Infinite-dimensional stochastic differential equations arising from random matrices

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- Random matrices /Airy RPFs
- Dynamical soft edge scaling / Infinite-dim SDEs of Airy RPFs
- General theorems of ISDEs on strong solutions:
- Dynamical rigidity of 2D Coulomb interacting Brownian motions

- The Gaussian unitary ensembles are Hermitian random matrices

$$M^N = \begin{pmatrix} m_{11} & \cdots & m_{1N} \\ \cdots & \cdots & \\ m_{N1} & \cdots & m_{NN} \end{pmatrix} \quad m_{ij} = \bar{m}_{ji}.$$

Here $m_{ij} = m_{ij}^1 + \sqrt{-1} m_{ij}^2$. $\{m_{ij}^1, m_{ij}^2, m_{kk}\}_{1 \le i \le j \le N, 1 \le k \le N}$ are independent real Gaussian r. v. with mean free, variance 1.

- The distribution of eigen values of M^N is given by ($\beta = 2$)

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z} \prod_{i<j}^{N} |x_i - x_j|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \tag{1}$$

- The dist of $N^{-1} \sum_{i=1}^N \delta_{x_i/\sqrt{N}}$ under m_β^N converges to

$$\varsigma(x)dx = \frac{1}{2\pi} \sqrt{4 - x^2} dx \qquad \text{(the semi-circle law)} . \tag{2}$$

Airy RPF: $\mu_{\mathsf{Ai},\beta}$

Take the scaling $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$ in $m_\beta^N(d\mathbf{x}_N)$. Then we have

$$\check{\mu}^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N} |s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{k=1}^{N}|2\sqrt{N}+N^{-1/6}s_k|^2} d\mathbf{s}_N.$$

Let $\mu^N_{\mathsf{Ai},\beta} = \check{\mu}^N_{\mathsf{Ai},\beta} \circ \mathfrak{u}^{-1}$. ($\mathfrak{u}(\mathbf{s}) = \sum_i \delta_{s_i}$ is the unlabeled map)

Then $\mu_{\mathsf{Ai},\beta}$ is the TDL of $\mu^N_{\mathsf{Ai},\beta}$:

$$\lim_{N\to\infty} \mu^N_{\mathsf{Ai},\beta} = \mu_{\mathsf{Ai},\beta} \qquad \text{(soft-edge scaling)}$$

We consider the natural SDE associated with

$$\check{\mu}^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^{N}|s_i - s_j|^{\beta} e^{-\frac{\beta}{4}\sum_{i=1}^{N}|2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N. \qquad (3)$$

We consider the energy form associated with (3). Then

$$\int_{\mathbb{R}^N}\frac{1}{2}\sum_{i=1}^{N}\frac{\partial f}{\partial s_i}\frac{\partial g}{\partial s_i}\, d\check{\mu}^N_{\mathsf{Ai},\beta} =$$

$$\int_{\mathbb{R}^N}\frac{1}{2}\{-\Delta f - \sum_{i=1}^{N}\frac{\partial f}{\partial s_i}(\sum_{j\neq i}^{N}\frac{\beta}{s_i - s_j} - \beta\{N^{\frac{1}{3}} + \frac{s_i}{2N^{\frac{1}{3}}}\})\}g\, d\check{\mu}^N_{\mathsf{Ai},\beta}$$

Hence we obtain the SDE of the natural N particle dynamics:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}(\{\sum_{j=1,j\neq i}^{N}\frac{1}{X_t^{N,i} - X_t^{N,j}}\} - \{N^{\frac{1}{3}} + \frac{1}{2N^{\frac{1}{3}}}X_t^{N,i}\})dt \qquad (4)$$

We will detect the limit ISDE of (4) and solve it.

Let $\varrho^N(x) = N^{\frac{1}{3}}\varsigma(xN^{-\frac{2}{3}}+2)$ be the rescaled semicircle. Then

$$\lim_{N\to\infty}\varrho^N(x) = \frac{\sqrt{-x}}{\pi}1_{(-\infty,0]}(x) =: \varrho(x)$$

$$N^{\frac{1}{3}} = \int_{\mathbb{R}} \frac{\varrho^N(x)}{-x}dx.$$

Hence we have, as $N\to\infty$,

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\{(\sum_{j=1,j\neq i}^{N} \frac{1}{X_t^{N,i}-X_t^{N,j}}) - (N^{\frac{1}{3}} + \frac{1}{2N^{\frac{1}{3}}}X_t^{N,i})\}dt$$

$$\sim dB_t^i + \frac{\beta}{2}\{(\sum_{j=1,j\neq i}^{N} \frac{1}{X_t^{N,i}-X_t^{N,j}}) - (\int_{\mathbb{R}} \frac{\varrho^N(x)}{-x}dx)\}dt$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\{(\sum_{|X_t^j|<r,j\neq i} \frac{1}{X_t^i-X_t^j}) - (\int_{|x|\leq r} \frac{\varrho(x)}{-x}dx)\}dt$$

