Strong solutions of infinite-dimensional stochastic differential equations and tail theorems.

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- Interacting Brownian motions in infinite-dimensions $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ are stochastic dynamics in $(\mathbb{R}^d)^{\mathbb{N}}$ given by ISDE

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j\in\mathbb{N},\, j\neq i} \nabla\Psi(X_t^i - X_t^j)dt \quad (i \in \mathbb{N})$$

Here Ψ is an interaction potential and β is inverse temperature.
This ISDE has been studied by Lang, Fritz, Tanemura, and others.
They construct strong solutions.

- So far Ψ is taken to be $C_0^3(\mathbb{R}^d)$ or exponential decay at infinity.
- Itô scheme (Picard approximation) is used here.

Intro.

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j \in \mathbb{N}, j \neq i} \nabla \Psi(X_t^i - X_t^j) dt \quad (i \in \mathbb{N})$$

- There many interesting potentials Ψ with polynomial decay or unbounded at infinity:
- These are excluded by the classical approach based on Itô scheme.
- In this talk, we present a new scheme applicable to polynomial decay or logatithmic potentials:

$$\Psi(x) = -\log|x|.$$

This appears in random matrix theory and vortex dynamics. If $d = 1$, $\beta = 2$, and Ψ is as above, then the ISDE is

$$dX_t^i = dB_t^i + \sum_{j \neq i} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{N}).$$

Intro.

- Itô scheme uses Lipschitz continuity of coefficients, which does not hold in infinite dimensions.
- We localize ISDE with increasing sets H_k and exit times τ_{H_k} such that coefficients are Lipschits continuous on each H_k and that

$$\lim_{k \to \infty} \tau_{H_k} = \infty.$$

- Since ISDEs like as

$$dX_t^i = dB_t^i + \sum_{j \neq i} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{N}).$$

are complicated, it is hard to find out such a sequence of subsets $\{H_k\}$. We give an algorithm to find out such sets by Dirichlet form theory and tail analysis. (In our theoem, exit times do not appear).

Intro.

- The purpose of the talk is to present a general theory to

  Unique, strong solutions of ISDEs of
  interacting Brownian motions in infinite-dimensions.

- Our scheme consists of

  Dirichlet form theory

  + Itô scheme for ∞ many finite dim SDEs with consistency
    (IFC solutions)

  + analysis of tail σ-fields (tail theorems)

- First construct weak solutions by Dirichlet form theory.
- Second, lift them strong solutions thrgouh
  "IFC solutions" and "tail theorems".

Examples.

We begin by showing examples which our theorem can apply to:

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Lennard-Jones 6-12 potential

Let $\Psi_{6,12}(x) = \beta\{|x|^{-12} - |x|^{-6}\}$, where $d = 3$ and $\beta > 0$ is a inverse temperature. $\Psi_{6,12}$ is called the Lennard-Jones 6-12 potential. The corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j\neq i}^{\infty} \{\frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}}\}dt \quad (i \in \mathbb{N}).$$

Coulomb like potentials (not Coulomb!)

Let $a > d$ and set $\Psi_a(x) = (\beta/a)|x|^{-a}$, where $\beta > 0$.

Then the corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j\neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}}dt \quad (i \in \mathbb{N}). \tag{1}$$

Sine rpf & Ginibre interacting Brownian motions in infinite-dimensions.

Let Ψ be the 2D Coulomb (logarithmic potential):

$$\Psi(x) = -\log |x|.$$

Sine$_\beta$ RPF: $d = 1$, $\beta = 1, 2, 4$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{j\neq i,\, |X_t^i - X_t^j|<r}^{\infty} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{N}).$$

Ginibre RPF: $d = 2$ and $\beta = 2$.

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{j\neq i,\, |X_t^i - X_t^j|<r}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N}). \tag{2}$$

We call $\mathbf{X} = (X_t^i)_{i\in\mathbb{N}}$ the Ginibre interacting Brownian motions (IBMs) in infinite-dimensions.

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Set up: Ginibre interacting Brownian motions in infinite-dimensions.

