Idea of "strong sol of ISDEs"

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- General theory to construct unique, strong solutions of infinite-dimensional stochastic differential equations

Strong solutions of ISDE: Non Markov type

$S = \mathbb{R}^d, [0,\infty), \mathbb{C}$

$W(S^{\mathbb{N}}) = C([0,T); S^{\mathbb{N}})$, $(0 < T < \infty)$ labeled path sp.

- a quadruplet $(\{\sigma^i\}, \{b^i\}, W_{\text{sol}}, \mathbf{S_0})$

W_{sol} : a Borel subset of $W(S^{\mathbb{N}})$ sp of soltions of ISDE

$\sigma^i, b^i : W_{\text{sol}} \to W(S^{\mathbb{N}})$ coefficients of ISDE

$\mathbf{S_0}$ be a Borel subset of $S^{\mathbb{N}}$ initial starting points of ISDE

- the ISDE on $S^{\mathbb{N}}$ of the form

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N}) \tag{1}$$

$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathbf{S_0} \tag{2}$$

$$\mathbf{X} \in W_{\text{sol}}. \tag{3}$$

- $\mathbf{X} = \{(X_t^i)_{i\in\mathbb{N}}\}_{t\in[0,T)} \in W_{\text{sol}}$
- $\mathbf{B} = (B^i)$ $(i \in \mathbb{N})$ is the $S^{\mathbb{N}}$-valued standard Br motion.

Strong solutions of ISDE: Assump (P1)

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N})$$
$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathbf{S_0}$$
$$\mathbf{X} \in W_{\mathsf{sol}}.$$

(P1) ISDE (1) has a solution $(\mathbf{X}, \mathbf{B})$. (not a strong sol!)

Here $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ is the Brownian motion on $S^{\mathbb{N}}$

Problem: Prove that $\mathbf{X}$ is a functional of the Br $\mathbf{B}$

Idea:
Strong solutions of Infinite-dimensional SDE
$\Leftrightarrow$
Infinite-many, finite-dimensional SDEs with consistency
$+$
Triviality of Tail σ-field of label pathes

Assump (P2) infinite-many, finite-dimensional SDEs with consistency

- $\bar{P}_{\mathbf{s}}$: a prob meas on $W(S^{\mathbb{N}}) \times W^{\mathbf{0}}(S^{\mathbb{N}})$
- $\bar{P}_{\mathbf{s},\mathbf{B}} = \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot | \mathbf{B})$: the regular conditional prob
- $\mathbf{P}_{\mathbf{s}} = \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot)$, $\quad P_{\mathsf{Br}}^{\infty} = \bar{P}_{\mathbf{s}}(\mathbf{B} \in \cdot)$

For $\mathbf{X} \in W_{\mathsf{sol}}$, $\mathbf{s} \in \mathbf{S_0}$, and $m \in \mathbb{N}$,

we introduce a new SDE (6) on $\mathbf{Y}^m = (Y_t^1, \dots, Y_t^m)$.

$$dY_t^i = \sigma^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dt \qquad (4)$$

$$\mathbf{Y}_0^m = (s_1, \dots, s_m) \in \mathsf{S}^m, \quad \text{where } \mathbf{s} = (s_i)_{i=1}^{\infty},$$

$$\mathbf{Y}^m + \mathbf{X}^{m*} \in W_{\mathsf{sol}}.$$

Here $\mathbf{X}^{m*} = (0, \dots, 0, X_t^{m+1}, X_t^{m+2}, \dots)$ and we set

$$\mathbf{Y}^m + \mathbf{X}^{m*} = (Y_t^1, \dots, Y_t^m, X_t^{m+1}, X_t^{m+2}, \dots). \qquad (5)$$

$\mathbf{X}^{m*}$ is interpreted as a part of the coefficients of the SDE (6).

