Palm resolution and restore density formulae of the Ginibre random point field

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• The Ginibre random point field is a probability measure on the configuration space over the complex plane ($\mathbb{R}^2$).

• It is rotation and translation invariant, and is the limit of the distributions of the eigenvalues of the non-hermitian Gaussian random matrices, called Ginibre ensembles.

• It describes the system of infinite-many particles interacting via the two dimensional Coulomb potential (logarithmic potential) with inverse temperature beta = 2.

$$\Psi(x) = -\log|x|.$$

• Since $2D$ Coulomb pot is very strong at infinity, phenomena different from usual Gibbs measures emerge.

Set up : Ginibre random point field

Let $S = \mathbb{C}$. We regard S as $\mathbb{R}^2$.

S: configuration space over S

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\, ;\, s_i \in S,\ \mathsf{s}(|s| < r) < \infty\ (\forall r \in \mathbb{N})\}$$

- S is a Polish space with the vague topology.
- A prob measure μ on S is called random point field (point process) on S.

- S is a set of unlabeled particles.
- $S^{\mathbb{N}}$ is the space of labeled particles.

Definition of Ginibre RPF

- A symmetric function ρ^n is called the n-correlation function of μ w.r.t. Radon m. m if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(\mathbf{x}_n) \prod_{i=1}^{n} m(dx_i) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!} d\mu$$

for any disjoint $A_i \in \mathcal{B}(S)$, $k_i \in \mathbb{N}$ s.t. $k_1 + \ldots + k_m = n$.

- μ is called the determinantal RPF generated by (K, m) if its n-correlation fun. ρ^n is given by

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{1\le i,j\le n}$$

- Ginibre RPF $S = \mathbb{C}$. μ_{gin} is generated by $(K_{\mathsf{gin}}, \mathrm{g})$

$$K_{\mathsf{gin}}(x,y) = e^{x\bar{y}} \qquad \mathrm{g}(dx) = \pi^{-1} e^{-|x|^2} dx$$

- A simulation is as follows:

Basic properties of Ginibre RPF

- μ_{gin} is rotation and translation invariant.
- μ_{gin} has a N-particle approximation μ_{gin}^N, i.e.

$$\lim_{N\to\infty} \mu_{\text{gin}}^N = \mu_{\text{gin}} \qquad \text{weakly}$$

and μ_{gin}^N is the RPF supported on the N-particles defined by the labeled density m^N

$$m^N(\mathbf{x}_N) = \frac{1}{\mathcal{Z}} \prod_{i<j}^{N} |x_i - x_j|^2 \prod_{k=1}^{N} \mathrm{g}(x_k) d\mathbf{x}_N$$

- μ_{gin}^N is the distribution of the eigen values of non-Hermitian Gaussian random matricies called Ginibre ensemble.

Two intuitive representations of Ginibre RPF

- From the N-particle density

$$m^N(\mathbf{x}_N) = \frac{1}{\mathcal{Z}} \prod_{i<j}^{N} |x_i - x_j|^2 \prod_{k=1}^{N} \mathrm{g}(x_k) \prod_{l=1}^{N} dx_l,$$

we have the first intuitive representation of Ginibre RPF:

$$\mu_{\mathsf{gin}} \sim \frac{1}{\mathcal{Z}} \prod_{i<j}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} \mathrm{g}(x_k) \prod_{l=1}^{\infty} dx_l. \tag{1}$$

- Taking the translation invariance of μ_{gin} into account, we have the second intuitive representation:

$$\mu_{\mathsf{gin}} \sim \frac{1}{\mathcal{Z}} \prod_{i<j}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} dx_k. \tag{2}$$

- These two informal expressions can be justified. This fact is one of geometric rigidities.

Absolute continuity & singularity of reduced Palm measures.

Another geometric rigidity of Ginibre RPF

- I talk about a property that Gibbs measures never have, and is very specific for Ginibre RPF.