- This is very complicated SDEs:

$$dX_t^1 = dB_t^1 + \lim_{r\to\infty} \sum_{j\neq 1,\, |X_t^1 - X_t^j| < r}^{\infty} \frac{X_t^1 - X_t^j}{|X_t^1 - X_t^j|^2} dt$$

$$dX_t^2 = dB_t^2 + \lim_{r\to\infty} \sum_{j\neq 2,\, |X_t^2 - X_t^j| < r}^{\infty} \frac{X_t^2 - X_t^j}{|X_t^2 - X_t^j|^2} dt$$

$$dX_t^3 = dB_t^3 + \lim_{r\to\infty} \sum_{j\neq 3,\, |X_t^3 - X_t^j| < r}^{\infty} \frac{X_t^3 - X_t^j}{|X_t^3 - X_t^j|^2} dt$$

$$dX_t^4 = dB_t^4 + \lim_{r\to\infty} \sum_{j\neq 4,\, |X_t^4 - X_t^j| < r}^{\infty} \frac{X_t^4 - X_t^j}{|X_t^4 - X_t^j|^2} dt$$

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Weak solutions of ISDEs: Quasi-Gibbs measures and Logarithmic derivative.

Ψ-Quasi-Gibbs meas.

- $S = \mathbb{R}^d$, $S_r = \{|x| \le r\}$, $\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i},\ \mathsf{s}(S_r) < \infty (\forall r)\}$
- $\pi_r, \pi_r^c : \mathsf{S} \to \mathsf{S}$, $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r)$, $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$

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- Let μ be a RPF over S.

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(S_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi))$$

- Let $\Psi : S \to \mathbb{R} \cup \{\infty\}$ (interaction).

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i - s_j)$$

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Def: μ is Ψ-quasi-Gibbs measure if $\exists$ $c_{r,\xi}^m$ s.t.

$${c_{r,\xi}^m}^{-1}e^{-\mathcal{H}_r}d\Lambda_r^m\le\mu_{r,\xi}^m\le c_{r,\xi}^m e^{-\mathcal{H}_r}d\Lambda_r^m$$

Here $\Lambda_r^m=\Lambda(\cdot|\mathsf{s}(S_r)=m)$ and Λ_r is the Poisson RPF with $1_{S_r}dx$.

- Gibbs measures $\Rightarrow$ Quasi-Gibbs measure .

Application of quasi-Gibbs property to dynamics

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ.

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

Here $S_r = \{|x| < r\}$, $\mathsf{S}_r^k = \{\mathsf{s}(S_r) = k\}$, σ_r^k is k-density fun on S_r.

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(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ. $\Rightarrow$ (closability)

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Let $\mathcal{D}_0$ be the set of local, smooth functions on S.

Let $\tilde{f}(s_1, \ldots) = f(\mathsf{s})$, where $\tilde{f}$ is symmetric, $\mathsf{s} = \sum \delta_{s_i}$.

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g]\mu(d\mathsf{s}), \;\; \mathbb{D}[f,g] = \frac{1}{2}\sum_i \nabla_i \tilde{f} \cdot \nabla_i \tilde{g}$$

$$\mathcal{D}_0^{\mu} = \{f \in \mathcal{D}_0; \mathcal{E}^{\mu}(f,f) < \infty, f \in L^2(\mu)\}$$

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Thm 1. (1) (A1) $\Rightarrow$ $(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$ *is closable on* $L^2(\mu)$.

(2) (A1), (A2) $\Rightarrow$ $\exists$ *diffusion* $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ *associated with*

$(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$ *on* $L^2(\mu)$.

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(A3) $\{X_t^i\}$ do not collide each other (non-collision)

(A4) each tagged particle X_t^i never explode (non-explosion)

By (A3) and (A4) the labeled dynamics

$$\mathbf{X}_t = (X_t^1, X_t^2, \ldots)$$

can be constructed from the unlabeled dynamics

$$\mathsf{X}_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i}.$$

Indeed, the particles keep the initial label forever.

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To represent $\mathbf{X}_t$ by ISDEs, we introduce the log derivative of μ.

