Geometric and dynamical properties of stochastic Coulomb systems in infinite dimensions

2013/2/15/Fri RIMS

- Coulomb RPFs in infinite volume
- Quasi-Gibbs meas. Log derivative
- Tail triviality • Palm singularity
- General theory for strong solutions of ISDEs
- Uniqueness of Dirichlet forms
- Soft edge scaling limit and Airy RPFs
- Application to interacting Brownian motions (IBMs)
- Examples: Sine, Bessel, Airy, Ginibre RPFs
- Homogenization & Phase transition conjecture of Ginibre IBMs

How to define Coulomb RPFs in infinite volume 1

Let Ψ_c be c-dim Coulomb potential:

$$\Psi_2(x) = -\log|x|, \quad (2-c)^{-1}\Psi_c(x) = |x|^{2-c} \ (c \neq 2)$$

Very loosely, translation invariant Coulomb random point fields μ_c with inverse temparature $\beta > 0$:

$$\mu_{c,\beta} \sim \frac{1}{Z} e^{-\beta \sum_{i<j}^{\infty} \Psi_c(x_i - x_j)} \prod_{k=1}^{\infty} dx_k \qquad (1)$$

In particular,

$$\mu_{2,\beta} \sim \frac{1}{Z} \prod_{i<j}^{\infty} |x_i - x_j|^{\beta} \prod_{k=1}^{\infty} dx_k \qquad (2)$$

(1) How to define Coulomb RPFs (No DLR eq.!)

How to solve c dim Coulomb infinte-dim SDEs:

- $(\mathbb{R}^d)^{\mathbb{N}}$-valued SDE: $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}}$
- Ψ_c is a c-dim Coulomb pot.
- Coulomb ISDE:

$$dX_t^i = dB_t^i - \frac{\beta}{2}\sum_{j=1,\ j\neq i}^{\infty} \nabla\Psi_c(X_t^i - X_t^j)$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j=1,\ j\neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^c}dt \quad \text{(c dim Coulomb).}$$

(2) How to solve Coulomb ISDEs (No Ito's scheme !)

I will show, if $(c,d,\beta) = (2,2,2), (2,1,1), (2,1,2), (2,1,4)$, then OK.

How to define Coulomb RPFs in infinite vol 2: Ψ-Quasi-Gibbs meas.

- $S = \mathbb{R}^d$, $S_r = \{|x| \le r\}$, $\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i},\ \mathsf{s}(S_r) < \infty (\forall r)\}$
- $\pi_r, \pi_r^c : \mathsf{S} \to \mathsf{S}$, $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r)$, $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$
- Let μ be a RPF over S.

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(S_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi))$$

- Let $\Psi : S \to \mathbb{R} \cup \{\infty\}$ (interaction).

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i - s_j)$$

Def: μ is Ψ-quasi-Gibbs measure if $\exists\ c_{r,\xi}^m$ s.t.

$${c_{r,\xi}^m}^{-1} e^{-\mathcal{H}_r} d\nu_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\nu_r^m$$

Here $\nu_r^m = \prod_{k=1}^m 1_{S_r}(s_k) ds_k$

- Gibbs measu $\Rightarrow$ Quasi-Gibbs measure .

Coulomb RPFs

$$c_{r,\xi}^{m}{}^{-1} e^{-\mathcal{H}_r} d\nu_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\nu_r^m \quad \text{(quasi-Gibbs property)}$$

Let Ψ_c is the c dim Coulomb potential as before.

- We say μ is Coulomb RPF if μ is Ψ_c-quasi-Gibbs meas.
- The case $(d \le c < d+2)$ is interesting.
- μ is called strict Coulomb RPF if $c = d$.

Thm 1 (O. AOP 13, O.-Honda, O.-Tanemura).

(1) *Ginibre RPF is a* $2\Psi_2$*-quasi Gibbs measure.*

(2) Sine$_\beta$ *RPF are* $\beta\Psi_2$*-quasi Gibbs m for* $\beta = 1, 2, 4$.

(3) Bessel$_2^a$ *RPF is a* $2\Psi_2$*-quasi Gibbs m.*

(4) Airy$_\beta$ *RPF are* $\beta\Psi_2$*-quasi Gibbs m for* $\beta = 1, 2, 4$.

- Conjecture: The following is a quasi Gibbs measure

(1)β-Sine, Bessel, Airy RPFs for all β.

(2) All determinantal RPFs. Zero points of GAFs.

Application of quasi-Gibbs property to dynamics

Let $\mathcal{D}_0$ be a local, smooth funs on S.