Thm 1 (O.-Tanemura). *Let* $\beta = 1, 2, 4$.

Consider the ISDE of $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}} \in \mathbb{R}^\infty$:

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{\substack{|X_t^j|<r, \\ j\neq i}} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

$$\mathbf{X}_0 = \mathbf{s}.$$

(1) *The ISDE has a unique, strong solution for* $\mu_{\mathsf{Ai},\beta} \circ \ell^{-1}$*-a.s.* $\mathbf{s}$.

Here ℓ *is a label, i.e.* $\ell : \sum_i \delta_{s_i} \longmapsto \mathbf{s} = (s_i) \in \mathbb{R}^\infty$.

(2) *Let* $\mathsf{X}_t = \sum_{i=1}^\infty \delta_{X_t^i}$ *be the associated unlabeled dynamics.*

Then X_t *is a* $\mu_{\mathsf{Ai},\beta}$*-reversible diffusion.*

- $\mathbf{X}_t^N = (X_t^{N,i})_{i=1}^N \in \mathbb{R}^N$ (labeled N particle dynamics):

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2} \{ N^{\frac{1}{3}} + \frac{1}{2N^{\frac{1}{3}}} X_t^{N,i} \} dt$$

- $\mathbf{X}_t = (X_t^i)_{i=1}^{\infty} \in \mathbb{R}^{\infty}$ (labeled ∞ particle dynamics):

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{|X_t^j|<r,\, j\neq i} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

Thm 2 (O.-Tanemura). *Let $\beta = 1,2,4$ and $m \in \mathbb{N}$.*
Take a label of $\mathbf{X}^N$ and $\mathbf{X}$ in decreasing order: $X_t^i > X_t^{i+1}$ for all i.
Denote by $\mathbf{X}^{N,m}$ and $\mathbf{X}^m$ the first m-components of $\mathbf{X}^N$ and $\mathbf{X}$.
Suppose $\mathfrak{u}(\mathbf{X}_0^N) \sim \mu_{\mathrm{Ai},\beta}^N$ and $\mathfrak{u}(\mathbf{X}_0) \sim \mu_{\mathrm{Ai},\beta}$. Then

$$\mathbf{X}^{N,m} \text{ converge to } \mathbf{X}^m \text{ weakly in } C([0,\infty);\mathbb{R}^m).$$

• Let $\beta = 1, 2, 4$. Suppose $\mathsf{X}_0 \sim \mu_{\mathsf{Ai},\beta}$.
Then the distribution of the top particle X^1 satisfy

$$X_t^1 \sim F_\beta$$

Here F_β is the β Tracy-Widom distribution.

• So far the stochastic dynamics related to Airy RPF was constructed only for $\beta = 2$ by Johansson, Spohn, and others by the method of space-time correlation functions. This stochastic dynamics is same as that of Thm 1.

• Our result is the first time to construct infinite dimensional dynamics for β other than 2.

• When $\beta = 2$, the top particle X^1 is called the Airy process.
The Airy process is a scaling limit of various models.

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General theorems of ISDEs on strong solutions: (1) Quasi-Gibbs, (2) Log derivative.

- For an interaction $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j=1,\, j\neq i}^{\infty} \nabla\Psi(X_t^i, X_t^j)dt \quad (i \in \mathbb{N})$$

- More generally, for $b : \mathbb{R}^d \times \mathsf{S} \to \mathbb{R}^d \cup \{\infty\}$

$$dX_t^i = dB_t^i + b(X_t^i, \mathsf{X}_t^{i*})dt \quad (i \in \mathbb{N})$$

Here X_t^{i*} denotes the unlabeled particles other than X_t^i:

$$\mathsf{X}_t^{i*} = \sum_{j\neq i} \delta_{X_t^j}$$

- The ISDEs describe the motion of infinitely many particles consisting of only one species. In fact, $b(x, \mathrm{s})$ is independent of i.