Log derivative of μ: precise correspondence between RPFs & potentials

- Let μ_x be the (reduced) Palm m. of μ conditioned at x

$$\mu_x(\cdot) = \mu(\cdot - \delta_x | \mathsf{s}(x) \geq 1)$$

- Let μ^1 be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^1(A \times B) = \int_A \rho^1(x) \mu_x(B) dx$$

- $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathsf{S}, \mu^1)$ is called the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^1 = - \int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}^\mu d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_\circ$$

Here ∇_x is the nabla on $\mathbb{R}^d$, $\mathcal{D}_\circ$ is the space of bounded, local smooth functions on S.

- Very informally

$$\mathsf{d}^\mu = \nabla_x \log \mu^1$$

Log derivative

- If $\mu^1(dxd\text{s}) = m(x, s_1, \ldots)dx \prod_i ds_i$, then

$$\begin{aligned} & -\int \nabla_x f(x, s_1, \ldots)\mu^1(dxds_1 \cdots) \\ &= -\int \nabla_x f(x, s_1, \ldots)m(x, s_1, \ldots)dx \prod_i ds_i \\ &= \int f(x, s_1, \ldots)\nabla_x m(x, s_1, \ldots)dx \prod_i ds_i \\ &= \int f(x, s_1, \ldots)\frac{\nabla_x m(x, s_1, \ldots)}{m(x, s_1, \ldots)} m(x, s_1, \ldots)dx \prod_i ds_i. \end{aligned}$$

Hence

$$\mathsf{d}^{\mu} = \frac{\nabla_x m(x, s_1, \ldots)}{m(x, s_1, \ldots)} = \nabla_x \log m(x, s_1, \ldots).$$

This is very informal calculation.

General theorems on infinite-dim SDEs

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Thm 2. (O.12(PTRF)) (A1)–(A5) $\Rightarrow$ $\exists \mathsf{S}_0 \subset \mathsf{S}$ *such that*

$$\mu(\mathsf{S}_0) = 1,$$

and that, for $\forall \mathsf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, $\exists \mathfrak{u}^{-1}(\mathsf{S}_0)$*-valued pr.* $(X_t^i)_{i\in\mathbb{N}}$ *and* $\exists S^{\mathbb{N}}$*-valued Brownian m.* $(B_t^i)_{i\in\mathbb{N}}$ *satisfying*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

Here $\mathfrak{u}\colon S^{\mathbb{N}} \to \mathsf{S}$ such that $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$.

General theorems on infinite-dim SDEs

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The solution $(\mathbf{X}, \mathbf{B})$ is not a strong solution.

We next construct a strong solution from a weak solution.

Strong solutions

- To construct strong solutions

we have two important geometric properties of RPFs.

Tail triviality & Tail decomposition

Let $\mathcal{T} = \mathcal{T}(\mathsf{S})$ be the tail σ field of S:

$$\mathcal{T}(\mathsf{S}) = \cap_{r=1}^{\infty}\sigma[\pi_r^c] \quad (\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)).$$

Lem 1. *Let μ be a det RPF. Then $\mathcal{T}(\mathsf{S})$ is μ-trivial.*

- Lem 1 is a generalization of that for the discrete determinantal RPFs due to Russel Lyons, Shirai-Takahashi.

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Lem 1 Let μ be a det RPF. Then $\mathcal{T}(\mathsf{S})$ is μ-trivial.

Lem 2. *Let μ be a quasi-Gibbs measure. Let $\mu(\cdot|\mathcal{T})$ be the regular conditional probability. Then*

$$\mu(\cdot) = \int_{\mathsf{S}} \mu(\cdot|\mathcal{T})(\xi)\mu(d\xi)$$

and, for μ-a.s. ξ,

$$\mu(A|\mathcal{T})(\xi) = 1_A(\xi) \qquad \textit{for any } A \in \mathcal{T}.$$

- Lem 2 is a generalization of that for the discrete Gibbs m due to Georgii.

existence of strong solution

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

We consider a condition such that the drifts $\mathsf{d}^{\mu}(x, \mathsf{s})$ are locally Lipschitz continuous in x.