Strong solutions of ISDE: (P2) seq of finite-dim SDEs with consistecy

$$dY_t^i = \sigma^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dt \qquad (6)$$
$$\mathbf{Y}_0^m = (s_1, \ldots, s_m) \in \mathsf{S}^m,$$
$$\mathbf{Y}^m + \mathbf{X}^{m*} \in W_{\mathsf{sol}}.$$

(P2) The SDE (6) has a unique, strong solution for each $\mathbf{s} \in \mathbf{S_0}$, $\mathbf{X} \in W_{\mathsf{sol}}^{\mathbf{s}}$, and $m \in \mathbb{N}$.

Strong solutions of ISDE: (P3) Tail triviality

Let $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ be the tail σ-field of $W(S^{\mathbb{N}})$; we set

$$\mathit{Tail}\,(W(S^{\mathbb{N}})) = \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}] \tag{7}$$

$$\mathit{Tail}^{\,[1]}(\mathbf{P}) = \{A \in \mathit{Tail}\,(W(S^{\mathbb{N}}))\,;\, \mathbf{P}(A) = 1\}.$$

Here $\mathbf{P}$ is a probability measure on $W(S^{\mathbb{N}})$.

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P}_{\mathbf{s}}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

(P1) ISDE (1) has a solution $(\mathbf{X}, \mathbf{B})$.

(P2) SDE (6) has a unique, strong solution for all $\mathbf{s}, \mathbf{X}, m$.

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P_s}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

Thm 1. *Assume* (P1)–(P3). *Then*

(1) *ISDE* (1)–(3) *has a strong solution for each* $\mathbf{s} \in \mathbf{S}_0$.

(2) *Let* $\mathbf{Y_s}$ *and* $\mathbf{Y}'_\mathbf{s}$ *be strong solutions of ISDE* (1)–(3) *starting at* $\mathbf{s} \in \mathbf{S}_0$ *defined on the same space of Brownian motions* $\mathbf{B}$. *Then* $\mathbf{Y_s} = \mathbf{Y}'_\mathbf{s}$ *a.s. if and only if*

$$\mathit{Tail}^{\,[1]}(\mathrm{Law}(\mathbf{Y_s})) = \mathit{Tail}^{\,[1]}(\mathrm{Law}(\mathbf{Y}'_\mathbf{s})). \tag{8}$$

Strong solutions of ISDE: Idea of Main Theorem 1 (1)

(P1) ISDE (1) has a solution $(\mathbf{X}, \mathbf{B})$.

(P2) SDE (6) has a unique, strong solution for all $\mathsf{s}, \mathbf{X}, m$.

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P}_{\mathsf{s}}$-trivial for each $\mathsf{s} \in \mathbf{S_0}$.

- $(\mathbf{X}, \mathbf{B})$: sol of ISDE by (P1). Let $(\mathbf{X}, \mathbf{B})$ be fixed.
- $\mathbf{Y}^m$ is a unique strong sol of SDE(5) by (P2)
- $\mathbf{Y}^m$ is $\sigma[\mathbf{B}] \bigvee \sigma[\mathbf{X}^{m*}]$-m'ble. $\mathbf{X}^{m*} = (X^n)_{m<n<\infty}$.
- $\mathbf{Y}^m = (X^1, \ldots, X^m)$. by (P2)
- $\mathbf{X}$ is $\sigma[\mathbf{B}] \bigvee \mathit{Tail}\,(W(S^{\mathbb{N}}))$-m'ble by $m \to \infty$.
- $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is trivial by (P3) $\Rightarrow$ $\mathbf{X}$ is a strong solution.

Strong solutions of ISDE: How to prove (P1)–(P3)

(P1) ISDE (1) has a solution $(\mathbf{X}, \mathbf{B})$.

(P2) SDE (6) has a unique, strong solution for all $\mathbf{s}, \mathbf{X}, m$.

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P}_{\mathbf{s}}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

- (P1) follows from a general theory of O..
- (P2) is classical.
- How to prove (P3) **?** $\Rightarrow$ Tail Theorems.

Thm 2. *Assume* $P_\mu \circ \mathsf{X}_t^{-1} \prec \mu$ *for all* t. *Assume* μ *is tail trivial. Then* (P3) *holds. Here*

$$\mathsf{X}_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i} \quad (\textit{unlabeled dynamics})$$

End

Let $S = \mathbb{R}^d$, $\mathbb{C}$, $[0, \infty)$.