- $S_r = \{s \in \mathbb{R}^d; |s| \le r\}$,
- $\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i},\ \mathsf{s}(S_r) < \infty (\forall r)\}$ (configuration space)

- S is the space of unlabeled particles (small ∞ dim)

$(\mathbb{R}^d)^{\mathbb{N}}$ is the space of labeled particles (large ∞ dim)

- $\pi_r, \pi_r^c : \mathsf{S} \to \mathsf{S}$, $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r)$, $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$.
- For a RPF μ we set the regular conditional probability:

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(S_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi))$$

- Hamiltonian on S_r:

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i - s_j)$$

Def: μ is Ψ-quasi-Gibbs measure if $\exists\ c_{r,\xi}^m$ s.t.

$$c_{r,\xi}^{m}{}^{-1} e^{-\mathcal{H}_r} d\Lambda_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\Lambda_r^m$$

Here Λ_r^m is the Poisson RPF with intensity $1_{S_r} ds$ on $\{\mathsf{s}(S_r) = m\}$.

That is, $\Lambda_r^m = \prod_{k=1}^m 1_{S_r}(s_k) ds_k/\text{permutation}$

- Gibbs measures $\Rightarrow$ Quasi-Gibbs measures.
- RPFs with log potentials are not Gibbs measures.

- μ is (canonical) Gibbs measures if

$$\mu_{r,\xi}^m = \frac{1}{\mathcal{Z}} e^{-\mathcal{H}_r - \sum_{s_i \in S_r,\, \xi_j \notin S_r} \Psi(s_i, \xi_j)} d\Lambda_r^m \tag{5}$$

If $\Psi(x, y) = -\log|x - y|$, then (5) does not make sense.

$$c_{r,\xi}^{m}{}^{-1} e^{-\mathcal{H}_r} d\Lambda_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\Lambda_r^m \quad \text{(quasi-Gibbs property)}$$

Lem 1 (O. 13 AOP, O. 13 SPA, O.-Tanemura).
(1) *Ginibre RPF is a* $2\log|x|$*-quasi Gibbs measure.*
(2) Airy$_\beta$ *RPF are* $\beta\log|x|$*-quasi Gibbs m for* $\beta = 1, 2, 4$.
(3) Sine$_\beta$ *RPF are* $\beta\log|x|$*-quasi Gibbs m for* $\beta = 1, 2, 4$.

• Conjecture: The following is a quasi Gibbs measure

(1) Airy$_\beta$, Sine$_\beta$, Bessel$_\beta$ for all $0 < \beta < \infty$.

(2) All determinantal RPFs.

(3) Zero points of GAFs.

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ.

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$.

Here $\mathsf{S}_r^k = \{\mathsf{s}(S_r) = k\}$, σ_r^k is k-density fun on S_r.

Thm 3 (O.96 CMP, O.13 AOP). *Assume* (A1) *and* (A2).
Then $\exists$ μ*-reversible diffusion* $\mathsf{X}_t = \sum_i \delta_{X_t^i}$.

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Thm 4 (O.96 CMP, O.13 AOP). *Assume* (A1) *and* (A2). *Then* $\exists$ *μ-reversible diffusion* $\mathsf{X}_t = \sum_i \delta_{X_t^i}$.

We next present an SDE representation of

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}.$$

We detect the ISDE that $\mathbf{X}_t = (X_t^i)$ satisfy.

The key notion is "log derivative d^{μ} of μ".

- ρ^1 is 1-correlation function of μ if

$$\int_{\mathsf{S}} \sum_i f(s_i)\mu(d\mathsf{s}) = \int_{\mathbb{R}^d} f(x)\rho^1(x)dx \qquad \text{for all } f.$$

- Let μ_x be the reduced Palm measure of μ conditioned at x

$$\mu_x(\cdot) = \mu(\cdot - \delta_x \,|\, \mathsf{s}(\{x\}) \geq 1)$$

- Let μ^1 be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^1(A \times B) = \int_{A \times B} \rho^1(x)\mu_x(d\mathsf{s})dx$$

- $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathsf{S}, \mu^1)$ is called the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}^\mu d\mu^1 = -\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_0$$

- Very informally $\mathsf{d}^\mu = \nabla_x \log \mu^1$.