Let $S_r = \{|x| < r\}$ and

$$\mathsf{H}(r,n) = \{\mathsf{s} = \sum_i \delta_{s_i}; |\nabla_x \mathsf{d}^{\mu}(s_i, \mathsf{s} - \delta_{s_i})| < n \text{ for } \forall i \text{ s.t. } s_i \in S_r\},$$

$$\mathsf{H} = \bigcap_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \mathsf{H}(r,n).$$

(A6) $\mathrm{Cap}^{\mu}(\mathsf{H}^c) = 0$.

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Thm 3 (O.-Tanemura). (A1)–(A6). $\Rightarrow$ (1) *The ISDE*

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has a strong solution for $\mathbf{s} = (s_i) \in S^{\mathbb{N}}$ *s.t.* $\sum_i \delta_{s_i} \in \mathsf{H}$.

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Thm 3[O.-Tanemura] (A1)–(A6). $\Rightarrow$ (1) The ISDE

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has a strong solution for $\mathbf{s} = (s_i) \in S^{\mathbb{N}}$ s.t. $\sum_i \delta_{s_i} \in \mathsf{H}$.
(2) The ass unlabeled diffusion $\mathsf{X} = \sum_i \delta_{X^i}$ satisfies

$$P_{\mu_\xi} \circ \mathsf{X}_t^{-1} \prec \mu_\xi \quad (\forall t) \quad \text{for } \mu\text{-a.s. } \xi$$

Here $\mu_\xi = \mu(\cdot|\mathcal{T}(\mathsf{S}))(\xi)$ in Lem 2.

By construction $\mu(\cdot|\xi)(A)$ are $\mathcal{T}$-measurable functions in a for each $A \in \mathcal{B}(S)$. By Lem 2 one can take a version of $\mu(\cdot|\xi)$ such that, for μ-a.s. $a \in S$,

$$\mu(\cdot|\xi)(A) = 1_A(a) \qquad \text{for all } A \in \mathcal{T}. \tag{3}$$

Let $\sim_{\mathcal{T}}$ be the equivalence relation such that if and only if

$$a \sim_{\mathcal{T}} b \Leftrightarrow 1_A(a) = 1_A(b) \qquad \text{for all } A \in \mathcal{T}. \tag{4}$$

From (3) we deduce that the set H in Thm 3 can be decomposed as a disjoint sum

$$H = \sum_{[a] \in H/\sim_{\mathcal{T}}} S_0^a \qquad \text{such that} \qquad \mu(\cdot|\xi)(S_0^a) = 1. \tag{5}$$

Thm 4 (O.-Tanemura). *Assume* (A1)–(A6).
Let $\mathbf{X} = (X^i)$ *and* $\hat{\mathbf{X}} = (\hat{X}^i)$ *be strong sol of the ISDE*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{\mathsf{j}\neq\mathsf{i}} \delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s} = (s_i)_{i\in\mathbb{N}}$$

on the same Br m. Let $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ *and* $\hat{\mathsf{X}}_t = \sum_i \delta_{\hat{X}_t^i}$.
Suppose, for μ*-a.s.* ξ,

$$P_{\mu_\xi} \circ \mathsf{X}_t^{-1} \prec \mu_\xi \text{ and } P_{\mu_\xi} \circ \hat{\mathsf{X}}_t^{-1} \prec \mu_\xi \ (\forall t)$$

Then

$$\mathbf{X} = \hat{\mathbf{X}} \text{ a.s.} \qquad \text{for } \mu\text{-a.s. } \mathsf{s} = \sum_{i=1}^{\infty} \delta_{s_i}$$

Thm 5 (O.-Tanemura). *Assume* (A1)–(A7). *Here*
(A7) μ *is tail trivial.*
Then the strong solution $\mathbf{X} = (X^i)$ *such that*

$$P_{\mu} \circ \mathsf{X}_t^{-1} \prec \mu \qquad \textit{for all } t$$

is unique for μ*-a.e.* $\mathsf{x} = \sum_i \delta_{x_i}$.
Here X *is the unlabeled dynamics of* $\mathbf{X}$*:*

$$\mathsf{X}_t = \sum_i^{\infty} \delta_{X_t^i}$$

Cor If μ is a determinantal RPF, then the strong, solution of the ISDE that is reversible w.r.t. μ is unique.

