabeled diffusion X_t.

Quasi-Gibbs measures:

- $\pi_r, \pi_r^c : \mathsf{S} \to \mathsf{S}$: projections

$$\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r), \qquad \pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$$

- For a RPF μ we set

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(S_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi))$$

- Let $\Psi : S \to \mathbb{R} \cup \{\infty\}$ (interaction).

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i - s_j)$$

- • •

Ψ-Quasi-Gibbs meas.

$$\mu_{r,\xi}^{m}(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(S_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi))$$

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i - s_j)$$

Def: μ is Ψ-quasi-Gibbs measure if $\exists$ $c_{r,\xi}^{m}$ s.t.

$$c_{r,\xi}^{m}{}^{-1} e^{-\mathcal{H}_r} d\Lambda_r^m \le \mu_{r,\xi}^{m} \le c_{r,\xi}^{m} e^{-\mathcal{H}_r} d\Lambda_r^m$$

Here $\Lambda_r^m = \Lambda(\cdot | \mathsf{s}(S_r) = m)$ and Λ_r is the Poisson RPF with $1_{S_r} dx$.

- The above definition is a simplified version.
- Gibbs measures $\Rightarrow$ Quasi-Gibbs measures: If

$$\mu_{r,\xi}^{m} = c_r^m e^{-\mathcal{H}_r - \sum_{x_i \in S_r, \xi_j \in S_r^c} \Psi(x_i, \xi_j)} d\Lambda_r^m, \qquad \text{(QG)}$$

then μ is a canonical Gibbs measure. (QG) does not make sense for

$$\Psi(x, y) = -\log |x - y|$$

$$\mu \Rightarrow (\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}, L^2(\mu)) \Rightarrow \mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i\in\mathbb{N}} \Rightarrow \text{ISDE}$$

Unlabeled diffusions

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ. $\Rightarrow$ (closability)

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$ $\Rightarrow$ (existence of diffusions)
Here $\mathsf{S}_r^k = \{\mathsf{s}(S_r) = k\}$, σ_r^k is k-density fun on S_r^k.

Thm 1 (O.'96 (CMP)). (1) (A1) $\Rightarrow$ $(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$ *is closable on* $L^2(\mu)$.
(2) (A1), (A2) $\Rightarrow$ $\exists$ *diffusion* $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ *associated with the closure*

$$(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}) \textit{ of } (\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}) \textit{ on } L^2(\mu).$$

Proof. Outline of (1): Let

$$\mathcal{E}^{\mu_{r,\xi}^m}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g] d\mu_{r,\xi}^m.$$

Then $(\mathcal{E}^{\mu_{r,\xi}^m}, \mathcal{D}_0^{\mu_{r,\xi}^m})$ is closable on $L^2(\mu_{r,\xi}^m)$ by (A1).
Hence $(\widehat{\mathcal{E}}_r^{\mu}, \mathcal{D}_0^{\mu})$ are closable on $L^2(\mu)$. Here

$$\widehat{\mathcal{E}}_r^{\mu}(f,g) = \int_{\mathsf{S}} \frac{1}{2} \sum_{s_i \in S_r} \frac{\partial \check{f}}{\partial s_i} \cdot \frac{\partial \check{g}}{\partial s_i} d\mu \quad \text{(reflecting BC)}.$$

By the monotone convergence theorem of closable forms we see

$$\widehat{\mathcal{E}}^{\mu}(f,f) = \lim_{r\to\infty} \widehat{\mathcal{E}}_r^{\mu}(f,f), \quad \widehat{\mathcal{D}}_0 = \{f; \lim_{r\to\infty} \widehat{\mathcal{E}}_r^{\mu}(f,f) < \infty\}$$

is closable. Hence $(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$ is closable.
(2) follows from a concrete construction of cut off function. □

Remark 1. In general, the closures of the limit Dirichlet forms

$$(\widehat{\mathcal{E}}^{\mu}, \widehat{\mathcal{D}}) \quad \text{and} \quad (\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$$

are not equal. We will prove the coincidence of these by using the strong uniqueness of the solutions of the associated ISDEs.
Lang's dynamics ('79) are given by the Dirichlet form $(\widehat{\mathcal{E}}^{\mu}, \widehat{\mathcal{D}})$.
O's ('96) dynamics are given by $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$.

Let $\Psi_2(x,y) = -\log|x-y|$ be the 2-dim Coulomb potential.

Thm 2 (O. AOP '13, O.-Honda (14), O.-Tanemura (14)).
(1) *Ginibre RPF is a* $2\Psi_2$*-quasi Gibbs measure.*
(2) Sine_β *RPF are* $\beta\Psi_2$*-quasi Gibbs m for* $\beta = 1, 2, 4$.
(3) Bessel_2^a *RPF is a* $2\Psi_2$*-quasi Gibbs m.*
(4) Airy_β *RPF are* $\beta\Psi_2$*-quasi Gibbs m for* $\beta = 1, 2, 4$.

$$\mu \Rightarrow (\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}, L^2(\mu)) \Rightarrow \mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i\in\mathbb{N}} \Rightarrow \text{ISDE}$$

Labeled dynamics

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ.
(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$
(A3) $\{X_t^i\}$ do not collide each other (non-collision)
(A4) each tagged particle X_t^i never explode (non-explosion)

By (A3) and (A4) the labeled dynamics

$$\mathbf{X}_t = (X_t^1, X_t^2, \ldots)$$

can be constructed from the unlabeled dynamics

$$\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}.$$

Indeed, the particles keep the initial label forever.

Sufficient condition of (A3) & (A4)

Let $\mathsf{S}_{s,i} = \mathsf{S}_s \cap \mathsf{S}_i$:

$$\mathsf{S}_s = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(\{x\}) = 0 \text{ for all } x \in S\}, \quad \mathsf{S}_i = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S) = \infty\}.$$

- (A3) is equaivalent to

$$\mathrm{Cap}^{\mu}(\mathsf{S}_{s,i}^c) = 0. \tag{3}$$

Let ρ^n be a n-correlation function of μ.

Lem 1. *Suppose μ is quasi-Gibbs with Ψ. Let ρ^2 be 2-correlation function of μ. Suppose one of the following holds. Then* (A3) *holds.*
(1) $d \ge 2$ and ρ^2 are locally bounded.
(2) $d = 1$ and

$$\rho^2(x, y) \le Ch(|x - y|) \textit{ locally near } \{x = y\}.$$

Here $h(t)$ such that

$$\int_{0+}^{1} \frac{1}{h(t)} dt = \infty.$$

Corollary 1. *Sine$_\beta$, Airy$_\beta$, Bessel$_\beta$ ($\beta \ge 1$), Ginibre RPFs satsfy* (A2).

- By (A3) we represent one-labeled process $(X_t^1, \sum_{j=2}^{\infty} \delta_{X_t^j})$ by the Dirichlet space

$$(\mathcal{E}^{\mu^{[1]}}, \mathcal{D}^{\mu^{[1]}}, L^2(\mu^{[1]})).$$

Applying Takeda criteria based on Lyons-Zheng decomposition we deduce (A4) from $\exists T > 0$

$$\liminf_{r\to\infty} \{ \int_{|x|\le r+R} \rho^1(x)dx \} \{ \int_{\frac{r}{\sqrt{(r+R)T}}} \mathrm{g}(u)du \} = 0 \quad \text{for all } T. \tag{4}$$

Lem 2. (A4) *follows from* (4).

$$\mu \Rightarrow (\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu}, L^2(\mu)) \Rightarrow \mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i \in \mathbb{N}} \Rightarrow \text{ISDE}$$

ISDE representation

Log derivative of μ: precise correspondence between RPFs & potentials

- Let μ_x be the (reduced) Palm m. of μ conditioned at x

$$\mu_x(\cdot) = \mu(\cdot - \delta_x | \mathsf{s}(x) \geq 1)$$

- Let μ^1 be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^1(A \times B) = \int_A \rho^1(x)\mu_x(B)dx$$

- $\mathsf{d}^\mu \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathsf{S}, \mu^1)$ is called the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^1 = -\int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}^\mu d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_0$$

  Here ∇_x is the nabla on $\mathbb{R}^d$.

- Very informally

$$\mathsf{d}^\mu = \nabla_x \log \mu^1$$

- A caluculation of log derivative of Gibbs measures are trivial.
  Indeed, it is immediate from DLR equation.
- This is not the case for RPFs appearing in RMT.
  We will give a sufficient condition later.

Log derivative

A very informal calculation shows:

- If $\mu^1(dxd\mathsf{s}) = m(x, s_1, \ldots)dx \prod_i ds_i$, then

$$
\begin{aligned}
& -\int \nabla_x f(x, s_1, \ldots)\mu^1(dx ds_1 \cdots) \\
&= -\int \nabla_x f(x, s_1, \ldots) m(x, s_1, \ldots) dx \prod_i ds_i \\
&= \int f(x, s_1, \ldots) \nabla_x m(x, s_1, \ldots) dx \prod_i ds_i \\
&= \int f(x, s_1, \ldots) \frac{\nabla_x m(x, s_1, \ldots)}{m(x, s_1, \ldots)} m(x, s_1, \ldots) dx \prod_i ds_i.
\end{aligned}
$$

Hence

$$
\mathsf{d}^\mu = \frac{\nabla_x m(x, s_1, \ldots)}{m(x, s_1, \ldots)} = \nabla_x \log m(x, s_1, \ldots).
$$

General theorems on infinite-dim SDEs

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ.