S: Configuration space over S

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\ s_i \in S,\ \mathsf{s}(|s| < r) < \infty \ (\forall r \in \mathbb{N})\}$$

μ: RPF on S. i.e. prob meas. on S.

Prob: (1) To construct a *natural* stochastic dynamics

$$\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}} \quad \text{(labeled dynamics)}$$

related to μ, *i.e.*

$$\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i} \quad \text{(unlabeled dynamics)}$$

is reversible w.r.t. μ.

(2) To find the ∞-dim. SDE that $\mathbf{X}_t$ satisfies.

- ρ^n is called the n-correlation function of μ w.r.t. Radon m. m if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(\mathbf{x}_n) \prod_{i=1}^{n} m(dx_i) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!} d\mu$$

for any disjoint $A_i \in \mathcal{B}(S)$, $k_i \in \mathbb{N}$ s.t. $k_1 + \ldots + k_m = n$.

- μ is called the determinantal RPF generated by (K, m) if its n-correlation fun. ρ^n is given by

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{1\le i,j\le n}$$

- Ginibre RPF $S = \mathbb{C}$. μ_{gin} is generated by $(K_{\mathsf{gin},2}, \mathrm{g})$

$$K_{\mathsf{gin},2}(x,y) = e^{x\bar{y}} \qquad \mathrm{g}(dx) = \pi^{-1} e^{-|x|^2} dx$$

Gibbs measure

Main theorems: Unlabeled level construction

Let $\mathbb{D}$ be the canonical square field on S: $\mathsf{s} = \sum_i \delta_{s_i}$, $\mathbf{s} = (s_i)$.

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_i \nabla_{s_i}\tilde{f}(\mathbf{s}) \cdot \nabla_{s_i}\tilde{g}(\mathbf{s})$$

Let $\mathcal{D}$ be the set of local smooth fun with $\mathcal{E}_1^{\mu}(f,f) < \infty$.

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g]d\mu$$

Thm 3. [O.96[CMP], 10[JMSJ], 12?[AOP]]

(1) *If μ is quasi-Gibbs with upper semi-cont potentials (Φ, Ψ), then $(\mathcal{E}^{\mu}, \mathcal{D}, L^2(\mathsf{S}, \mu))$ is closable.*

(2) *If $(\mathcal{E}^{\mu}, \mathcal{D}, L^2(\mathsf{S}, \mu))$ is closable & all correlation fun are loc bounded, then a diffusion X_t associated with the closure $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$ exists.*

If μ is Poisson rpf with Lebesgue intensity, then $\mathsf{X}_t = \sum_i \delta_{B_t^i}$.

Main theorems: Infinite-dim SDE

(A1) μ is a quasi-Gibbs measure. (closability)

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$ (quasi-regular)

Here $S_r = \{|x| < r\}$, $\mathsf{S}_r^k = \{\mathsf{s}(S_r) = k\}$, σ_r^k is k-density fun on S_r.

(A3) The log derivative $\mathsf{d}_\mu \in L^1_{\mathrm{loc}}(\mu^1)$ exists (SDE rep)

(A4) $\{X_t^i\}$ do not collide each other (non-collision)

(A5) each tagged particle X_t^i never explode (non-explosion)

Let $\mathfrak{u}: S^{\mathbb{N}} \to \mathsf{S}$ such that $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$.

Thm 4. (O.12(PTRF)) (A1)–(A5) $\Rightarrow \exists \mathsf{S}_0 \subset \mathsf{S}$ *such that*

$$\mu(\mathsf{S}_0) = 1, \tag{9}$$

and that, for $\forall \mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, $\exists \mathfrak{u}^{-1}(\mathsf{S}_0)$*-valued pr.* $(X_t^i)_{i\in\mathbb{N}}$ *and* $\exists S^{\mathbb{N}}$*-valued Brownian m.* $(B_t^i)_{i\in\mathbb{N}}$ *satisfying*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}_{\mu}(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s} \tag{10}$$

Main theorems: labeled diffusions

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}_\mu(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

Thm 5 (O. (JMSJ 10)). *The family of processes* $\{(X_t^i)_{i\in\mathbb{N}}\}$ *is a diffusion with state space* $\mathfrak{u}^{-1}(\mathsf{S}_0) \subset S^{\mathbb{N}}$.