- Tail σ-fields of Airy, Sine, Ginibre RPFs with $\beta = 2$ are trivial.

Let $\mathcal{D}^{\mu}_{\text{poly}}$ be the closure of polynomials on S s.t. $\mathcal{E}^{\mu}_{1}(f,f) < \infty$. Then

$$\mathcal{D}^{\mu}_{\text{poly}} \subset \mathcal{D}^{\mu}$$

because polynomials are local and smooth.

Thm 6 (O.-Tanemura '14). *Assume* (A1)–(A7). *Then quasi-regular Dirichlet forms that are extension of* $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}_{\text{poly}})$ *are unique.*

In particular, $\mathcal{D}^{\mu}_{\text{poly}} = \mathcal{D}^{\mu}$, and Lang's construction and O.'s construction are same.

Remark 1. (1) Dirichlet forms here are same as those constructed by Albeverio-Kondratiev-Röckner, and Yoshida.
(2) If (A5) (non-explosion) does not hold. Then Thm 6 does not hold. This is very natural theorem that says the uniqueness of Dirichlet forms is related to the non-explosion problem of tagged problem.
(3) Proof follows from the uniqueness of solutions of ISDEs. This idea originates from Tanemura (98. PTRF).
We thus prove static property (uniqueness of q.r. Dirichlet forms) from dynamical property (uniqueness of strong solutions of ISDEs).

Outline of the proof.

Our approach consists of 6 steps:

- By the first three steps we construct weak solutions.
- By the next three steps we lift them to strong solutions and prove the pathwise uniqueness of ISDEs.

Idea to solve ISDE: $\mathsf{S} \Rightarrow C([0,\infty);\mathsf{S}) \Rightarrow C([0,\infty);S^{\mathbb{N}})$

(Step 1) • We start with a random point field μ (a probability measure on configuration space S).

• We construct μ-reversible unlabeled diffusions X by Dirichlet forms.

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}.$$

For this we introduce the map from RPF μ on S to bilinear forms :

$$\mu \mapsto \mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g]d\mu \quad \text{on } L^2(\mathsf{S},\mu).$$

Here $\mathbb{D}$ is the standard square field on S:

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_{i=1}^{\infty} \frac{\partial \tilde{f}}{\partial s_i} \cdot \frac{\partial \tilde{g}}{\partial s_i}$$

Here f is a local and smooth function on S, and $\tilde{f}(s_1,\ldots,)$ is a symmetric function such that $f(\mathsf{s}) = \tilde{f}(s_1,\ldots,)$, where $\mathsf{s} = \sum_{i=1}^{\infty} \delta_{s_i}$.

• If μ is the Poisson RPF $= \Lambda$ with Lebesgue intensity, then the associated diffusion X_t is S-valued Brownian motion $\mathsf{B}_t = \sum_{i=1}^{\infty} \delta_{B_t^i}$, which is a reason we call $\mathbb{D}$ the standard square field.
Thus this Dirichlet space is a distorted Brownian motion on S although μ does not have a density with respect to Λ usually.

• We assume:

μ is a Ψ-quasi-Gibbs measure.

Roughly speaking, quasi-Gibbs means that μ has a local density conditioned out side. Gibbs measures are of course quasi-Gibbs, and there exist RPF that are quasi-Gibbs for logarithmic potential Ψ.

• Assume that μ is Ψ-quasi-Gibbs with upper semicontinuous Ψ, and that $\sum_{m=1}^{\infty} m\mu(\mathsf{S}_r^m) < \infty$ ($\mathsf{S}_r^m = \{\mathsf{s};\mathsf{s}(S_r) = m\}$), and that m-density functions on S_r are in $L^2(S_r^m)$ for all $r, m \in \mathbb{N}$. Here $S_r = \{|s| < r\}$.