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

(A3) $\{X_t^i\}$ do not collide each other

(A4) each tagged particle X_t^i never explode

(A5) The log derivative $\mathsf{d}^{\mu} \in L^1_{\text{loc}}(\mu^1)$ exists $\Rightarrow$(SDE representation)

General theorems on infinite-dim SDEs

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(A5) The log derivative $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mu^1)$ exists $\Rightarrow$(SDE representation)

Thm 3. (O.12(PTRF)) (A1)–(A5) $\Rightarrow$ $\exists \mathsf{S}_0 \subset \mathsf{S}$ *such that* $\mu(\mathsf{S}_0) = 1$, *and that, for* $\forall \mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, *there exists a solution* $(\mathbf{X}, \mathbf{B})$ *satisfying*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^\mu(X_t^i, \sum_{j\neq i} \delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

$$\mathbf{X}_t \in \mathfrak{u}^{-1}(\mathsf{S}_0) \qquad \textit{for all } t$$

Here $\mathfrak{u}: S^{\mathbb{N}} \to \mathsf{S}$ such that $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$.

Corollary 2. *Suppose that there exists a RPF* μ *satisfying* (A1)–(A4) *and*

$$\nabla_x \log \mu^{[1]}(x, \mathsf{s}) = 2b(x, \mathsf{s}).$$

Then ISDE (1) *has a weak solution.*

Proof:

- $S^{\mathbb{N}}$ does not have good measures $\Rightarrow$ no Dirichlet forms on $S^{\mathbb{N}}$ $\Rightarrow$

Introduce a sequence of spaces with Campbel measures $\mu^{[M]}$:

$$S^M \times \mathsf{S}, \quad d\mu^{[M]} = \rho^M(\mathbf{x}_M)\mu_{\mathbf{x}_m}(d\mathsf{s})d\mathbf{x}_M$$

Here ρ^M is a M-correlation function of μ and $\mu_{\mathbf{x}_m}$ is the reduced Palm measure conditioned at $\mathbf{x}_M$.

Let $\mathbb{D}^{[M]}$ be the natural square field of $S^M \times \mathsf{S}$. Let

$$\mathcal{E}^{[M]}(f,g) = \int_{S^M \times \mathsf{S}} \mathbb{D}^{[M]}[f,g]d\mu^{[M]},$$

$$L^2(\mu^{[M]}), \quad C_0^\infty(S^M) \otimes \mathcal{D}_\circ.$$

Lem 3. *These bilinear forms are closable, and their closures are quasi-regular Dirichlet forms. Hence associated diffusion* $(\mathbf{X}_t^M, \mathsf{X}_t^{M*})$ *exists:*

$$(\mathbf{X}_t^M, \mathsf{X}_t^{M*}) = (X_t^{M,1}, \ldots, X_t^{M,M}, \sum_{i=M+1}^{\infty} \delta_{X_t^{M,i}})$$

Coupling of Dirichlet forms:

- Let fix a label ℓ. Let

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$$

be the unlabeld diffusion associated with the original unlabeled Dirichlet form

$$(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}, L^2(\mu)).$$

Thm 4. *Associated diffusions have consistency*

$$(X_t^{M,1}, \dots, X_t^{M,M}, X_t^{M,M+1}, \dots) = (X_t^1, \dots, X_t^M, X_t^{M+1}, \dots) \quad \textit{in law}$$

or equivalently

$$(\mathbf{X}_t^M, \mathsf{X}_t^{M*}) = (X_t^1, \dots, X_t^M, \sum_{i=M+1}^{\infty} \delta_{X_t^i}) \quad \textit{in law}$$

From this coupling and Fukushima decomposition (Itô formula) we prove that (X_t^i) satisfies the ISDE. We use the M-labeled process $(\mathbf{X}_t^M, \mathsf{X}_t^{M*})$, to apply Itô formula to coordinate functions $x_1, \dots, x_M$.

Coupling of Dirichlet forms:

- The key point here is that, instead of large space

$$S^{\mathbb{N}}$$

we use a system of countably infinite *good* infinite dimensional sapce

$$S^1 \times \mathsf{S},\ S^2 \times \mathsf{S},\ S^3 \times \mathsf{S},\ S^4 \times \mathsf{S},\ S^5 \times \mathsf{S},\ S^6 \times \mathsf{S},\ S^7 \times \mathsf{S},\ \cdots$$

- By the diffusion X on the original unlabeled space

$$\mathsf{S},$$

we construct a coupling of diffusions $(\mathbf{X}^M, \mathsf{X}^{M*})$ on these inifinite many spaces $S^M \times \mathsf{S}$.

- From this coupling, we have the ISDE representation. Indeed, we can apply Itoô formula to each coordinate functions $f(\mathbf{x}) = x_k$. We use $\mathcal{E}^{[M]}(f, g)$ for $1 \le k \le M$.

Log derivative of μ: precise correspondence between RPFs & potentials

- The log derivative gives the precise correspondence between RPFs μ and potentials (Φ, Ψ).

- We next give examples of logarithmic derivatives

$\mathsf{d}^{\mu} = \nabla_x \log \mu$[1]

Thm 5 (O. PTRF 12).

(1) *Let* μ_{gin} *be the Ginibre RPF. Then*

$$\mathsf{d}^{\mu_{\mathsf{gin}}}(x, \mathsf{s}) = \lim_{r\to\infty} 2 \sum_{|x-s_i|<r} \frac{x - s_i}{|x - s_i|^2}$$

$$\mathsf{d}^{\mu_{\mathsf{gin}}}(x, \mathsf{s}) = -2x + \lim_{r\to\infty} 2 \sum_{|s_i|<r} \frac{x - s_i}{|x - s_i|^2}$$

(2) *Let* $\mu_{\mathsf{sin},\beta}$ *be the* Sine_{β} *RPF. Suppose* $\beta = 1, 2, 4$. *Then*

$$\mathsf{d}^{\mu_{\mathsf{sin},\beta}}(x, \mathsf{s}) = \lim_{r\to\infty} \beta \sum_{|x-s_i|<r} \frac{1}{x - s_i}$$

Thm 6 (O.-Honda). *Let* $\mu_{\mathsf{bes},2}^{a}$ *be the* Bessel_2^a *RPF. Then*

$$\mathsf{d}^{\mu_{\mathsf{bes},2}^{a}}(x, \mathsf{s}) = \frac{a}{x} + 2 \sum_{|x-s_i|<r} \frac{1}{x - s_i}$$

Thm 7 (O.-Tanemura). [*Airy RPFs:* $\mu_{\mathsf{Ai},\beta}$]
Let $\beta = 1, 2, 4$. *Then the log derivative* $\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}$ *is*

$$\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}(x, \mathsf{s}) = \beta \lim_{r\to\infty} \{ (\sum_{|x-s_i|<r} \frac{1}{x - s_i}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \}$$

Here

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0)}(x)$$

A criteria of Quasi-Gibbs property.

For $\Phi : S \to \mathbb{R} \cup \{\infty\}$ and $\Psi : S \times S \to \mathbb{R} \cup \{\infty\}$, let

$$\mathcal{H}_A^{\Phi,\Psi}(\mathsf{x}) = \sum_{x_i \in A} \Phi(x_i) + \sum_{x_i, x_j \in A, i<j} \Psi(x_i, x_j), \qquad \text{where } \mathsf{x} = \sum_i \delta_{x_i}.$$

We assume $\Phi < \infty$ almost everywhere (a.e.) to avoid triviality.
We set

$$\mathcal{H}_r(\mathsf{x}) = \mathcal{H}_{S_r}^{\Phi,\Psi}(\mathsf{x}). \tag{5}$$

For a subset $A \subset S$, we define the map $\pi_A : \mathsf{S} \to \mathsf{S}$ by $\pi_A(\mathsf{s}) = \mathsf{s}(A \cap \cdot)$.
Let Λ be the Poisson RPF for which the intensity is the Lebesgue measure on S. We set

$$\Lambda_r = \Lambda(\cdot \cap \mathsf{S}_r^m).$$

We write $\nu_1 \le \nu_2$ if $\nu_1(A) \le \nu_2(A)$ for all $A \in \mathcal{B}$. Here ν_1, ν_2 are measures on $(\Omega, \mathcal{B})$.

A criteria of Quasi-Gibbs property

Definition 1. A RPF μ is called a (Φ, Ψ)-quasi-Gibbs measure if
(1) There exists an increasing sequence $\{b_r\} \subset \mathbb{N}$ such that, for each $r, m \in \mathbb{N}$, there exists a sequence of Borel subsets $\mathsf{S}_{r,k}^m$ satisfying

$$\mathsf{S}_{r,k}^m \subset \mathsf{S}_{r,k+1}^m \subset \mathsf{S}_r^m \text{ for all } k, \tag{6}$$

$$\lim_{k\to\infty} \mu_{r,k}^m = \mu_r^m \quad \text{weakly}, \tag{7}$$

where $\mu_{r,k}^m = \mu(\cdot \cap \mathsf{S}_{r,k}^m)$ and $\mu_r^m = \mu(\cdot \cap \mathsf{S}_r^m)$.
(2) For all $r, m, k \in \mathbb{N}$ and $\mu_{r,k}^m$-a.e. $\mathsf{s} \in \mathsf{S}$,

$$\frac{1}{C} e^{-\mathcal{H}_r(\mathsf{x})} 1_{\mathsf{S}_r^m}(\mathsf{x}) \Lambda_r^m(d\mathsf{x}) \leq \mu_{r,k,\mathsf{s}}^m(d\mathsf{x}) \leq C e^{-\mathcal{H}_r(\mathsf{x})} 1_{\mathsf{S}_r^m}(\mathsf{x}) \Lambda_r^m(d\mathsf{x}). \tag{8}$$

Here, $C = C(r, m, k, \pi_{S_r^c}(\mathsf{s}))$ is a positive constant and $\mu_{r,k,\mathsf{s}}^m$ is the regular conditional probability measure of $\mu_{r,k}^m$ defined as

$$\mu_{r,k,\mathsf{s}}^m(d\mathsf{x}) = \mu_{r,k}^m(\pi_{S_r} \in d\mathsf{x} |\ \pi_{S_r^c}(\mathsf{s})). \tag{9}$$

A criteria of Quasi-Gibbs property

We give a set of conditions for the quasi-Gibbs property.