Remark 1. (1) (A1)–(A5) can be checked for Ginibre RPF ($\beta = 2$), Sine RPFs, Airy RPFs and Bessel RPFs ($\beta = 1, 2, 4$).
(2) We can calculate the log derivatives of these measures.
(3) We have general theorems for quasi-Gibbs property and the log derivatives (O. PTRF12, to appear in AOP, preprint). The statements are too messy to be omitted here.

$$\mathsf{H}^1 = \{(x, \mathsf{s}) \in S \times \mathsf{S}\,;\, \mathsf{d}_\mu(x, \mathsf{s}) \text{ is locally Lips cont. } \}$$

Here "locally" means we regard $\mathsf{d}_\mu(x, \mathsf{s})$ as symmetric fun on S_r with fixed particles outside S_r^c for $\forall r$ except a capacity zero set (non-single points, say).

Let $\mathsf{H} = \{\delta_x + \mathsf{s}\,;\, (x, \mathsf{s}) \in \mathsf{H}^1\}$ Assume

(A6) $\mathrm{Cap}^\mu(\mathsf{H}^c) = 0$.

Thm 6 (with Tanemura). *Assume (A1)–(A6). Then the SDE has a unique, strong solution for initial starting points $(s_i) \in S^{\mathbb{N}}$ such that $\sum_i \delta_{s_i} \in \mathsf{H}$ q.e..*

Remark: (1) It is quite likely that all determinantal rpfs in continuous spaces satisfy (A1)–(A6).

(2) It is likely that the conclusion of Thm 6 holds for all initial points $\mathbf{s} = (s_i)$ such that $\sum_i \delta_{s_i} \in \mathsf{H}$. (in progress)

Let $\mathcal{D}^{\mu}_{\text{poly}}$ be the closure of the set of polynomials on S such that $\mathcal{E}^{\mu}_{1}(f,f) < \infty$. Then

$$\mathcal{D}^{\mu}_{\text{poly}} \subset \mathcal{D}^{\mu}$$

because polynomials are local and smooth.

Thm 7 (with Tanemura). *Assume* (A1)–(A6)*. Then the Dirichlet form that are extension of* $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}_{\text{poly}})$ *is unique.*

In particular, $\mathcal{D}^{\mu}_{\text{poly}} = \mathcal{D}^{\mu}$, and Lang's construction and Osada's construction are same.

Remark 2. If (A5) (non-explosion) does not hold. Then Thm 7 does not hold. This is very natural theorem that says the uniqueness of Dirichlet forms is related to the non-explosion problem of tagged problem.

All example below satisfy (A1)–(A6). Hence by Thm 6 we have a unique, strong solution.

Gibbs measures :

- All Gibbs measures with Ruelle's class potentials (smooth outside the origin) satisfy the assumptions (A.1)–(A.6).

Non-collision (A4) does not hold in general. But it always holds for $d \geq 2$ and, for repulsive interaction Ψ in $d = 1$.

- In this case, the SDEs become

$$dX_t^i = dB_t^i - \frac{1}{2}\nabla\Phi(X_t^i)dt - \frac{1}{2}\sum_{j\neq i}\nabla\Psi(X_t^i - X_t^j)dt. \qquad (11)$$

Examples: Ruelle's class potentials

Lennard-Jones 6-12 potential

Let $\Phi_{6,12}(x) = c\{|x|^{-12} - |x|^{-6}\}$, where $d = 3$ and $c > 0$ is a constant. $\Phi_{6,12}$ is called the Lennard-Jones 6-12 potential. The corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{c}{2} \sum_{j=1, j\neq i}^{\infty} \{\frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}}\}dt \quad (i \in \mathbb{N})$$

Coulomb like potentials (not Coulomb!)