(reduced) Palm meas. For a set of m-points $\mathbf{x} = \{x_1, \dots, x_{\mathsf{m}}\}$ let

$$\mu_{\mathbf{x}} := \mu(\,\cdot - \sum_{l=1}^{\mathsf{m}} \delta_{x_l} \mid \mathsf{s}(\{x_l\}) \geq 1 \quad (l = 1, \dots, \mathsf{m}))$$

Thm 1 (O.-Shirai 14). *Let* $\mathsf{m}, \mathsf{n} \in \{0\} \cup \mathbb{N}$. *Then*

(1) *If* $\mathsf{m} = \mathsf{n}$, *then* $\mu_{\mathrm{gin},\mathbf{x}}$ *and* $\mu_{\mathrm{gin},\mathbf{y}}$ *are mutually ab. cont..*

(2) *If* $\mathsf{m} \neq \mathsf{n}$, *then* $\mu_{\mathrm{gin},\mathbf{x}}$ *and* $\mu_{\mathrm{gin},\mathbf{y}}$ *are singular each other.*

- (2) shows a special property of Ginibre rpf. Indeed,

Λ Poisson rpf $\Rightarrow \Lambda_{\mathbf{x}} = \Lambda$

ν Gibbs meas with Ruelle's class potentials $\Rightarrow \nu_{\mathbf{x}} \prec \nu$

- ν periodic rpf $\Rightarrow$ (2) holds

Thm 2 (O.-Shirai). *Let* $\mathsf{m} = \mathsf{n}$. *Then for* $\mu_{\mathsf{gin},\mathbf{y}}$*-a.s.* s

$$\frac{d\mu_{\mathsf{gin},\mathbf{x}}}{d\mu_{\mathsf{gin},\mathbf{y}}} = \frac{1}{\mathcal{Z}_{\mathbf{xy}}} \lim_{r\to\infty} \prod_{|s_i|<b_r} \frac{|\mathbf{x}-s_i|^2}{|\mathbf{y}-s_i|^2} \quad (\mathsf{s} = \sum_i \delta_{s_i}) \tag{3}$$

compact uniformly in $\mathbf{x} \in \mathbb{C}^{\mathsf{m}}$, $\mathbf{y} \in \mathbb{C}^{\mathsf{m}} \backslash \{s_1, \ldots, s_{\mathsf{m}}\}$

$$\mathcal{Z}_{\mathbf{xy}} = \frac{\Delta^2(\mathbf{y}) \det[K_{\mathsf{gin}}(x_i, x_j)]_{i,j=1}^{\mathsf{m}}}{\Delta^2(\mathbf{x}) \det[K_{\mathsf{gin}}(y_i, y_j)]_{i,j=1}^{\mathsf{m}}}$$

$$\Delta^2(\mathbf{x}) = \prod_{i<j}^{\mathsf{m}} |x_i - x_j|^2, \qquad |\mathbf{x} - s_i| = \prod_{m=1}^{\mathsf{m}} |x_m - s_i|$$

$$\{b_r\}_{r\in\mathbb{N}} : \qquad b_r \uparrow \infty$$

Let $D_{\sqrt{q}} = \{z \in \mathbb{C}\,;\, |z| < \sqrt{q}\}$,

$$F_r(\mathsf{s}) = \frac{1}{r}\sum_{q=1}^{r}(\mathsf{s}(D_{\sqrt{q}}) - q). \tag{4}$$

By definition $\mathsf{s}(D_{\sqrt{q}})$ is the number of particles $\mathsf{s} = \sum_i \delta_{s_i}$ in the disk $D_{\sqrt{q}}$.

Thm 3. *Let* $\mathbf{x} = (x_1, \ldots, x_{\mathsf{m}})$*.*

$$\lim_{r\to\infty} F_r(\mathsf{s}) = -\mathsf{m} \qquad \textit{weakly in } L^2(\mathsf{S}, \mu_{\mathbf{x}}) \tag{5}$$

- The 3 means we can determine the number of missing particles.