(H.1) The measure μ has a locally bounded, n-correlation function ρ^n for each $n \in \mathbb{N}$.

(H.2) $\exists$ probability measures $\{\mu^N\}_{N\in\mathbb{N}}$ on S such that:

(1) The n-correlation functions ρ^n_N of μ^N satisfy

$$\lim_{N\to\infty} \rho^n_N(\mathbf{x}_n) = \rho^n(\mathbf{x}_n) \quad \text{a.e.} \quad \text{for all } n \in \mathbb{N}, \tag{10}$$

$$\sup\{\rho^n_N(\mathbf{x}_n); N \in \mathbb{N}, \mathbf{x}_n \in S^n_r\} \le \{Cn^\delta\}^n \quad \text{for all } n, r \in \mathbb{N}, \tag{11}$$

where $C = C(r) > 0$, and $\delta = \delta(r) < 1$.

(2) $\mu^N(\mathsf{s}(S) = \mathsf{n}_N) = 1$ for each N, where $\mathsf{n}_N \uparrow\in \mathbb{N}$.

[A good finite-particle approximation $\{\mu^N\}_{N\in\mathbb{N}}$]

(3) μ^N is a (Φ^N, Ψ^N)-canonical Gibbs measure.

(4) There exists a sequence $\{\mathfrak{m}_\infty^N\}_{N\in\mathbb{N}}$ in $\mathbb{R}^d$ such that

$$\lim_{N\to\infty}\{\Phi^N(x) - \mathfrak{m}_\infty^N \cdot x\} = \Phi(x) \quad \text{for a.e. } x, \tag{12}$$

$$\inf_{N\in\mathbb{N}} \inf_{x\in S}\{\Phi^N(x) - \mathfrak{m}_\infty^N \cdot x\} > -\infty.$$

(5) The interaction potentials $\Psi^N : S\times S \to \mathbb{R}\cup\{\infty\}$ satisfies

$$\lim_{N\to\infty} \Psi^N = \Psi \text{ compactly and uniformly in } C^1(S\times S\backslash\{x=y\}), \tag{13}$$

$$\inf_{N\in\mathbb{N}} \inf_{x,y\in S_r} \Psi^N(x,y) > -\infty \quad \text{for all } r\in\mathbb{N}.$$

[Airy RPF (soft edge scaling limit)]

Remark 2.

- For the GUE soft-edge (the Airy RPF), we take $\mathfrak{m}_\infty^N = N^{1/3}$.
- In fact, in this case, the limit of Φ^N diverges.
- Hence, we substitute $\mathfrak{m}_\infty^N \cdot x$ from $\Phi^N(x)$ to make the limit finite.
- We see that the terms $\mathfrak{m}_\infty^N \cdot x$ are cancelled by the interaction terms.

$$-\frac{\beta}{4}\sum_{i=1}^{N}|2\sqrt{N}+N^{-1/6}x_i|^2 = -\frac{\beta}{4}\sum_{i=1}^{N}\{4N+N^{-1/3}|x_i|^2+4N^{1/3}x_i\}.$$

[Ψ-tightness]

- The next assumption (H.3) is a tightness condition on $\{\mu^N\}$ according to the interaction Ψ^N.
- Let $\mathsf{x} = \sum \delta_{x_i}$, $\mathsf{y} = \sum \delta_{y_j} \in \mathsf{S}$, $S_{rs} = S_s \backslash S_r$, and $S_{r\infty} = S_r^c$.

For $r < s \le t < u \le \infty$, we set

$$\Psi^N_{rs,tu}(\mathsf{x},\mathsf{y}) = \sum_{x_i \in S_{rs},\ y_j \in S_{tu}} \Psi^N(x_i, y_j) \quad . \tag{14}$$

We write $\Psi^N_{0r,rs}(\mathsf{x},\mathsf{y}) = \Psi^N_{0r,rs}(x,\mathsf{y})$ if $\mathsf{x} = \delta_x$.

$$\widetilde{\Psi}^N_{rs,tu}(\mathsf{x},\mathsf{y}) = \Psi^N_{rs,tu}(\mathsf{x},\mathsf{y}) + \{ \sum_{x_i \in S_{rs}} x_i \} \cdot (\mathfrak{m}^N_t - \mathfrak{m}^N_u). \tag{15}$$

For $\{\Psi^N\}$, $r, k \in \mathbb{N}$, and $\{\mathfrak{m}^N_s\}$

$$\mathsf{H}^N_{r,k} = \{\mathsf{y} \in \mathsf{S}\,;\, \mathsf{y}(S) = \mathsf{n}_N,\ \{ \sup_{r<s\in\mathbb{N}} \sup_{\substack{x,w \in S_r \\ x \neq w}} \frac{|\widetilde{\Psi}^N_{0r,rs}(x,\mathsf{y}) - \widetilde{\Psi}^N_{0r,rs}(w,\mathsf{y})|}{|x-w|} \} \le k \}.$$

[A sufficient condition of quasi-Gibbs property]

We define $\mathsf{H}_{r,k}$ as

$$\mathsf{H}_{r,k} = \sum_{N=1}^{\infty} \mathsf{H}_{r,k}^{N}. \tag{16}$$

(H.3) There exists a sequence $\{\mathfrak{m}_s^N\}$ in $\mathbb{R}^d$ such that the set $\mathsf{H}_{r,k}$ satisfies the following:

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mu^N(\mathsf{H}_{r,k}^c) = 0 \quad \text{for all } r \in \mathbb{N}, \tag{17}$$

$$\lim_{s\to\infty} \mathfrak{m}_s^N = \mathfrak{m}_\infty^N, \tag{18}$$

$$\sup_{N\in\mathbb{N}} |\mathfrak{m}_s^N| < \infty \quad \text{for all } s \in \mathbb{N}. \tag{19}$$

Thm 8. *Assume* (H.1), (H.2) *and* (H.3). *Then* μ *is a* (Φ,Ψ)*-quasi-Gibbs measure.*

$[d = 1, 2]$

- We next assume $d = 1, 2$. To unify these two cases, we set $S = \mathbb{C}$ and regard here $\mathbb{R}^2$ as $\mathbb{C}$.
- We assume Ψ^N is independent of N and of the form

$$\Psi(x, y) := \Psi^N(x, y) = -\beta \log |x - y| \qquad (\beta \in \mathbb{R}). \tag{20}$$

- We give a suff condition of (H.3) through correlation functions.
- Let $\mathsf{x} = \sum_i \delta_{x_i}$ and $\tilde{S}_{rs} = \tilde{S}_s \backslash S_r$, where $S_r = \{s \in S; \ |s| < r\}$, as before. For $1 \le r < s \le \infty$ let $\mathsf{v}_{\ell,rs}\colon \mathsf{S} \to \mathbb{C}$ such that

$$\mathsf{v}_{\ell,rs}(\mathsf{x}) = \beta \{ \sum_{x_i \in \tilde{S}_{rs}} \frac{1}{x_i^\ell} \} \quad (\ell \ge 2) \tag{21}$$

$$\mathsf{v}_{1,rs}^N(\mathsf{x}) = \beta \{ \sum_{x_i \in \tilde{S}_{rs}} \frac{1}{x_i} \} + \bar{\mathsf{m}}_r^N - \bar{\mathsf{m}}_s^N \quad (\ell = 1). \tag{22}$$

Here $\bar{\mathsf{m}}_r^N = \mathsf{m}_{r,1}^N - \sqrt{-1}\mathsf{m}_{r,2}^N$ is the complex conjugate of m_r^N.

[d=1,2]

Now the key assumption is as follows.

(H.4) There exists an ℓ_0 such that $2 \le \ell_0 \in \mathbb{N}$ and that

$$\sup_{N\in\mathbb{N}} \{ \int_{1\le|x|<\infty} \frac{1}{|x|^{\ell_0}} \rho_N^1(x)dx \} < \infty \tag{23}$$

and that, for each $1 < \ell < \ell_0$,

$$\sup_{N\in\mathbb{N}} \|\mathsf{v}_{\ell,rs}\|_{L^1(\mu^N)} < \infty \text{ for all } r < s \in \mathbb{N}, \tag{24}$$

$$\lim_{s\to\infty} \sup_{N\in\mathbb{N}} \|\mathsf{v}_{\ell,s\infty}\|_{L^1(\mu^N)} = 0 \tag{25}$$

and that, for each $\ell = 1$,

$$\sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} \mathsf{v}_{\ell,rs}^M \|_{L^1(\mu^N)} < \infty \text{ for all } r < s \in \mathbb{N}, \tag{26}$$

$$\lim_{s\to\infty} \sup_{N\in\mathbb{N}} \| \sup_{M\in\mathbb{N}} \mathsf{v}_{\ell,s\infty}^M \|_{L^1(\mu^N)} = 0. \tag{27}$$

Thm 9. *Assume* (20) *and* $S = \mathbb{C}$. *Assume* (H.1), (H.2) *and* (H.4). *Assume* (18). *Then* μ *is a* (Φ, Ψ)*-quasi-Gibbs measure.*

Calculation of logarithmic derivative

- Assume that n-point cor funs $\{\rho^{N,n}\}$ satisfy for each $r, n \in \mathbb{N}$

$$\lim_{N\to\infty} \rho^{N,n}(\mathbf{x}) = \rho^n(\mathbf{x}) \quad \text{uniformly on } S_r^n, \tag{28}$$

$$\sup_{N\in\mathbb{N}} \sup_{\mathbf{x}\in S_r^n} \rho^{N,n}(\mathbf{x}) \le C_1^{-n} n^{C_2 n}, \quad 0 < C < \infty, 0 < C_2 < 1, \ . \tag{29}$$