• With these assumption, the bilinear form is closable and its closure is a quasi-regular Dirichlet form.

• We thus have unlabeled diffusions.

$\mathsf{S} \Rightarrow C([0,\infty);\mathsf{S}) \Rightarrow C([0,\infty);S^{\mathbb{N}})$

(Step 2) • Assuming non-collision and non-explosion of tagged particles, we can construct labeled dynamics.
• The difficulty to construct $S^{\mathbb{N}}$-valued diffusion, there is no good measure on $S^{\mathbb{N}}$. (Hence no associated Dirichlet forms).
Even if Brownian motions, the measure should be $dx^{\mathbb{N}}$!
Hence we consider m-Campbell measure $\mu^{[\mathsf{m}]}$ of μ.
Introduce the countable family of Dirichlet forms:

$$(\mathcal{E}^{\mu^{[\mathsf{m}]}}, L^2(S^{\mathsf{m}} \times \mathsf{S}, \mu^{[\mathsf{m}]})), \quad \mathbf{X}^{[\mathsf{m}]} := (X^{\mathsf{m},1}, \ldots, X^{\mathsf{m},\mathsf{m}}, \sum_{i=\mathsf{m}+1}^{\infty} \delta_{X^{\mathsf{m},i}})$$

There is natural coupling associated diffusions. $\Rightarrow$
$X^{\mathsf{m},i}$ are independent of m. $\Rightarrow$
From this consistency we can construct the labeled diffusion on $S^{\mathbb{N}}$.

• We use unlabeled diffusion X_t to couple with these $\mathbf{X}^{[\mathsf{m}]}$.

(Step 3) Calculate the logarithmic derivative d^{μ}. ISDE becomes

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i} \delta_{X_t^j})dt$$

In the case of Ginibre, Sine$_\beta$(Dyson), Bessel, and Gibbs measures:

$$\beta\nabla\Phi(x) + \beta \lim_{r\to\infty} \sum_{j\neq i, |x-s_j|<r} \nabla\Psi(x - s_j)$$

Then we have the ISDE (weak solution):

$$dX_t^i = dB_t^i - \frac{\beta}{2}\nabla\Phi(X_t^i) - \frac{\beta}{2} \lim_{r\to\infty} \sum_{j\neq i, |X_t^i - X_t^j|<r} \nabla\Psi(X_t^i - X_t^j)dt$$

To calculate the logarithmic derivative we use finite particle approximation. In particular, orthogonal polynomials.

The shape of Airy RPF is different.

(Step 4) Introduce:

The infinite system of finite-dimensional SDEs with consistency (IFC):

Let $(\mathbf{X}, \mathbf{B})$ be a weak solution.

We regard $\mathbf{X}$ as a part of coefficients of SDEs.

For each m consider SDE of $\mathbf{Y}^{\mathsf{m}} = (Y^{\mathsf{m},1}, \dots, Y^{\mathsf{m},\mathsf{m}})$:

$$
\begin{aligned}
dY_t^{\mathsf{m},i} =& dB_t^i - \frac{\beta}{2}\nabla\Phi(Y_t^{\mathsf{m},i}) \\
& - \frac{\beta}{2}\sum_{j=1, j\neq i}^{\mathsf{m}} \nabla\Psi(Y_t^{\mathsf{m},i} - Y_t^{\mathsf{m},j})dt - \frac{\beta}{2}\sum_{j=\mathsf{m}+1}^{\infty} \nabla\Psi(Y_t^{\mathsf{m},i} - X_t^j)dt.
\end{aligned}
$$

These (time inhomogeneous, finite-dimensional) SDEs have unique strong solution (under suitable assumptions). Hence

$$
\mathbf{Y}^{\mathsf{m}} = \mathbf{X}^{\mathsf{m}} := (X^1, \dots, X^{\mathsf{m}})
$$

- We solve infinite-many finite-dimensional SDEs with consistency in stead of solving a single ISDE.