(A3) $\{X_t^i\}_{i\in\mathbb{N}}$ do not collide each other

$$\Leftrightarrow \mathrm{Cap}^{\mu}(\{\mathsf{s}\in\mathsf{S};\ \exists x\in\mathbb{R}^d \text{ s.t. } \mathsf{s}(\{x\})\ge 2\})=0$$

(A4) $\int_{\mathbb{R}^d}\rho^1(x)e^{-|x|^{2-c}}dx<\infty$ for some $c>0$.

(A5) The log derivative $\mathsf{d}^{\mu}\in L^1_{\mathrm{loc}}(\mu^1)$ exists.

Thm 5. (O.12(PTRF)) (A1)–(A5) $\Rightarrow$ $\exists\mathsf{S}_0\subset\mathsf{S}$ *such that*

$$\mu(\mathsf{S}_0)=1, \tag{6}$$

and that, for $\forall\mathbf{s}\in\mathfrak{u}^{-1}(\mathsf{S}_0)$, *there exists* $(\mathbf{X},\mathbf{B})$, *where* $\mathbf{X}=(X^i)$ *and* $\mathbf{B}=(B^i)$, *satisfying*

$$dX_t^i=dB_t^i+\frac{1}{2}\mathsf{d}^{\mu}(X_t^i,\sum_{j\neq i}\delta_{X_t^j})dt,\quad (X_0^i)_{i\in\mathbb{N}}=\mathbf{s} \tag{7}$$

Here $\mathfrak{u}\colon S^{\mathbb{N}}\to\mathsf{S}$ such that $\mathfrak{u}((s_i))=\sum_i\delta_{s_i}$.

Let $\mathcal{T} = \mathcal{T}(\mathsf{S})$ be the tail σ field of S:

$$\mathcal{T} = \bigcap_{r=1}^{\infty} \sigma[\pi_r^c].$$

Lem 2. *$\mathcal{T}$ is μ-trivial for any determinantal RPF μ.*

Let $\mu(\cdot|\mathcal{T})$ be the regular conditional probability. Then

$$\mu(\cdot) = \int_{\mathsf{S}} \mu(\cdot|\mathcal{T})(\xi)\mu(d\xi)$$

Lem 3. *Let μ be a quasi-Gibbs measures.*
Then $\mathcal{T}$ is $\mu(\cdot|\mathcal{T})(\xi)$-trivial for μ-a.s. ξ.

existence of strong solution

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

Let $\mathsf{d}^{\mu}(x, \mathsf{s}) = \check{\mathsf{d}}^{\mu}(x, \mathbf{s}_m, \mathsf{s}_m^*)$,
where $\mathbf{s}_m = (s_1, \ldots, s_m)$, $\mathsf{s}_m^* = \sum_{l>m} \delta_{s_l}$ for $\mathsf{s} = \sum_i \delta_{s_i}$,
and $\check{\mathsf{d}}^{\mu}(x, \mathbf{s}_m, \mathsf{s}_m^*)$ is symmetric in $\mathbf{s}_m$.

$$\mathsf{H}_k^m = \{\{(x, \mathbf{s}_m, \mathsf{s}_m^*) \in (\mathbb{R}^d)^{1+m} \times \mathsf{S} ;$$
$$|\check{\mathsf{d}}^{\mu}(x, \mathbf{s}_m, \mathsf{s}_m^*) - \check{\mathsf{d}}^{\mu}(\hat{x}, \hat{\mathbf{s}}_m, \mathsf{s}_m^*)| \le c_k^m |(x, \mathbf{s}_m) - (\hat{x}, \hat{\mathbf{s}}_m)|\}$$

$$\mathsf{H}^m = \cup_{k=1}^{\infty} \mathsf{H}_k^m, \quad \mathsf{H}_k^m \subset \mathsf{H}_{k+1}^m \cdots$$

(A6) $\mathrm{Cap}^{\mu^m}((\mathsf{H}^m)^c) = 0$ for all $m = 0, 1, \ldots$

i.e. $P_{(x,\mathbf{s}_m,\mathsf{s}_m^*)}((X_t, \mathbf{X}_t^m, \mathsf{X}_t^{m*}) \in \mathsf{H}^m$ for all $t) = 1$.

Here μ^m is the m-Campbell measure of μ.

Thm 6 (O.-Tanemura). *Assume* (A1)–(A6). *Then:*

(1) $\exists \mathsf{S}_0$ *such that* $\mu(\mathsf{S}_0) = 1$ *and that the ISDE*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{\mathsf{j}\neq\mathsf{i}} \delta_{X_t^j})dt,$$

$$(X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

has a strong solution for $\mathbf{s} = (s_i) \in S^{\mathbb{N}}$ *s.t.* $\sum_i \delta_{s_i} \in \mathsf{S}_0$.