Calculation of logarithmic derivative

- We assume that μ^N have log derivative d^N such that

$$\mathsf{d}^N(x,\mathsf{y}) = u^N(x) + \mathsf{g}_s^N(x,\mathsf{y}) + w_s^N(x,\mathsf{y}) \tag{30}$$

Here g, g^N, v, $v^N : S^2 \to \mathbb{R}^d$ and $w : S \to \mathbb{R}^d$ and set ($\mathsf{y} = \sum_i \delta_{y_i}$)

$$\mathsf{g}_s(x,\mathsf{y}) = \int_{|x-y|<s} v(x,y)dy + \sum_{|x-y_i|<s} g(x,y_i),$$

$$\mathsf{g}_s^N(x,\mathsf{y}) = \int_{|x-y|<s} v^N(x,y)dy + \sum_{|x-y_i|<s} g^N(x,y_i),$$

$$w_s^N(x,\mathsf{y}) = \int_{s\le|x-y|} v^N(x,y)dy + \sum_{s\le|x-y_i|} g^N(x,y_i) \in L^{\hat{p}}_{\mathrm{loc}}(\mu^1).$$

Calculation of logarithmic derivative

- Let $1 < p < \widehat{p} < \infty$. Assume that

$$\limsup_{N\to\infty} \int_{S_r\times \mathsf{S}} |\mathsf{d}^N - u^N|^{\widehat{p}} d\mu^{N,1} < \infty \quad \text{for all } r \in \mathbb{N} \tag{31}$$

$$\lim_{N\to\infty} u^N = u \quad \text{in } L^{\widehat{p}}_{\text{loc}}(S, dx) \tag{32}$$

$$\lim_{N\to\infty} \mathsf{g}_s^N = \mathsf{g}_s \quad \text{in } L^{\widehat{p}}_{\text{loc}}(\mu^1) \quad \text{for all } s, \tag{33}$$

$$\lim_{s\to\infty} \limsup_{N\to\infty} \int_{S_r\times \mathsf{S}} |w_s^N(x,\mathsf{y}) - w(x)|^{\widehat{p}} d\mu^{N,1} = 0. \tag{34}$$

Recall that

$$\mathsf{g}_s(x,\mathsf{y}) = \int_{|x-y|<s} v(x,y)dy + \sum_{|x-y_i|<s} g(x,y_i)$$

Thm 10. *Assume (28)–(34). Then* d^μ *exists in* $L^p_{\text{loc}}(\mu^1)$ *given by*

$$\mathsf{d}^\mu(x,\mathsf{y}) = u(x) + \lim_{s\to\infty} \mathsf{g}_s(x,\mathsf{y}) + w(x). \tag{35}$$

Calculation of logarithmic derivative

Recall that

$$\mathsf{g}_s(x,\mathsf{y}) = \int_{|x-y|<s} v(x,y)dy + \sum_{|x-y_i|<s} g(x,y_i)$$

Thm 10 The log derivative d^μ exists in $L^p_{\text{loc}}(\mu^1)$ and is given by

$$\mathsf{d}^\mu(x,\mathsf{y}) = u(x) + \lim_{s\to\infty} \mathsf{g}_s(x,\mathsf{y}) + w(x). \tag{36}$$

Example 1. In the case of Ginibre RPF, we take

$$u^N(x) = u(x) = -2x, \quad w(x) = 2x,$$
$$v^N(x,y) = v(x,y) = 0,$$
$$g^N(x,y) = g(x,y) = \frac{2(x-y)}{|x-y|^2}.$$

Example 2. In the case of Airy RPF, we take

$$u^N(x) = \beta\{\int_{\mathbb{R}} \frac{\rho^{N,1}_{\beta,x}(y)}{x-y}dy\} - N^{1/3} - \frac{N^{-1/3}}{2}x$$

$$u(x) = \beta \lim_{s\to\infty}\{\int_{|s|<s} \frac{\rho^{1}_{\beta,x}(y)}{x-y}dy - \int_{|y|<s} \frac{\varrho(y)}{-y}dy\}$$

$$w(x) = 0$$

$$v^N(x,y) = -\beta\frac{\rho^{N,1}_{\beta,x}(y)}{x-y}$$

$$v(x,y) = -\beta\frac{\rho^{1}_{\beta,x}(y)}{x-y}$$

$$g^N(x,y) = g(x,y) = \frac{\beta}{x-y}.$$

Strong solutions and pathwise uniqueness

2014/9/1/Mon–2014/9/5/Wed Warwick

UK-Japan Stochastic Analysis School (JSPS Core-to-Core programme)

Outline:

- Unique strong solutions of ISDEs (general theorems)
- Triviality of tail σ-fields of labeled path spaces.
- Applications to interacting Brownian motions in infinite dimensions.

General theorems on infinite-dim SDEs

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ.
(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$
(A3) $\{X_t^i\}$ do not collide each other
(A4) each tagged particle X_t^i never explode
(A5) The log derivative $\mathsf{d}^{\mu} \in L^1_{\mathrm{loc}}(\mu^1)$ exists

General theorems on infinite-dim SDEs

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(A5) The log derivative $\mathsf{d}^{\mu} \in L^1_{\mathrm{loc}}(\mu^1)$ exists

l:isde

Thm 1. (O.12(PTRF)) (A1)–(A5) $\Rightarrow \exists \mathsf{S}_0 \subset \mathsf{S}$ *such that*

$$\mu(\mathsf{S}_0) = 1,$$

and that, for $\forall \mathsf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, $\exists \mathfrak{u}^{-1}(\mathsf{S}_0)$*-valued pr.* $(X_t^i)_{i\in\mathbb{N}}$ *and* $\exists S^{\mathbb{N}}$*-valued Brownian m.* $(B_t^i)_{i\in\mathbb{N}}$ *satisfying*

:isde

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathsf{s} \tag{1}$$

Here $\mathfrak{u}: S^{\mathbb{N}} \to \mathsf{S}$ *such that* $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$.

- The solution $(\mathbf{X}, \mathbf{B})$ is not a strong solution.
- In this talk we construct a strong solution from a weak solution, and prove pathwise uniqueness.

Tail σ-field of configuration space S

- To construct strong solutions, we use two geometric properties of RPFs. : Tail triviality & Tail decomposition
- Let $\pi_r^c : \mathsf{S} \to \mathsf{S}$ such that $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$, where $S_r = \{|s| < r\}$.
- Let $\mathcal{T} = \mathcal{T}(\mathsf{S})$ be the tail σ field of S:

$$\mathcal{T}(\mathsf{S}) = \bigcap_{r=1}^{\infty} \sigma[\pi_r^c].$$

l:tail

Thm 2. l:tail *Let μ be a determinantal RPF. Then $\mathcal{T}(\mathsf{S})$ is μ-trivial.*

- Thm 2 is a generalization of the result for the discrete determinantal RPFs due to Russel Lyons, Shirai-Takahashi.
- In general, quasi-Gibbs measures μ are not tail trivial. Hence we introduce the tail decomposition of μ.

• • •

Let $\mathcal{T} = \mathcal{T}(\mathsf{S})$ be the tail σ field of S as above.
Let $\mu(\cdot|\mathcal{T})$ be the regular conditional probability.
Then by construction

$$\mu(\cdot) = \int_{\mathsf{S}} \mu(\cdot|\mathcal{T})(\xi)\mu(d\xi)$$

and, for any $A \in \mathcal{T}$,

$$\mu(A|\mathcal{T})(\xi) = 1_A(\xi) \qquad \text{for } \mu\text{-a.s.}\,\xi.$$

We can interchange the roll of "for any $A \in \mathcal{T}$" and "for μ-a.s. ξ".

l:decom

Thm 3. *Let μ be a quasi-Gibbs measure. Then for μ-a.s. ξ,*

$$\mu(A|\mathcal{T})(\xi) = 1_A(\xi) \qquad \textit{for any } A \in \mathcal{T}.$$

l:decom

- Thm 3 is a generalization of the result for the discrete Gibbs m due to Georgii.
- With this, the assumption of tail triviality of μ turn out to be not an essential restriction.

existence of strong solution

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^\mu(X_t^i, \sum_{\mathsf{j}\neq\mathsf{i}}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

We introduce the condition such that the drifts $\mathsf{d}^\mu(x,\mathsf{s})$ are locally Lipschitz continuous in x for fixed outside $\pi_r^c(\mathsf{s})$.

Let $S_r = \{|x| < r\}$ and

$$\mathsf{H}(r,n) = \{\mathsf{s} = \sum_i \delta_{s_i}; |\nabla_x \mathsf{d}^\mu(s_i, \mathsf{s}-\delta_{s_i})| < n \text{ for } \forall i \text{ s.t. } s_i \in S_r\},$$

$$\mathsf{H} = \bigcap_{r=1}^{\infty}\bigcup_{n=1}^{\infty}\mathsf{H}(r,n).$$

(A6) $\mathrm{Cap}^\mu(\mathsf{H}^c) = 0$ + marginal assumption

- We pose in (A6) a condition that the coefficients $\mathsf{d}^\mu(x, \mathsf{X}_t^{i\diamondsuit})$ in x are Lipschitz constinuous in each $\mathsf{H}(r,n)$. Here $\mathsf{X}_t^{i\diamondsuit} = \sum_{j\neq i}\delta_{X_t^j}$.

existence of strong solution

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ.

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

(A3) $\{X_t^i\}$ do not collide each other

(A4) each tagged particle X_t^i never explode

(A5) The log derivative $\mathsf{d}^{\mu} \in L^1_{\mathrm{loc}}(\mu^1)$ exists

l:strong (A6) $\mathrm{Cap}^{\mu}(\mathsf{H}^c) = 0$.