Let $a > d$ and set $\Phi_a(x) = (c/a)|x|^{-a}$, where $c > 0$. Then the corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{c}{2} \sum_{j=1, j\neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}} dt \quad (i \in \mathbb{N}). \qquad (12)$$

Examples: Ruelle's class potentials

Coulomb like potentials (not Coulomb!)

Let $a > d$ and set $\Phi_a(x) = (c/a)|x|^{-a}$, where $c > 0$. Then the corresponding ISDE is:

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At first glance the ISDE (13) resembles Ginibre IBMs, because these corresponds to the case $a = 0$ in (13). The sums in the drift terms, however, converge absolutely, unlike Coulomb (log) potentials. We emphasize that the structures of the dynamics given by the solutions of (13) and Ginibre IBMs are completely different from each other.

Ginibre rpf: $\Psi(x) = -\beta \log|x|$ $d = 2$, $\beta = 2$. If $\mu = \mu_{\mathrm{gin},2}$,

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r, j \neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \tag{14}$$

and also

$$dX_t^i = dB_t^i - X_t^i dt + \lim_{r\to\infty} \sum_{|X_t^j| < r \ \ j \neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{15}$$

This comes from the plural expressions of $\mathrm{d}_{\mu_{\mathrm{gin},2}}$.
For finite N, these SDEs give different solution.
But in the limit $N \to \infty$ give the same solution if the initial distribution is closed to Ginibre rpf.

Examples: Bessel rpf–hard edge scaling limit

Bessel RPF (joint work with Honda):

$S = [0, \infty)$, $\beta = 2$, $a > 1$

$$dX_t^i = dB_t^i + \frac{a}{2X_t^i}dt + \lim_{r\to\infty}\frac{\beta}{2}\sum_{\substack{|X_t^j|<r\\ j\neq i}}\frac{1}{X_t^i - X_t^j}dt$$

$\beta = 1, 4$ are in progress.

Examples: sine rpf (Dyson's model)–bulk scaling limit

Sine$_\beta$ RPF: $S = R$, $\beta = 1, 2, 4$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \to \infty} \sum_{|X_t^i - X_t^j| < r,\ j \neq i} \frac{1}{X_t^i - X_t^j} dt$$

Spohn (1987) considered the case $\beta = 2$:

$$dX_t^i = dB_t^i + \sum_{j \neq i} \frac{1}{X_t^i - X_t^j} dt$$

He constructed the dynamics as a Markov semigr by Dirichlet form.

The def of $\mu = \mu_{\sin,\beta}$:

$\beta = 2 \Rightarrow \mu_{\sin,\beta}$ is the det rpf generated by $(K_{\sin}, dx)$:

$$K_{\sin}(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}$$

$\beta = 1, 4 \Rightarrow$ the correlation funs are given by quaternion det.

Thm 8 (with Tanemura). *Let* $\beta = 1, 2, 4$. *Then:*

- *The log derivative* $\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}$ *is*

$$\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}(x, \mathsf{s}) = \beta \lim_{r\to\infty} \{ (\sum_{|x-s_i|<r} \frac{1}{x - s_i}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \}$$

Here

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x)$$

- *Airy rpf* $\mu_{\mathsf{Ai},\beta}$ *satisfy* (A1)–(A6) *and the limit ISDE is*

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

- The key idea is to take the rescaled semi-circle law ς, as the first approximation of the 1-correlation fun $\rho^{N,1}_{\mathsf{Ai},\beta}$.
- Our method can be applied to other soft edge scaling. Our result is the first time to clarify the SDE describing the limit infinite system for the soft edge.

Thm 9 (with Tanemura). *Assume* $\beta = 2$.

Let us label $X_t^i > X_t^{i+1}$ $(\forall i)$.