(2) *The ass unlabeled diffusion* $\mathsf{X} = \sum_i \delta_{X^i}$ *satisfies the absolutely continuity condition:*

$$P_{\mu_\xi} \circ \mathsf{X}_t^{-1} \prec \mu_\xi \qquad (\forall t) \quad \textit{for } \mu\textit{-a.s. } \xi$$

Here $\mu_\xi = \mu(\cdot|\mathcal{T})(\xi)$ *is the tail decomposition in Lem 3.*

Thm 7 (O.-Tanemura). *Assume* (A1)–(A6).
Let $\mathbf{X} = (X^i)$ *and* $\hat{\mathbf{X}} = (\hat{X}^i)$ *be strong sol of the ISDE*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i} \delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s} = (s_i)_{i\in\mathbb{N}}$$

on the same Br m. Let $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ *and* $\hat{\mathsf{X}}_t = \sum_i \delta_{\hat{X}_t^i}$.
Suppose that the absolutely continuity condition is satisfied for μ*-a.s.* ξ:

$$P_{\mu_\xi} \circ \mathsf{X}_t^{-1} \prec \mu_\xi \text{ and } P_{\mu_\xi} \circ \hat{\mathsf{X}}_t^{-1} \prec \mu_\xi \ (\forall t)$$

Then

$$\mathbf{X} = \hat{\mathbf{X}} \text{ a.s.} \qquad \text{for } \mu\text{-a.s. } \mathsf{s} = \sum_{i=1}^{\infty} \delta_{s_i}$$

Cor 1 *Assume* (A1)–(A6). *Assume*
(A7) μ is tail trivial.
Then the strong solution $\mathbf{X} = (X^i)$ *satisfying*

$$P_\mu \circ \mathsf{X}_t^{-1} \prec \mu \qquad \textit{for all } t$$

is unique for μ*-a.e.* $\mathsf{x} = \sum_i \delta_{x_i}$.

- Recall that tail σ-fields of determinantal RPFs, in particular, Airy, Sine, Ginibre RPFs with $\beta = 2$ are trivial.

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• Recall that tail σ-fields of determinantal RPFs, in particular, Airy, Sine, Ginibre RPFs with $\beta = 2$ are trivial.

Cor 2 *The ISDE*

$$dX_t^i = dB_t^i + b(X_t^i, \mathsf{X}_t^{i*})dt \quad (i \in \mathbb{N}) \tag{8}$$

has a strong solution if the differential equation of RPFs μ

$$\frac{1}{2}\nabla_x \log \mu(x, \mathsf{s}) = b(x, \mathsf{s}) \tag{9}$$

has a solution μ *satisfying* (A1)–(A4), *and* (A6).
(The tail σ-field of S is "the boundary condition" of (8), (9)).

Strong solutions of ISDEs: Remarks 1/2

Remark 1 Short history of strong solutions of the ISDEs

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{i=1, j\neq i}^{\infty} \nabla\Psi(X_t^i - X_t^j)dt$$

- Lang ('78, '79, ZWVG) : $\Psi \in C_0^3(\mathbb{R}^d)$, μ is grand canonical Gibbs m. "stationary Markov solutions". Lippner, Rost (1D)
- Fritz ('87, AOP): $\Psi \in C_0^3(\mathbb{R}^d)$, μ is grand canonical Gibbs m. $d \leq 4$, "Non-equilibrium solutions" and "strong Markov".
- Tamemura ('96 PTRF):Hard core balls, "strong Markov"
- Rölly-Tanemura: hard core + "exponential decay Ψ"

- All authors above used "Ito scheme" in infinite-dimension space. i.e., they use the "Lipschitz continuity" of the coefficients. This makes their proof very hard.

Strong solutions of ISDEs: Remarks 2/2

Remark 2: • Our method is as follows:

1) Construct (not strong) solution $(\mathbf{X}, \mathbf{B})$ by Thm 5.

2) Introduce infinite-many, finite-dimensional SDEs with consistency by using $(\mathbf{X}, \mathbf{B})$.

3) By (A6), the first m-components $\mathbf{X}^m$ of $\mathbf{X}$ is a unique strong solution of the m'th finite dimensional SDE.