Thm 4 (O.-Tanemura). (A1)–(A6). $\Rightarrow$ (1) *The ISDE*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i} \delta_{X_t^j})dt, \ (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

has a strong solution for $\mathbf{s} = (s_i) \in S^{\mathbb{N}}$ *s.t.* $\sum_i \delta_{s_i} \in \mathsf{H}$.

• • •

existence of strong solution

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ.

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

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(A6) $\mathrm{Cap}^{\mu}(\mathsf{H}^c) = 0$. l:strong

Thm 4[O.-Tanemura] (A1)–(A6). $\Rightarrow$ (1) The ISDE

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i} \delta_{X_t^j})dt, \ (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

has a strong solution for $\mathbf{s} = (s_i) \in S^{\mathbb{N}}$ s.t. $\sum_i \delta_{s_i} \in \mathsf{H}$.

(2) The ass unlabeled diffusion $\mathsf{X} = \sum_i \delta_{X^i}$ satisfies

$$P_{\mu_\xi} \circ \mathsf{X}_t^{-1} \prec \mu_\xi \quad (\forall t) \quad \text{for } \mu\text{-a.s. } \xi$$ l:decom

Here $\mu_\xi = \mu(\cdot|\mathcal{T}(\mathsf{S}))(\xi)$ in Thm 3.

Decomposition of unlabeled state space of strong solutions 1

- By construction $\mu(\cdot|\xi)(\mathsf{A})$ are $\mathcal{T}$-measurable in ξ for each $\mathsf{A} \in \mathcal{B}(\mathsf{S})$.
l:decom
- By Thm 3, we take a version of $\mu(\cdot|\xi)$ such that, for μ-a.s. $\mathsf{a} \in \mathsf{S}$,
:22w
$$\mu(\cdot|\xi)(\mathsf{A}) = 1_{\mathsf{A}}(\mathsf{a}) \quad \text{for all } \mathsf{A} \in \mathcal{T}. \tag{2}$$
- Let $\sim_{\mathcal{T}}$ be the equivalence relation such that $\mathsf{a} \sim_{\mathcal{T}} \mathsf{b}$ if and only if
:22x
$$1_{\mathsf{A}}(\mathsf{a}) = 1_{\mathsf{A}}(\mathsf{b}) \quad \text{for all } \mathsf{A} \in \mathcal{T}. \tag{3}$$
:22w l:strong
- From (2) we deduce that the set H in Thm 4 can be decomposed as a disjoint sum
:22y
$$\mathsf{H} = \sum_{[\xi] \in \mathsf{H}/\sim_{\mathcal{T}}} \mathsf{H}^{\xi} \quad \text{such that} \quad \mu(\cdot|\xi)(\mathsf{H}^{\xi}) = 1. \tag{4}$$

l:strong
The solution in Thm 4 satisfy for μ_{ξ}-a.s. $\mathsf{s} \in \mathsf{H}^{\xi}$
$$P_{\mathsf{s}}(\mathsf{X}_t \in \mathsf{H}^{\xi} \ \text{ for all } t) = 1.$$

Uniqueness of strong solutions 1

l:strong0

Thm 5 (O.-Tanemura). *Assume* (A1)–(A6).
Let $\mathbf{X} = (X^i)$ *and* $\hat{\mathbf{X}} = (\hat{X}^i)$ *be strong sol of the ISDE*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{\mathsf{j}\neq\mathsf{i}} \delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s} = (s_i)_{i\in\mathbb{N}}$$

on the same Brownian motion $\mathbf{B} = (B_t^i)_{i\in\mathbb{N}}$. *Let*

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \quad \text{and} \quad \hat{\mathsf{X}}_t = \sum_{i=1}^{\infty} \delta_{\hat{X}_t^i}.$$

Suppose, for μ*-a.s.* ξ,

$$P_{\mu_\xi} \circ \mathsf{X}_t^{-1} \prec \mu_\xi \text{ and } P_{\mu_\xi} \circ \hat{\mathsf{X}}_t^{-1} \prec \mu_\xi \ (\forall t).$$

Then

$$\mathbf{P}_{\mathbf{s}}(\mathbf{X} = \hat{\mathbf{X}}) = 1 \quad \text{for } \mu\text{-a.s. } \mathsf{s} = \sum_{i=1}^{\infty} \delta_{s_i}$$

.

Uniqueness of strong solutions

l:strongx

Thm 6 (O.-Tanemura). *Assume* (A1)–(A7). *Here*
(A7) μ *is tail trivial.*
Then the strong solution $\mathbf{X} = (X^i)$ *such that*

$$P_\mu \circ \mathsf{X}_t^{-1} \prec \mu \qquad \textit{for all } t$$

is unique for μ*-a.e.* $\mathsf{x} = \sum_i \delta_{x_i}$.
Here X *is the unlabeled dynamics of* $\mathbf{X}$*:*

$$\mathsf{X}_t = \sum_i^{\infty} \delta_{X_t^i}$$

Cor If μ is a determinantal RPF, then the associated ISDE has a unique strong solution that is reversible w.r.t. μ.

• Tail σ-fields of Airy, Sine, Ginibre RPFs with $\beta = 2$ and all other determinantal RPFs are trivial.

Uniqueness of Dirichlet forms

Let $\mathcal{D}^{\mu}_{\text{poly}}$ be the closure of the set of polynomials on S such that $\mathcal{E}^{\mu}_{1}(f,f) < \infty$. Then

$$\mathcal{D}^{\mu}_{\text{poly}} \subset \mathcal{D}^{\mu}$$

l:strong2 because polynomials are local and smooth.

Thm 7 (O.-Tanemura '14). *Assume* (A1)–(A7). *Then quasi-regular Dirichlet forms that are extension of* $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}_{\text{poly}})$ *are unique.*

In particular, $\mathcal{D}^{\mu}_{\text{poly}} = \mathcal{D}^{\mu}$, and Lang's construction and O.'s construction are same.

r:df

Remark 1. (1) Dirichlet forms here are same as those constructed by Albeverio-et al, and Yoshida.

l:strong2

(2) If (A5) (non-explosion) does not hold. Then Thm 7 does not hold. This is very natural theorem that says the uniqueness of Dirichlet forms is related to the non-explosion problem of tagged problem.

Idea of "strong sol of ISDEs"

- General theory to construct unique, strong solutions of infinite-dimensional stochastic differential equations

- Weak solution: (O. JPSJ 10, PTRF 12, AOP 13, SPA 13)
- logarithmic derivative d^{μ}: Very informally,

$$\mathsf{d}^{\mu}(x, \mathsf{y}) = \nabla_x \log \mu^{[1]}$$

Here $\mu^{[1]}$ is a 1-Campbell measure of μ.
- μ is quasi-Gibbs with upper semi-continuous potential Ψ.
- mariginal assumptions

Then ISDE has a weak solution $(\mathbf{X}, \mathbf{B})$:

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i}^{\infty} \delta_{X_t^j})dt \quad (i \in \mathbb{N})$$

Strong solutions of ISDE: Non Markov type

- Strong solutions and uniqueness:

- We lift weak solutions to strong solutions.

- IFC solutions.
- Tail analysis.

- The key idea is the following:
- We interpret single ISDE as an infinite system of finite dimensional SDEs with consistency (IFC).
- We regard the tail σ-field of the labeled path spaces as boundary condition of ISDEs.

Strong solutions of ISDE: Non Markov type

- We consider non-Markov SDEs because the arguement is general.

$S = \mathbb{R}^d, [0,\infty), \mathbb{C}$, e.t.c.. (the space where particles move),

$W(S^{\mathbb{N}}) = C([0,T]; S^{\mathbb{N}})$, $(0 < T < \infty)$ (labeled path spaces)

- a quadruplet $(W_{\mathsf{sol}}, \mathbf{S_0}, \{\sigma^i\}, \{b^i\})$

W_{sol} : a Borel subset of $W(S^{\mathbb{N}})$ (space of solutions of ISDE)

$\mathbf{S_0}$: a Borel subset of $S^{\mathbb{N}}$ (initial starting points of ISDE)

$\sigma^i, b^i : W_{\mathsf{sol}} \to W(S^{\mathbb{N}})$ (coefficients of ISDE)

- the ISDE on $S^{\mathbb{N}}$ of the form

:q0a

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N}) \tag{5}$$

:q0b

$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathbf{S_0} \tag{6}$$

:q0c

$$\mathbf{X} \in W_{\mathsf{sol}}. \tag{7}$$

- $\mathbf{X} = \{(X_t^i)_{i\in\mathbb{N}}\}_{t\in[0,T]}$
- $\mathbf{B} = (B_t^i)$ $(i \in \mathbb{N})$ is the $S^{\mathbb{N}}$-valued standard Brownian motion.

Strong solutions of ISDE: Assump (P1)

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N})$$
$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathbf{S_0}$$
$$\mathbf{X} \in W_{\mathsf{sol}}.$$

(P1) ISDE (5) :q0a has a weak solution $(\mathbf{X}, \mathbf{B})$. (not a strong solution!)