So

$$\infty - \mathsf{m} \neq \infty$$

Palm decomposition

- Set the Palm measure conditioned at $\mathsf{a} = \sum_{i=1}^{\mathsf{m}} \delta_{a_i}$ by

$$\mu_{\mathsf{gin},\mathsf{a}} = \mu_{\mathsf{gin}}(\cdot - \sum_{i=1}^{\mathsf{m}} \delta_{a_i} | \mathsf{s}(\{a_i\}) \geq 1).$$

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- From Thm 1, we see $\exists \mathsf{S}^{(\mathsf{m})}$ $(\mathsf{m} \in \{0\} \cup \mathbb{N})$ such that

$$\mu_{\mathsf{gin,a}}(\mathsf{S}^{(\mathsf{m})}) = 1 \qquad \text{if } \#\mathsf{a} = \mathsf{m}$$
$$\mathsf{S}^{(\mathsf{m})} \cap \mathsf{S}^{(\mathsf{n})} = \emptyset \qquad \text{if } \mathsf{m} \neq \mathsf{n}.$$

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- So we have a Palm decomposition of the space as

$$\mathsf{S} = \sum_{\mathsf{m}=0}^{\infty} \mathsf{S}^{(\mathsf{m})} + \text{residual part}$$

Thm 4 (O.). *Palm resolution: Let* $\sharp \mathsf{a} = \mathsf{m}$ *and* $\mathsf{m} \in \mathbb{N}$.

(1) *There exists a probability kernel* $\tau_{\mathsf{a}}^{\mathsf{m}}(d\mathbf{x}, \mathsf{s})$ *s.t.*

$$\int_{(\mathbb{R}^2)^{\mathsf{m}}} \tau_{\mathsf{a}}^{\mathsf{m}}(d\mathbf{x}, \mathsf{s}) d\mathbf{x} = 1 \qquad \text{for } \mu_{\mathsf{gin},\mathsf{a}}\text{-a.s. } \mathsf{s}, \tag{6}$$

and that the prob m ν_{a} *on* $(\mathbb{R}^2)^{\mathsf{m}} \times \mathsf{S}$ *defined as*

$$\nu_{\mathsf{a}}(d\mathbf{x} d\mathsf{s}) = \tau_{\mathsf{a}}^{\mathsf{m}}(d\mathbf{x}, \mathsf{s}) \mu_{\mathsf{gin},\mathsf{a}}(d\mathsf{s}) \tag{7}$$

satisfies $(\mathbf{x} = (x_1, \ldots, x_{\mathsf{m}}))$

$$\mu_{\mathsf{gin}} = \text{ Law of } \{\sum_{i=1}^{\mathsf{m}} \delta_{x_i} + \mathsf{s}\} \qquad \text{under } \nu_{\mathsf{a}}. \tag{8}$$

(2) $\tau_{\mathsf{a}}^{\mathsf{m}}(d\mathbf{x}, \mathsf{s})$ *is unique up to* $\mu_{\mathsf{gin},\mathsf{a}}(d\mathsf{s})$*-a.e..*

Thm 5 (O.). *Let $\bar{\tau}_{\mathsf{a}}^{\mathsf{m}}(d\mathbf{x})$ be the averaged kernel:*

$$\bar{\tau}_{\mathsf{a}}^{\mathsf{m}}(d\mathbf{x}) = \int_{\mathsf{S}} \tau_{\mathsf{a}}^{\mathsf{m}}(d\mathbf{x}, \mathsf{s}) \mu_{\mathsf{gin,a}}(d\mathsf{s})$$

Let $\mathsf{g}(x) = \frac{1}{\pi} e^{-|x|^2}$, *and* $\rho_{\mathsf{a}}^{\mathsf{m}}$ *is the* m*-correlation fun of* $\mu_{\mathsf{gin,a}}$. *Then* $\bar{\tau}_{\mathsf{a}}^{\mathsf{m}}(d\mathbf{x})$ *has a density* $\bar{\tau}_{\mathsf{a}}^{\mathsf{m}}(d\mathbf{x}) = \bar{\tau}_{\mathsf{a}}^{\mathsf{m}}(\mathbf{x}) d\mathbf{x}$ *satisfying*

$$\bar{\tau}_{\mathsf{a}}^{\mathsf{m}}(\mathbf{x}) = \{\rho^{\mathsf{m}}(\mathbf{x}) - \rho_{\mathsf{a}}^{\mathsf{m}}(\mathbf{x})\} \mathsf{g}^{\mathsf{m}}(\mathbf{x}). \tag{9}$$

In particular, if $\mathsf{a} = \mathsf{m}\delta_0$, *then*

$$\bar{\tau}_{\mathsf{m}\delta_0}(\mathbf{x}) = \mathsf{g}^{\mathsf{m}}(\mathbf{x})$$

Remark 1. (1) Translation invariant Gibbs measures with Ruelle's class potentials and, in particular, Poisson RPFs with Lebesgue intensity, do not have such a decomposition.
(2) Goldman (2010, AAP) proves an existence of such a kernel for the case $\mathsf{m} = 1$.
(3) Thms 4 implies the coupling (stochastic domination) between $\mu_{\mathsf{gin,a}}$ and μ_{gin}.