Let $\tilde{f}(s_1,\ldots) = f(\mathsf{s})$, where $\tilde{f}$ is symmetric, $\mathsf{s} = \sum \delta_{s_i}$.

$$\mathbb{D}[f,g] = \frac{1}{2}\sum_i \nabla_i \tilde{f} \cdot \nabla_i \tilde{g}$$

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g]\mu(d\mathsf{s}) \text{ : bilinear form on } \mathcal{D}_0^{\mu}$$

$$\mathcal{D}_0^{\mu} = \{f \in \mathcal{D}_0; \mathcal{E}^{\mu}(f,f) < \infty, f \in L^2(\mu)\}$$

Thm 2. *Let μ be Ψ-quasi-Gibbs with upper semi-conti Ψ. Then*

(1)$(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$ *is closable on $L^2(\mu)$.*

(2) $\exists$ *diffusion $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ associate with $(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$.*

Log derivative of μ: precise correspondence between RPFs & potentials

- Let μ_x be the (reduced) Palm m. of μ conditioned at x
$$\mu_x(\cdot) = \mu(\cdot - \delta_x | \mathsf{s}(x) \geq 1)$$
- Let μ^1 be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:
$$\mu^1(A \times B) = \int_A \rho^1(x)\mu_x(B)dx$$
- $\mathsf{d}^\mu \in L^1(\mathbb{R}^d \times \mathsf{S}, \mu^1)$ is called the log derivative of μ if
$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^1 = -\int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}^\mu d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}$$

Here ∇_x is the nabla on $\mathbb{R}^d$, $\mathcal{D}$ is the space of bounded, local smooth functions on S.

- Very informally
$$\mathsf{d}^\mu = \nabla_x \log \mu^1$$

$\mathsf{d}^{\mu} = \nabla_x \log \mu^1$

Thm 3 (O. PTRF 12).

(1) *Let* μ_{gin} *be the Ginibre RPF. Then*

$$\mathsf{d}^{\mu_{\mathrm{gin}}}(x, \mathsf{s}) = \lim_{r\to\infty} 2 \sum_{|x-s_i|<r} \frac{x - s_i}{|x - s_i|^2}$$

$$\mathsf{d}^{\mu_{\mathrm{gin}}}(x, \mathsf{s}) = -2x + \lim_{r\to\infty} 2 \sum_{|s_i|<r} \frac{x - s_i}{|x - s_i|^2}$$

(2) *Let* $\mu_{\mathrm{sin},\beta}$ *be the* Sine_β *RPF. Suppose* $\beta = 1, 2, 4$. *Then*

$$\mathsf{d}^{\mu_{\mathrm{sin},\beta}}(x, \mathsf{s}) = \lim_{r\to\infty} \beta \sum_{|x-s_i|<r} \frac{1}{x - s_i}$$

Thm 4 (O.-Honda). *Let* $\mu^a_{\mathrm{bes},2}$ *be the* Bessel$^a_2$ *RPF. Then*

$$\mathsf{d}^{\mu^a_{\mathrm{bes},2}}(x,\mathsf{s}) = \frac{a}{x} + \lim_{r\to\infty} 2 \sum_{|x-s_i|<r} \frac{1}{x-s_i}$$

Thm 5 (O.-Tanemura). [*Airy RPFs:* $\mu_{\mathrm{Ai},\beta}$]

Let $\beta = 1,2,4$. *Then the log derivative* $\mathsf{d}^{\mu_{\mathrm{Ai},\beta}}$ *is*

$$\mathsf{d}^{\mu_{\mathrm{Ai},\beta}}(x,\mathsf{s}) = \beta \lim_{r\to\infty} \{ (\sum_{|x-s_i|<r} \frac{1}{x-s_i}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \}$$

Here

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x)$$

Let $\mathcal{T} = \mathcal{T}(\mathsf{S})$ be the tail σ field of S:

$$\mathcal{T}(\mathsf{S}) = \cap_{r=1}^{\infty} \sigma[\pi_r^c].$$

Thm 6. *Let μ be a det RPF. Then $\mathcal{T}(\mathsf{S})$ is μ-trivial.*

• Thm 6 is a generalization of that for the discrete determinantal RPFs due to Russel Lyons, Shirai-Takahashi.

Thm 7. *Let μ be a quasi-Gibbs measure. Let $\mu(\cdot|\mathcal{T})$ be the regular conditional probability. Then*

$$\mu(\cdot) = \int_{\mathsf{S}} \mu(\cdot|\mathcal{T})(\xi)\mu(d\xi)$$

and, for μ-a.s. ξ,

$$\mu(A|\mathcal{T})(\xi) = 1_A(\xi) \qquad \text{for any } A \in \mathcal{T}.$$

• Thm 7 is a generalization of that for the discrete Gibbs m due to Georgii.