Here $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ is Brownian motion on $S^{\mathbb{N}}$

Problem: Prove that $\mathbf{X}$ is a functional of the Brownian motion $\mathbf{B}$

Idea:
Strong solutions of Infinite-dimensional SDE
$\Leftrightarrow$
Infinite-many, Finite-dimensional SDEs with Consistency (IFC)
$+$
Tail Triviality of Labeled path spaces w.r.t. the label (TTL)

Assump (P2) infinite-many, finite-dimensional SDEs with consistency

- $\bar{P}_{\mathbf{s}}$: a prob meas on $W(S^{\mathbb{N}}) \times W^{\mathbf{0}}(S^{\mathbb{N}})$: dist. of weak sol $(\mathbf{X}, \mathbf{B})$.
- $\mathbf{P}_{\mathbf{s}} = \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot)$
- $P_{\mathsf{Br}}^{\infty} = \bar{P}_{\mathbf{s}}(\mathbf{B} \in \cdot)$.
- For each $\mathbf{X} \in W_{\mathsf{sol}}$, $\mathbf{s} \in \mathbf{S_0}$, and $m \in \mathbb{N}$, we define the new SDE of

$$\mathbf{Y}^m = (Y_t^{m,1}, \dots, Y_t^{m,m})$$

such that

:q1b

$$\begin{aligned} dY_t^{m,i} &= \sigma^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dt \\ \mathbf{Y}_0^m &= (s_1, \dots, s_m) \in \mathsf{S}^m, \quad \text{where } \mathbf{s} = (s_i)_{i=1}^{\infty}, \\ (\mathbf{Y}^m, \mathbf{X}^{m*}) &\in W_{\mathsf{sol}}. \end{aligned} \tag{8}$$

Here we set

$$\mathbf{X}^{m*} = (X_t^{m+1}, X_t^{m+2}, \dots)$$

$$(\mathbf{Y}^m, \mathbf{X}^{m*}) = (Y_t^{m,1}, \dots, Y_t^{m,m}, X_t^{m+1}, X_t^{m+2}, \dots).$$

$\mathbf{X}^{m*}$ is interpreted as a part of the coefficients of the SDE (8):q1b.

In fact, we regard (8):q1b as SDEs of $\mathbf{Y}$ such as

For $m = 1$

$$dY_t^{1,1} = \sigma^1(\mathbf{Y}^1, \mathbf{X}^{1*})_t dB_t^1 + b^1(\mathbf{Y}^1, \mathbf{X}^{1*})_t dt. \qquad (m = 1)$$

For $m = 2$

$$dY_t^{2,1} = \sigma^1(\mathbf{Y}^2, \mathbf{X}^{2*})_t dB_t^1 + b^1(\mathbf{Y}^2, \mathbf{X}^{2*})_t dt \qquad (m = 2)$$

$$dY_t^{2,2} = \sigma^2(\mathbf{Y}^2, \mathbf{X}^{2*})_t dB_t^2 + b^2(\mathbf{Y}^2, \mathbf{X}^{2*})_t dt.$$

For $m = 3$

$$dY_t^{3,1} = \sigma^1(\mathbf{Y}^3, \mathbf{X}^{3*})_t dB_t^1 + b^1(\mathbf{Y}^3, \mathbf{X}^{3*})_t dt \qquad (m = 3)$$

$$dY_t^{3,2} = \sigma^2(\mathbf{Y}^3, \mathbf{X}^{3*})_t dB_t^2 + b^2(\mathbf{Y}^3, \mathbf{X}^{3*})_t dt$$

$$dY_t^{3,3} = \sigma^3(\mathbf{Y}^3, \mathbf{X}^{3*})_t dB_t^3 + b^3(\mathbf{Y}^3, \mathbf{X}^{3*})_t dt.$$

For $m = 4$

$$
\begin{aligned}
dY_t^{4,1} &= \sigma^1(\mathbf{Y}^4, \mathbf{X}^{4*})_t dB_t^1 + b^1(\mathbf{Y}^4, \mathbf{X}^{4*})_t dt \\
dY_t^{4,2} &= \sigma^2(\mathbf{Y}^4, \mathbf{X}^{4*})_t dB_t^2 + b^2(\mathbf{Y}^4, \mathbf{X}^{4*})_t dt \\
dY_t^{4,3} &= \sigma^3(\mathbf{Y}^4, \mathbf{X}^{4*})_t dB_t^3 + b^4(\mathbf{Y}^4, \mathbf{X}^{4*})_t dt \\
dY_t^{4,4} &= \sigma^4(\mathbf{Y}^4, \mathbf{X}^{4*})_t dB_t^4 + b^4(\mathbf{Y}^4, \mathbf{X}^{4*})_t dt.
\end{aligned}
\tag{$m=4$}
$$

For $m = 5$

$$
\begin{aligned}
dY_t^{5,1} &= \sigma^1(\mathbf{Y}^5, \mathbf{X}^{5*})_t dB_t^1 + b^1(\mathbf{Y}^5, \mathbf{X}^{5*})_t dt \\
dY_t^{5,2} &= \sigma^2(\mathbf{Y}^5, \mathbf{X}^{5*})_t dB_t^2 + b^2(\mathbf{Y}^5, \mathbf{X}^{5*})_t dt \\
dY_t^{5,3} &= \sigma^3(\mathbf{Y}^5, \mathbf{X}^{5*})_t dB_t^3 + b^5(\mathbf{Y}^5, \mathbf{X}^{5*})_t dt \\
dY_t^{5,4} &= \sigma^4(\mathbf{Y}^5, \mathbf{X}^{5*})_t dB_t^4 + b^4(\mathbf{Y}^5, \mathbf{X}^{5*})_t dt \\
dY_t^{5,5} &= \sigma^5(\mathbf{Y}^5, \mathbf{X}^{5*})_t dB_t^5 + b^5(\mathbf{Y}^5, \mathbf{X}^{5*})_t dt.
\end{aligned}
\tag{$m=5$}
$$

• • •

Strong solutions of ISDE: (P2) seq of finite-dim SDEs with consistecy

$$dY_t^{m,i} = \sigma^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dt \qquad (8)$$
$$\mathbf{Y}_0^m = (s_1, \dots, s_m) \in \mathsf{S}^m,$$
$$(\mathbf{Y}^m, \mathbf{X}^{m*}) \in W_{\mathsf{sol}}.$$

:q1b

(P2) The SDE (8) has a unique, strong solution for each $\mathbf{s} \in \mathbf{S_0}$, $\mathbf{X} \in W_{\mathsf{sol}}^{\mathbf{s}}$, and $m \in \mathbb{N}$.

- (P2) is a reasonable assumption. Since $\mathbf{X} \in W_{\mathsf{sol}}$ and W_{sol} is a nice subset of W, we can assume this for the weak solution we have constructed. Here we use Dirichlet form theory again to prove $\mathbf{X}$ stay W_{sol}.
- In the case of Dyson Brownian motions with $m = 1$, we see that

$$b^1(x, \mathsf{y}) = \frac{1}{2}\mathsf{d}^{\mu}(x, \mathsf{y}) = \lim_{r\to\infty} \sum_{j=1, |y_j|<r}^{\infty} \frac{1}{x - y_j}.$$

Hence we see that $b^1(x, \mathbf{X}_t^{1*})$ is locally Lipschitz continuous in x for fixed weak solution $\mathbf{X}$.

Strong solutions of ISDE: (P3) Tail triviality

Let $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ be the tail σ-field of $W(S^{\mathbb{N}})$; we set

:q0y

$$\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}}) = \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}]. \tag{9}$$

(P3) $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ is $\mathbf{P}_{\mathbf{s}}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

Strong solutions of ISDE: Theorem 8

(P1) ISDE (5) has a solution $(\mathbf{X}, \mathbf{B})$.
(P2) SDE (8) has a unique, strong solution for all $\mathbf{s}, \mathbf{X}, m$.
(P3) $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ is $\mathbf{P_s}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

Thm 8. *Assume (P1)–(P3). Then*
(1) *ISDE (5)–(7) has a strong solution for each $\mathbf{s} \in \mathbf{S}_0$.*
(2) *Let $\mathbf{Y_s}$ and $\mathbf{Y}'_\mathbf{s}$ be strong solutions of ISDE (5)–(7) starting at $\mathbf{s} \in \mathbf{S}_0$ defined on the same space of Brownian motions $\mathbf{B}$. Then*

$$\mathbf{Y_s} = \mathbf{Y}'_\mathbf{s} \ \text{a.s.}$$

if and only if

$$\mathbf{Y_s}|_{\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})} = \mathbf{Y}'_\mathbf{s}|_{\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})} \qquad \text{in Law..} \tag{10}$$

• Since $\mathbf{Y_s}$ is strong solution, $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ is trivial w.r.t. $\mathbf{Y_s}$. Hence (10) is equivalent to

$$\mathit{Tail}^{[1]}(\mathbf{Y_s}) = \mathit{Tail}^{[1]}(\mathbf{Y}'_\mathbf{s})$$

Here

$$\mathit{Tail}^{[1]}(\mathbf{Y_s}) = \{\mathbf{A} \in \mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})\,;\, P(\mathbf{Y_s} \in \mathbf{A}) = 1\}. \tag{11}$$

Strong solutions of ISDE: Idea of Theorem 8 l:q1

(P1) ISDE (5) :q0a has a solution $(\mathbf{X}, \mathbf{B})$.

(P2) SDE (8) :q1b has a unique, strong solution for all $\mathbf{s}, \mathbf{X}, m$.

(P3) $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is $\mathbf{P}_{\mathbf{s}}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

- $(\mathbf{X}, \mathbf{B})$: sol of ISDE by (P1). Let $(\mathbf{X}, \mathbf{B})$ be fixed.
- $\mathbf{Y}^m$ is a unique strong sol of SDE(8) :q1b by (P2)
- $\mathbf{Y}^m$ is $\sigma[\mathbf{B}] \bigvee \sigma[\mathbf{X}^{m*}]$-m'ble. $\mathbf{X}^{m*} = (X^n)_{m<n<\infty}$.
- $\mathbf{Y}^m = (X^1, \ldots, X^m)$. by (P2)
- $\mathbf{X}$ is $\sigma[\mathbf{B}] \bigvee \mathcal{T}_{\text{path}}(S^{\mathbb{N}})$-m'ble by $m \to \infty$.
- $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is trivial by (P3) $\Rightarrow$ $\mathbf{X}$ is a strong solution.

Let $(\mathbf{X},\mathbf{B})$ be a weak solution.