Application of Palm resolution to the explicit representation of global Palm densities of μ_{gin}

For $\mathsf{s} \in \mathsf{S}^{(\mathsf{m})}$ let $\sigma^{\mathsf{m}}(d\mathsf{x}, \mathsf{s})$ be a regular conditional dist.

$$\sigma^{\mathsf{m}}(d\mathsf{x}, \mathsf{s}) = \mu_{\mathsf{gin}}(d\mathsf{x} - \mathsf{s} | \mathsf{s}(\{s_i\}) \geq 1) \qquad \text{for } \mathsf{s} = \sum_i^{\infty} \delta_{s_i}$$

By Palm resolution $\sigma^{\mathsf{m}}(d\mathsf{x}, \mathsf{s})$ supported on the set:

$$\mathsf{x}(\mathbb{R}^2) = \mathsf{m} \qquad \text{for } \sigma^{\mathsf{m}}(d\mathsf{x}, \mathsf{s})\text{-a.s..}$$

We prove σ^{m} has a label density $\sigma^{\mathsf{m}}(\mathbf{x}, \mathsf{s})$ on $\mathbb{R}^{2\mathsf{m}}$.

$$\sigma^{\mathsf{m}}(d\mathsf{x},\mathsf{s})dx = \mu_{\mathsf{gin}}(d\mathsf{x}-\mathsf{s}|\mathsf{s}(\{y_i\}) \geq 1\ (i \in \mathbb{N})) \qquad \text{for } \mathsf{s} = \sum \delta_{s_i}$$

Thm 6 (O.). *The label density* $\sigma^{\mathsf{m}}(\mathbf{x},\mathsf{s})$ *exists, and given by*

$$\sigma^{\mathsf{m}}(\mathbf{x},\mathsf{s}) = \frac{1}{\mathcal{Z}(\mathsf{s})} \lim_{r\to\infty} \{ \prod_{|s_i|\leq r} |1-\frac{\mathbf{x}}{s_i}|^2 \} \Delta^2(\mathbf{x}) \mathrm{g}^{\mathsf{m}}(\mathbf{x}) \tag{10}$$

for $\mu_{\mathsf{gin,a}}$*-a.s.* s *and strongly in* $L^2(\mu_{\mathsf{gin,a}})$ *for all* $\mathbf{x} \in (\mathbb{R}^2)^{\mathsf{m}}$, *and also compact uniformly in* $\mathbf{x} = (x_n)$ *for* $\mu_{\mathsf{gin,a}}$*-a.s.* s. *Here*

$$0 < \mathcal{Z}(\mathsf{s}) < \infty \tag{11}$$

$$|1-\frac{\mathbf{x}}{s_i}| = \prod_{n=1}^{\mathsf{m}} |1-\frac{x_n}{s_i}|, \qquad \Delta^2(\mathbf{x}) = \prod_{i<j}^{\mathsf{m}} |x_i - x_j|^2.$$

- $\sigma^{\mathsf{m}}(\mathbf{x},\mathsf{s})$ *are locally Lipschitz cont in* $\mathbf{x} \in (\mathbb{R}^2)^{\mathsf{m}}$ *for* $\mu_{\mathsf{gin,a}}$*-a.s.* s.
- $\sigma^{\mathsf{m}}(\mathbf{x},\mathsf{s})$ *are translation invariant:*

$$\sigma^{\mathsf{m}}(\mathbf{x},\mathsf{s}) = \sigma^{\mathsf{m}}(\theta_h\mathbf{x},\theta_h\mathsf{s}) \qquad \text{for all } h \in \mathbb{R}^2. \tag{12}$$

Here θ_h *denotes the translations on* $(\mathbb{R}^2)^{\mathsf{m}}$ *and* S.

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