Palm sigularity: Def of Ginibre RPF

Let $K(x, y)$ be a kernel, m be a meas.

- ν is called a determinantal rpf generated by (K, m) if its n correaltion fun ρ^n w.r.t. m is given by

$$\rho^n(\mathbf{x}_n) = \mathrm{det}[K(x_i, x_j)]_{1 \le i,j \le n} \tag{3}$$

- Ginibre rpf μ is the det rpf generated by $(K_{\mathrm{gin}}, \mathrm{g})$:

$$K_{\mathrm{gin}}(x, y) = e^{x\bar{y}} \qquad \mathrm{g}(dx) = \pi^{-1} e^{-|x|^2} dx$$

- Ginibre RPF is translation invariant.

Problem:

- Let ν be a translation invariant rpf on $\mathbb{C}$.
- Let $\mathsf{s} = \sum_i \delta_{s_i}$ be a sample point under ν.
- Remove a finite number of particles from the sample points $\{s_i\}$.
- Can one detect the number of the removed particles?

Answers

Problem:

- Let ν be a translation invariant rpf on $\mathbb{C}$.
- Let $\mathsf{s} = \sum_i \delta_{s_i}$ be a sample point under ν.
- Remove a finite number of particles from the sample points $\{s_i\}$.
- Can one detect the number of the missing particles?

If ν is a perioduc rpf, then "Yes".

If ν is a Poisson rpf, then "No".

The Ginibre rpf μ has a property between periodic and Poisson.

- Yes! for this problem. So Ginibre is simialr to periodic RPF rather than Poisson RPFs.
- The quasi-Gibbs property implies Ginibre is similar to Poisson RPFs rather than periodic RPFs.

Palm meas. For a set of m-points $\mathbf{x} = \{x_1, \dots, x_{\mathsf{m}}\}$ let

$$\mu_{\mathbf{x}} := \mu(\cdot - \sum_{l=1}^{\mathsf{m}} \delta_{x_l} \mid \mathsf{s}(\{x_l\}) \geq 1 \quad (l = 1, \dots, \mathsf{m}))$$

Thm 8 (O.-Shirai). *Let* $\mathsf{m}, \mathsf{n} \in \{0\} \cup \mathbb{N}$. *Then*

(1) *If* $\mathsf{m} = \mathsf{n}$, *then* $\mu_{\mathbf{x}}$ *and* $\mu_{\mathbf{y}}$ *are mutually ab. cont..*

(2) *If* $\mathsf{m} \neq \mathsf{n}$, *then* $\mu_{\mathbf{x}}$ *and* $\mu_{\mathbf{y}}$ *are singular each other.*

• (2) shows a special property of Ginibre rpf. Indeed,

Λ Poisson rpf $\Rightarrow \Lambda_{\mathbf{x}} = \Lambda$

ν Gibbs meas with Ruelle's class potentials $\Rightarrow \nu_{\mathbf{x}} \prec \nu$

• ν periodic rpf $\Rightarrow$ (2) holds

Thm 9 (O.-Shirai). *Let* $\mathsf{m} = \mathsf{n}$. *Then for* $\mu_{\mathbf{y}}$*-a.s.* s

$$\frac{d\mu_{\mathbf{x}}}{d\mu_{\mathbf{y}}} = \frac{1}{Z_{\mathbf{xy}}} \lim_{r\to\infty} \prod_{|s_i|<b_r} \frac{|\mathbf{x}-s_i|^2}{|\mathbf{y}-s_i|^2} \quad (\mathsf{s} = \sum_i \delta_{s_i}) \tag{4}$$

compact uniformly in $\mathbf{x} \in \mathbb{C}^{\mathsf{m}}$, $\mathbf{y} \in \mathbb{C}^{\mathsf{m}} \backslash \{s_1, \ldots, s_{\mathsf{m}}\}$

$$Z_{\mathbf{xy}} = \frac{\Delta(\mathbf{y}) \det[K_{\mathsf{gin}}(x_i, x_j)]_{i,j=1}^{\mathsf{m}}}{\Delta(\mathbf{x}) \det[K_{\mathsf{gin}}(y_i, y_j)]_{i,j=1}^{\mathsf{m}}}$$

$$\Delta(\mathbf{x}) = \prod_{i<j}^{\mathsf{m}} |x_i - x_j|^2, \qquad |\mathbf{x} - s_i| = \prod_{m=1}^{\mathsf{m}} |x_m - s_i|$$

$$\{b_r\}_{r\in\mathbb{N}} : \qquad b_r \uparrow \infty$$

Let $D_{\sqrt{q}} = \{z \in \mathbb{C}\,;\, |z| < \sqrt{q}\}$,

$$F_r(\mathsf{s}) = \frac{1}{r}\sum_{q=1}^{r}(\mathsf{s}(D_{\sqrt{q}}) - q). \tag{5}$$