(Step 5)

- Let $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ be the tail σ-field of labeled path space w.r.t. label.

$$\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}}) = \bigcap_{\mathsf{m}=1}^{\infty} \sigma[X^{\mathsf{m}}, \ldots,].$$

- $\mathbf{Y}^{\mathsf{m}}$ is a functional of $(\mathbf{B}, (X^{\mathsf{m}+1}, \ldots,))$.

$\Rightarrow$ If $\lim_{\mathsf{m}\to\infty} \mathbf{Y}^{\mathsf{m}}$ exists, then $\sigma[\mathbf{B}] \vee \mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$-measurable.

$\Rightarrow$ Since $\lim_{\mathsf{m}\to\infty} \mathbf{Y}^{\mathsf{m}} = \mathbf{X}$, $\mathbf{X}$ is $\sigma[\mathbf{B}] \vee \mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$-measurable.

$\Rightarrow$ If $\mathcal{T}_{\mathsf{path}}(S^{\mathbb{N}})$ is trivial, then $\mathbf{X}$ is a strong solution.

- Since we see in the (Step 5) that

$$\mathbf{Y}^{\mathsf{m}} = \mathbf{X}^{\mathsf{m}} := (X^1, \ldots, X^{\mathsf{m}}),$$

$\mathbf{Y}^{\mathsf{m}}$ satisy these.

(Step 6) • We say unlabeled diffusion satisfies the absolutely continuity condition (ACC) if

$$P_{\mu}(\mathsf{X}_t \in \cdot) \prec \mu \quad \text{for all } t. \tag{ACC}$$

Thm 7 (Tail theorem 2)**.** *Assume* (ACC) *and that* S *is* μ*-tail trivial. Then* $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ *is* $P_{\mathbf{s}}(\mathbf{X} \in \cdot)$*-trivial for* μ^{ℓ}*-a.s.* $\mathbf{s}$*.*

Here ℓ *is a label, and* $P_{\mathbf{s}}(\mathbf{X} \in \cdot)$ *is the distribution of labeled path starting at* $\mathbf{s}$ *(solution of ISDE startng at* $\mathbf{s}$*).*

- Tail triviality of RPF $\Rightarrow$ tail tiriviality of labeled path space.
- We regard $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ as a *boundary condition of ISDE*.

So if it is trivial and unique, then the solution of ISDE is unique.

• Our pathwise uniqueness does not exclude the posiibility of the existence of a *tail moving* or *shock* solution. It is related to the uniqueness of Dirichlet forms (domain choice).

• We have not yet solve the non-equilibrium problem. We have not yet fully utilize the property of this method, and expect that with this we can solve the non-equilibrium problem at the lebel of Fritz (1987).

Tail triviality of μ is not a real restriction. Indeed,

Prop 1. *Determinantal RPFs (in continuous spaces) are tail trivial. In particular, Ginibre RPF is tail trivial.*

This result is a generalization of Shirai-Talahashi, and Russel Lyons for discrete spaces.

Note that RPFs appearing in random matrix theory are determinantal random point fields if $\beta = 2$. So our results provide the uniqueness for these.

Even if μ is not tail trivial, we can still apply our results to quasi-Gibbs measures because of the following result.

Prop 2. *Quasi-Gibbs measures* μ *have decomposition w.r.t. their tail* σ*-fields* $\mathcal{T}(\mathsf{S})$ *such that each components are tail trivial: For* μ*-a.s.* s

$$\mu(\mathsf{A}|\mathcal{T}(\mathsf{S}))(\mathbf{s}) = 1_{\mathsf{A}}(\mathbf{s}) \qquad \textit{for all } \mathsf{A} \in \mathcal{T}(\mathsf{S}).$$

This is an analogy of the result of Georgii on Gibbs measures on discrete spaces.

Summary:

- We regard the tail σ-field of labeled path spaces as boundary condition of ISDEs.

Hence if it is trivial, then there exists a strong solution.

- In addition, two solutions are equal if and only if their distributions on the tail σ-field are equal.
- The tail triviality of labeled path spaces follows from the tail triviality of configuration spaces.

END