4) $\mathbf{X}^m$ is $\sigma[\mathbf{B}, \mathbf{X}^{m*}]$-measurable. Here $\mathbf{X}^{m*} = (X^{m+1}, \dots)$

5) Hence the limit $\mathbf{X}$ is $\sigma[\mathbf{B}] \vee \cap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}]$-measurable.

6) From the μ-tail triviality of S, we deduce the triviality of the tail σ-field $\cap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}]$ of the labeled path space $C([0, \infty) : (\mathbb{R}^d)^{\mathbb{N}})$.

Examples: Gibbs measures

- All example below satisfy (A1)–(A6).

Gibbs measures :

- All Gibbs measures with Ruelle's class potentials Ψ (smooth outside the origin). Non-collision (A3) does not hold in general. But it always holds for $d \geq 2$ and, for repulsive interaction Ψ in $d = 1$.
- In this case, the SDEs become

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j \neq i} \nabla \Psi (X_t^i - X_t^j) dt. \tag{10}$$

Lennard-Jones 6-12 potential: Set $d = 3$.

$$\Psi_{6,12}(x) = |x|^{-12} - |x|^{-6}$$

The corresponding ISDE is: ($i \in \mathbb{N}$)

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^{\infty} \{ \frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}} \} dt \ .$$

Sine$_\beta$ RPF: $d = 1,\ \beta = 1, 2, 4$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{1}{X_t^i - X_t^j} dt$$

The definition of $\mu = \mu_{\mathrm{sin},\beta}$:

$$\mu_{\mathrm{sin},\beta} = \lim_{N\to\infty} \mu_{\mathrm{sin},\beta}^N \quad \text{(bulk scaling)}$$

where

$$\mu_{\mathrm{sin},\beta}^N = \frac{1}{\mathcal{Z}_N} \{\prod_{i<j}^N |s_i - s_j|\}^\beta e^{-\frac{\beta}{4N}\sum_{k=1}^N |s_k|^2} d\Lambda_N.$$

Examples: Ginibre RPF

Ginibre RPF: $d = 2$, $\Psi(x) = -2\log|x|$.

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \qquad (11)$$

- $\mu_{\mathrm{gin},2}$ is the TD limit of

$$\mu_{\mathrm{gin},2}(d\mathsf{s}) = \lim_{N\to\infty} \frac{1}{\mathcal{Z}_N} \prod_{i=1,\, i<j}^{N} |s_i - s_j|^2 \prod_{k=1}^{N} \frac{1}{\pi} e^{-|s_k|^2} \Lambda_N$$

- $\mu_{\mathrm{gin},2}$ is rotation and translation invariant.

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Ginibre RPF: $d = 2$, $\Psi(x) = -2\log|x|$.

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{11}$$

- $\mu_{\mathrm{gin},2}$ is the TD limit of

$$\mu_{\mathrm{gin},2}(ds) = \lim_{N\to\infty} \frac{1}{\mathcal{Z}_N} \prod_{i=1,\, i<j}^{N} |s_i - s_j|^2 \prod_{k=1}^{N} \frac{1}{\pi} e^{-|s_k|^2} \Lambda_N$$

- $\mu_{\mathrm{gin},2}$ is rotation and translation invariant.

Dynamical rigidity of the Ginibre RPF

Ginibre RPF: $d = 2$, $\Psi(x) = -2\log|x|$.

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \qquad (11)$$

Thm 8 (O.12 PTRF). *The solution* (X^i) *of* (11) *satisfies*

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^j|<r,\ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt - X_t^i dt. \qquad (12)$$

Remark: • Two ISDEs (11) and (12) have common unique, strong solutions.

- Solutions are trapped very thin subset of $(\mathbb{R}^2)^{\mathbb{N}}$.

Ginibre RPF: $d = 2$, $\Psi(x) = -2 \log |x|$.

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r,\, j \neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \qquad (11)$$

Thm 9. *Each tagged particles are sub diffusive:*

$$\lim_{\varepsilon\to 0} \varepsilon X^i_{t/\varepsilon^2} = 0 \qquad \text{for all } i \in \mathbb{N} \qquad (13)$$

weakly in $C([0,\infty) : \mathbb{R}^2)$ *in* $\mu_{\mathrm{gin},2}$*-measure.*

- This result is strikingly different from that of interacting Brownian motions with Ruelle's class potentials. Tagged particles are always diffusive for Ruelle's class potentials for $d \geq 2$.