- For $(\mathbf{X},\mathbf{B})$ we introduce IFC: $(m\in\mathbb{N})$

$$dY_t^{m,i} = \sigma^i(\mathbf{Y}^m,\mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m,\mathbf{X}^{m*})_t dt \tag{8}$$
$$\mathbf{Y}_0^m = (s_1,\ldots,s_m)\in \mathsf{S}^m,$$
$$(\mathbf{Y}^m,\mathbf{X}^{m*})\in W_{\mathsf{sol}}.$$

:q1b

- We emphasize IFC is a collection of infinitely many finite dimensional equations:
- This system is equivalent to a single ISDE:

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i\in\mathbb{N}) \tag{5}$$
$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathbf{S_0} \tag{6}$$
$$\mathbf{X}\in W_{\mathsf{sol}}. \tag{7}$$

:q0a :q0b :q0c

- We show that these two are equivalent, and by using IFC we give a notion of strong solution and pathwise uniqueness in terms of tail σ-field of the labeled path space.

IFC solutions

- We set

$$\mathcal{S} = W_{\mathsf{sol}} \times W(S^{\mathbb{N}}).$$

- Let $F^m : \mathcal{S} \to W_{\mathsf{sol}}$ the map defined as

$$F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}) = (\mathbf{Y}^m, \mathbf{X}^{m*}).$$

Here $\mathbf{Y}^m$ is a unique strong solution of (8) :q1b. (by (P2)).

- Let $\bar{P}_{\mathbf{s}}$ be a probability on $\mathcal{S}$ such that $\bar{P}_{\mathbf{s}}(\mathbf{X}_0 = \mathbf{s}) = 1$. We say

$$\lim_{m\to\infty} F^m = F^\infty \quad \text{in } W_{\mathsf{sol}} \text{ under } \bar{P}_{\mathbf{s}}$$

if for $\bar{P}_{\mathbf{s}}$-a.s. $\mathbf{s}$ and $\forall i \in \mathbb{N}$

$$\lim_{m\to\infty} F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}) = F^\infty(\mathbf{s}, \mathbf{X}, \mathbf{B}) \in W_{\mathsf{sol}},$$

$$\lim_{m\to\infty} \int_0^t \sigma^i(F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u dB_u^i = \int_0^t \sigma^i(F^\infty(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u dB_u^i,$$

$$\lim_{m\to\infty} \int_0^t b^i(F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u du = \int_0^t b^i(F^\infty(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u du.$$

IFC solutions

1:ifc Let $\bar{P}_{\mathbf{s}}$ be an IFC solution, and

Lem 1. (8):q1b *has an IFC solution* $\bar{P}_{\mathbf{s}}$ *iff* (5):q0a *has a weak solution* $(\mathbf{X}, \mathbf{B})$.

Proof. • Set

$$\mathbf{Y}^{\infty} = F^{\infty}(\mathbf{s}, \mathbf{X}, \mathbf{B}).$$

Then $(\mathbf{Y}^{\infty}, \mathbf{B})$ under $\bar{P}_{\mathbf{s}}$ is a weak solution.

• Let $(\mathbf{X}, \mathbf{B})$ be a weak solution with distribution $\bar{P}_{\mathbf{s}}$. Since $\mathbf{X}$ is a fix point of F^{∞}, $\bar{P}_{\mathbf{s}}$ is an IFC solution. □

l:ifc Let $\bar{P}_{\mathbf{s},\mathbf{B}} = \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot | \mathbf{B})$: the regular conditional probability.

Thm 9. (1) $(\mathbf{Y}, \mathbf{B})$ *under* $\bar{P}_{\mathbf{s}}$ *is a strong solution of* (5) :q0a *iff* $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ *is* $\bar{P}_{\mathbf{s},\mathbf{B}}$*-trivial for* P^{∞}_{Br} $\mathbf{B}$.

(2) *Let* $\mathbf{X}_{\mathbf{s}}$ *and* $\mathbf{X}'_{\mathbf{s}}$ *be strong solutions defined on the same Brownian motion starting at* $\mathbf{s}$. *Then*

:q1dd

$$\mathbf{X}_{\mathbf{s}} = \mathbf{X}'_{\mathbf{s}} \text{ for a.s. } \Longleftrightarrow \mathbf{X}_{\mathbf{s}}|_{\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})} = \mathbf{X}'_{\mathbf{s}}|_{\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})} \text{ in Law.}. \tag{12}$$

l:q1

Application of Thm 8 to interacting Brownian motions.

Application of Thm 8 [1:q1] to interacting Brownian motions.

- We apply Thm 8 [1:q1] to ISDE (1) [:isde]:

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s} \tag{1}$$

:isde

or more generally

$$dX_t^i = \sigma(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dB_t^i + b(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt. \tag{1}$$

:isde

Here $a = \sigma^t\sigma$ and

$$b(x,\mathsf{y}) = \frac{1}{2}\{\nabla a(x,\mathsf{y}) + a(x,\mathsf{y})\mathsf{d}^{\mu}(x,\mathsf{y})\}dt \tag{13}$$

$\mathsf{d}^{\mu}(x,\mathsf{y})$ is the logarithmic derivative (informally) defined as

$$\nabla_x \log \mu^{[1]}$$

with 1-Campbel measure $\mu^{[1]}$ of μ:

$$d\mu^{[1]} = \rho^1(x)\mu_x(d\mathsf{y})dx$$

Strong solutions of ISDE:

We check (P1)–(P3) for (1). :isde

(P1) ISDE (5) :q0a has a solution $(\mathbf{X}, \mathbf{B})$. (by Thm 1 l:isde)

(P2) SDE (8) :q1b has a unique, strong solution for all $\mathsf{s}, \mathbf{X}, m$.

(P3) $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ is $\mathbf{P}_{\mathsf{s}}$-trivial for each $\mathsf{s} \in \mathbf{S_0}$.

- How to prove (P2) **?** $\Rightarrow$ $\nabla_x \mathsf{d}^{\mu} \in \mathcal{D}_{\mathsf{loc}}^{\mu^{[1]}}$
- How to prove (P3) **?** $\Rightarrow$ Tail Theorems.

Strong solutions of ISDE: How to prove (P3)

- We give a sufficient condition of (P3):

(P3) $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is $\mathbf{P}_{\mathbf{s}}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

- The following implies (P3):

(Q1) μ is tail trivial.
(Q2) $P_\mu \circ \mathsf{X}_t^{-1} \prec \mu$ for all t.
(Q3) $P_\mu(\cap_{r=1}^{\infty}\{\mathsf{m}_r(\mathsf{X}) < \infty\}) = 1$.
Here $S_r = \{|x| < r\}$, $\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i}$, $X^i = \{X_t^i\}$,

$$\mathsf{m}_r = \inf\{m \in \mathbb{N}; X^i \in C([0,T]; S_r^c) \text{ for } m < \forall i \in \mathbb{N}\}.$$

l:tail3

Thm 10. *Assume* (Q1)–(Q3). *Then* (P3) *holds.*

- (Q2) is trivial because the unlabeled dynamics is μ-reversible.
- (Q3) is immediate from Lyons-Zheng decomposition.

1:tail3

Out line of the proof of Thm 10 :

- Let $\mathbf{T} = \{\mathbf{t} = (t_1, \ldots, t_m)\,;\, t_i \in [0,T], m \in \mathbb{N}\}$.
- $\tilde{\mathcal{T}}_{\mathsf{path}}(\mathsf{S})$ is the cylindrical tail σ-field of the unlabeled path space

:40a

$$\tilde{\mathcal{T}}_{\mathsf{path}}(\mathsf{S}) = \bigvee_{\mathbf{t}\in\mathbf{T}} \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}(\mathsf{X}_{\mathbf{t}})]. \tag{14}$$

- $\tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}})$ is the cylindrical tail σ-field of the labeled path space:

:40a

$$\tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}}) = \bigvee_{\mathbf{t}\in\mathbf{T}} \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_{\mathbf{t}}^{n*}]. \tag{15}$$

Here $\mathbf{X}_{\mathbf{t}}^{n*} = (\mathbf{X}_{t_1}^{n*}, \ldots, \mathbf{X}_{t_m}^{n*})$.

- We will deduce the triviality of $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ from that of S. We will do this along the scheme: $(P_\mu = \int_{\mathsf{S}} P_{\mathsf{s}} d\mu)$.

$\mathcal{T}(\mathsf{S})\Rightarrow$	$\tilde{\mathcal{T}}_{\mathsf{path}}(\mathsf{S})\Rightarrow$	$\tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}})\Rightarrow$	$\tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}})\Rightarrow$	$\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$
P_μ	P_μ	$\mathbf{P}_{\mu^\ell}$	$\mathbf{P}_{\mathsf{s}}$ a.s. s	$\mathbf{P}_{\mathsf{s}}$ a.s. s

l:tail3

Out line of the proof of Thm 10 :

Below we assume (Q1): μ is tail trivial.

$$\mathcal{T}(\mathsf{S}) \Rightarrow \tilde{\mathcal{T}}_{\mathsf{path}}(\mathsf{S}) \Rightarrow \tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}}) \Rightarrow \tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}}) \text{ a.s.} \Rightarrow \mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}}) \text{ a.s.}$$

l:tt0

Lem 2. *$\mathcal{T}(\mathsf{S})$ is $P_{\mathsf{s}}(\mathsf{X}_t \in \cdot)$-trivial for μ-a.s.* s.