- *The top particle* X_t^1 *is the Airy process* $\mathcal{A}(t)$ *in the sense of Spohn.*
- *The infinite dim stochastic dynamics constructed by Spohn, Johansson & others by the space-time correlation fun is a solution of the prescribed SDE:*

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

Examples: Airy rpf – Soft edge scaling limit

- The SDE gives a kind of Girsanov formula.
- These examples are the first time that the infinite dynamics are constructed for rpf appeared in random matrix theory with $\beta = 1, 4$ even if the bulk and the hard edge as well as the soft edge scaling

  In one dimensional system, the method of space-time correlation functions are available (Nagao, Katori-Tanemura, Spohn, and others), but this method is restricted to $\beta = 2$.
- By construction, if the total system start from the Airy_β rpf $\mu_{\text{Ai},\beta}$, then the distribution of the top particle X_t^1 equals $F_{\beta,edge}(x)$, the β Tracy-Widom distribution, where $\beta = 1, 2, 4$.

To sum up

Thm 10. *Ginibre RPF ($\beta = 2$), Sine RPFs, Airy RPFs ($\beta = 1, 2, 4$) and Bessel RPFs ($\beta = 2$) are quasi-Gibbs m. for $\Psi(x) = -\beta \log |x|$, and the log derivative can be calculated. The associated ISDE has a unique, strong solution.*

Remark 3. Virág et all have been constructed the RPF for all β on Dyson, Airy and Bessel RPFs (called β ensemble). It is quite likely that these RPFs satisfy our assumptions (A1)–(A6). But unfortunately, they have not yet prove the existence of correlation functions for these models. Only an existence of TDL has been established!

It is important to prove these are quasi-Gibbs measures and to calculate the log derivative.

Thank You !

To sum up

- The key point of the proof is to use the small fluctuation property (SFP) of linear statistics for these measures.
- SFP was established by Soshnikov (Sine, Airy, Bessel RPFs), Shirai (Ginibre RPF).
- Proof consists of several parts:

(1) To find a good finite particle approximation $\{\mu^N\}$

(2) To prove uniform *small fluctuation* of $\{\mu^N\}$

(3) To prove uni bounds of 1 & 2 cor funs of $\{\mu^N\}$

(4) To carry out the limiting procedure of $\mathtt{d}_{\mu^N}$ & quasi-Gibbs property by using general theorems. (O. 11,12)

Derivation of (**??**): $(\mathcal{E}^{\mu^N_{\mathsf{sin},\beta}}, L^2(\mu^N_{\mathsf{sin},\beta}))$

$$\mathcal{E}^{\mu^N_{\mathsf{sin},\beta}}(f,g) = \int \mathbb{D}[f,g]\mu^N_{\mathsf{sin},\beta}(d\mathbf{x}), \quad \mathbb{D}[f,g] = \frac{1}{2}\sum_{i=1}^{N} \frac{\partial f}{\partial s_i}\frac{\partial g}{\partial s_i}$$

$$\mathcal{E}^{\mu^N_{\mathsf{sin},\beta}}(f,g) = \int \mathbb{D}[f,g]\frac{1}{Z}\sum_{i<j}^{N}|s_i - s_j|^{\beta}\prod_{k=1}^{N} e^{-\beta|s_k|^2/4N} d\mathbf{s}_N$$

$$= -\frac{1}{2}\int\{\Delta f + \sum_{i\neq j}^{N}\frac{\beta}{s_i - s_j}\frac{\partial f}{\partial s_i} - \sum_{k=1}^{N}\frac{\beta s_k}{2N}\frac{\partial f}{\partial s_k}\}g$$

$$\frac{1}{Z}\sum_{i<j}^{N}|s_i - s_j|^{\beta}\prod_{k=1}^{N} e^{-\beta|s_k|^2/4N} d\mathbf{s}_N$$

$$= -\int\{\frac{1}{2}\Delta f + \frac{\beta}{2}\sum_{i=1}^{N}[(\sum_{j\neq i}^{N}\frac{1}{s_i - s_j}) - \frac{s_i}{2N}]\frac{\partial f}{\partial s_i}\}g \ \mu^N_{\mathsf{sin},\beta}(d\mathbf{s})$$