Proof. Since X_t is μ-reversible, $P_{\mu}(\mathsf{X}_t \in \cdot) = \mu$.
Hence by (Q1) we see that $\mathcal{T}(\mathsf{S})$ is l:tt0 $P_{\mu}(\mathsf{X}_t \in \cdot)$-trivial.
From this we easily obtain Lem 2. □

l:tt1

Lem 3. *$\tilde{\mathcal{T}}_{\mathsf{path}}(\mathsf{S})$ is P_{μ}-trivial and P_{s}-trivial for μ-a.s.* s.

l:tt0

Proof. From Lem 2 and the Markov property of unlabeled dynamics yields Lem 3. □

l:tt1

$\mathcal{T}(\mathsf{S}) \Rightarrow \tilde{\mathcal{T}}_{\mathsf{path}}(\mathsf{S}) \Rightarrow \tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}}) \Rightarrow \tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}})$ a.s. $\Rightarrow \mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ a.s.

l:tt2

Lem 4. $\tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}})$ *is* $\mathbf{P}_{\mu^\ell}$*-trivial.*

Proof. For a label ℓ, we have a natural map $\ell_{\mathsf{path}} : W(\mathsf{S}) \to W(S^{\mathbb{N}})$:

$$\ell_{\mathsf{path}}(\mathsf{X}) = (X^1, X^2 \dots).$$

For each $\mathbf{A} \in \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_{\mathbf{t}}^{n*}]$ for some $\mathbf{t} \in \mathbf{T}$, we see

$$\begin{aligned} \ell_{\mathsf{path}}^{-1}(\mathbf{A}) &= \bigcap_{r=1}^{\infty} [\ell_{\mathsf{path}}^{-1}(\mathbf{A}) \bigcap \{\mathsf{m}_r(\mathsf{X}) < \infty\}] \\ &\subset \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}(\mathsf{X}_{\mathbf{t}})] \bigcap \{\mathsf{m}_r(\mathsf{X}) < \infty\} \\ &= \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}(\mathsf{X}_{\mathbf{t}})] \subset \tilde{\mathcal{T}}_{\mathsf{path}}(\mathsf{S}) \end{aligned}$$

$\square$

l:tail3

Out line of the proof of Thm 10 :

$$\mathcal{T}(\mathsf{S}) \;\Rightarrow\; \tilde{\mathcal{T}}_{\mathsf{path}}(\mathsf{S}) \Rightarrow \tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}}) \Rightarrow \tilde{\mathcal{T}}_{\mathsf{path}}(S^{\mathbb{N}}) \text{ a.s.} \Rightarrow \mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}}) \text{ a.s.}$$

l:tt3

Lem 5. $\tilde{\mathcal{T}}_{\mathsf{path}}(\mathsf{S})$ *is* $\mathbf{P}_{\mathbf{s}}$*-trivial for* μ^{ℓ}*-a.s.* $\mathbf{s}$.

Proof. Easy. □

l:tail3

Out line of the proof of Thm 10 :

$$\mathcal{T}(\mathsf{S}) \;\Rightarrow\; \tilde{\mathcal{T}}_{\text{path}}(\mathsf{S}) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \text{ a.s.} \Rightarrow \mathcal{T}_{\text{path}}(S^{\mathbb{N}}) \text{ a.s.}$$

l:tt4

Thm 11. *$\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is $\mathbf{P}_{\mathbf{s}}$-trivial for μ^{ℓ}-a.s. $\mathbf{s}$.*

The difficulty is that the σ-field $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is not countbly determined because tail fields are not topologcally well behaved. But if we restrict the support of F^{∞}, then $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is coutably determined under $\mathbf{P}_{\mathbf{s}}$.

- Recall the map

$$F^{\infty}(\mathbf{s}, \mathbf{X}, \mathbf{B}) = \mathbf{Y}$$

given by the IFC solution of ISDE.

- Then $\mathbf{X}$ is a weak solution.
- Hence $\mathbf{X}$ is a fix point:

$$F^{\infty}(\mathbf{s}, \mathbf{X}, \mathbf{B}) = \mathbf{X}.$$

1:tail3

Out line of the proof of Thm 10 :

For a measurable space $(U,\mathcal{U})$ we call a subset $\mathcal{V} \subset \mathcal{U}$ a determination class of $(U,\mathcal{U})$ if any two probability measures P and Q on $(U,\mathcal{U})$ are equal if and only if $P(A) = Q(A)$ for all $A \in \mathcal{V}$.

1:ifc2

Lem 6. *Let $\mathcal{V} = \{V_n\}_{n\in\mathbb{N}}$ be a countable determination class of $(U,\mathcal{U})$. Let m be a probability measure on $(U,\mathcal{U})$. Then*

$$m(V_n) \in \{0,1\} \text{ for all } V_n \in \mathcal{V} \Rightarrow m(A) \in \{0,1\} \text{ for all } A \in \mathcal{U}.$$

Proof. Let $N(1) = \{n \in \mathbb{N}; m(V_n) = 1\}$. If $N(1) = \emptyset$, then m is the zero measure. If $N(1) \neq \emptyset$, then we take

$$V = \Big(\bigcap_{n\in N(1)} V_n\Big) \bigcap \Big(\bigcap_{n\notin N(1)} V_n^c\Big).$$

Clearly, we obtain $m(V) = 1$.

Let $A \in \mathcal{U}$. Suppose that $V \cap A \notin \{\emptyset, V\}$. Then we can not determine the value of $m(V \cap A)$ by the value of $m(V_n)$ $(n \in \mathbb{N})$. This yields contradiction. Hence $V \cap A \in \{\emptyset, V\}$. If $V \cap A = \emptyset$, then $m(A) = 0$. If $V \cap A = V$, then $m(A) \geq m(V) = 1$. $\square$

l:tail3

Out line of the proof of Thm 10 :

- Since $S^{\mathbb{N}}$ is Polish, $\exists$ a countable dense set $S_0^{\mathbb{N}} = \{\mathbf{s}_k\}$. Let

$$\mathcal{U} = \cup_{k=1}^{\infty} \mathcal{A}[U_r(\mathbf{s}_k); 0 < r \in \mathbb{Q},\ k \in \mathbb{N}]$$

Here $U_r(\mathbf{s})$ is a ball centered at $\mathbf{s}$ with radius r, $\mathcal{A}[\cdot]$ denotes the algebra generated by $\cdot$. Let

$$\mathcal{V} = \cup_{j=1}^{\infty} \{(\mathbf{X}_{\mathbf{t}})^{-1}(\mathbb{A})\,;\ \mathbf{A} \in \mathcal{U}^j,\ \mathbf{t} \in \{\mathbf{Q} \cap [0,T]\}^j\}.$$

l:ifc3 Then $\mathcal{V}$ becomes a countable determination class of $(W(S^{\mathbb{N}}), \mathcal{B}(W(S^{\mathbb{N}})))$.

Lem 7. *Then, for each* $\mathbf{V} \in \mathcal{V}$,

$$F^{\infty}(\mathbf{s}, \cdot, \mathbf{B})^{-1}(\mathbf{V}) \bigcap W_{T,\text{fix}} \in \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \bigcap W_{T,\text{fix}} \qquad \textit{for } P_{\text{Br}}^{\infty}\textit{-a.s. } \mathbf{B}$$

l:ifc4

Lem 8. *For each* $\mathbf{A} \in \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}})$,

$$\bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) \in \{0,1\} \qquad \textit{for } P_{\text{Br}}^{\infty}\textit{-a.s. } \mathbf{B}$$

Proof of Thm 11: l:tt4

Thm 11 l:tt4 $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ is $\mathbf{P}_{\mathbf{s}}$-trivial for μ^{ℓ}-a.s. $\mathbf{s}$.

Proof. From $F^{\infty}(\mathbf{s}, \mathbf{X}, \mathbf{B}) = \mathbf{X}$, we deduce that, for P^{∞}_{Br}-a.s. $\mathbf{B}$, :45a

$$\bar{P}_{\mathbf{s},\mathbf{B}} \circ F^{\infty}(\mathbf{s}, \cdot, \mathbf{B})^{-1} = \bar{P}_{\mathbf{s},\mathbf{B}}. \tag{16}$$

- From Lem 7 l:ifc3, Lem 8 l:ifc4, and $F^{\infty}(\mathbf{s}, \cdot, \mathbf{B})(W_{T,\mathrm{fix}}) = 1$, for all $\mathbf{V} \in \mathcal{V}$,

$$\bar{P}_{\mathbf{s},\mathbf{B}} \circ F^{\infty}(\mathbf{s}, \cdot, \mathbf{B})^{-1}(\mathbf{V}) \in \{0, 1\} \quad \text{for } P^{\infty}_{\mathrm{Br}}\text{-a.s. } \mathbf{B}.$$

Since $\mathcal{V}$ is countable, we deduce that, for P^{∞}_{Br}-a.s. $\mathbf{B}$, :45b

$$\bar{P}_{\mathbf{s},\mathbf{B}} \circ F^{\infty}(\mathbf{s}, \cdot, \mathbf{B})^{-1}(\mathbf{V}) \in \{0, 1\} \quad \text{for all } \mathbf{V} \in \mathcal{V}. \tag{17}$$

Since $\mathcal{V}$ l:ifc2 is a countable determination class, we obtain from (17) :45b and Lem 6 that :45c

$$\bar{P}_{\mathbf{s},\mathbf{B}} \circ F^{\infty}(\mathbf{s}, \cdot, \mathbf{B})^{-1}(\mathbf{A}) \in \{0, 1\} \quad \text{for all } \mathbf{A} \in \mathcal{B}(W(S^{\mathbb{N}})). \tag{18}$$

Hence we deduce that

$$\bar{P}_{\mathbf{s},\mathbf{B}} \circ F^{\infty}(\mathbf{s}, \cdot, \mathbf{B})^{-1} = \delta_{\mathbf{X}} \text{ for some } \mathbf{X} = \mathbf{X}(\mathbf{s}, \mathbf{B}) \in W(S^{\mathbb{N}}).$$

In particular, $\mathbf{X}$ is a function of $(\mathbf{s}, \mathbf{B})$. This combined with (16) :45a implies that $\bar{P}_{\mathbf{s},\mathbf{B}} = \delta_{\mathbf{X}(\mathbf{s},\mathbf{B})}$, and $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ is $\mathbf{P}_{\mathbf{s}}$-trivial for μ^{ℓ}-a.s.. □

END