By definition $\mathsf{s}(D_{\sqrt{q}})$ is the number of particles $\mathsf{s} = \sum_i \delta_{s_i}$ in the disk $D_{\sqrt{q}}$.

Thm 10 (O.-Shirai). *Let* $\mathbf{x} = (x_1, \dots, x_{\mathsf{m}})$.

$$\lim_{r\to\infty} F_r(\mathsf{s}) = -\mathsf{m} \qquad \textit{weakly in } L^2(\mathsf{S}, \mu_{\mathbf{x}}) \tag{6}$$

- Thm 10 means we can determine the number of missing particles:

$$\infty - \mathsf{m} \neq \infty$$

Dynamical Theory: infinite dimentional SDEs

General theorems on infinite-dim SDEs

(A1) μ is a Ψ-quasi-Gibbs m with upper-semicont Ψ. $\Rightarrow$ (closability)

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$ $\Rightarrow$ (existence of diffusions)

Here $S_r = \{|x| < r\}$, $\mathsf{S}_r^k = \{\mathsf{s}(S_r) = k\}$, σ_r^k is k-density fun on S_r.

(A3) The log derivative $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mu^1)$ exists $\Rightarrow$(SDE representation)

(A4) $\{X_t^i\}$ do not collide each other $\Rightarrow$ (non-collision)

(A5) each tagged particle X_t^i never explode $\Rightarrow$ (non-explosion)

Thm 11. (O.12(PTRF)) (A1)–(A5) $\Rightarrow$ $\exists \mathsf{S}_0 \subset \mathsf{S}$ *such that*

$$\mu(\mathsf{S}_0) = 1, \tag{7}$$

and that, for $\forall \mathsf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, $\exists \mathfrak{u}^{-1}(\mathsf{S}_0)$*-valued pr.* $(X_t^i)_{i\in\mathbb{N}}$ *and* $\exists S^{\mathbb{N}}$*-valued Brownian m.* $(B_t^i)_{i\in\mathbb{N}}$ *satisfying*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^\mu(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathsf{s} \tag{8}$$

Here $\mathfrak{u}: S^{\mathbb{N}} \to \mathsf{S}$ such that $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$.

Main theorems: labeled diffusions

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{\mathsf{j}\neq\mathsf{i}} \delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

Thm 12 (O. (JMSJ 10)). *The family of processes* $\{(X_t^i)_{i\in\mathbb{N}}\}$ *is a diffusion with state space* $\mathfrak{u}^{-1}(\mathsf{S}_0) \subset S^{\mathbb{N}}$.

Remark 1. (1) (A1)–(A5) can be checked for Ginibre RPF ($\beta = 2$), Sine RPFs, Airy RPFs and Bessel RPFs ($\beta = 1, 2, 4$).
(2) We can calculate the log derivatives of these measures.
(3) We have general theorems for quasi-Gibbs property and the log derivatives (O. PTRF12, to appear in AOP, preprint). The statements are too messy to be omitted here.

existence of strong solution

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{\mathsf{j}\neq\mathsf{i}} \delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

$$\mathsf{H}^1 = \{(x, \mathsf{s}) \in S \times \mathsf{S}\,;\, \mathsf{d}^{\mu}(x, \mathsf{s}) \text{ is locally Lips cont. }\}$$

Here "locally" means we regard $\mathsf{d}^{\mu}(x, \mathsf{s})$ as symmetric fun on S_r with fixed particles outside S_r^c for $\forall r$ except a capacity zero set. (non-single points, say).

(A6) $\mathrm{Cap}^{\mu}(\mathsf{H}^c) = 0$. Here $\mathsf{H} = \{\delta_x + \mathsf{s}\,;\, (x, \mathsf{s}) \in \mathsf{H}^1\}$

Thm 13 (O.-Tanemura). *Assume (A1)–(A6). Then: The SDE has a strong solution for ini cond* $(s_i) \in S^{\mathbb{N}}$ *s. t.* $\sum_i \delta_{s_i} \in \mathsf{H}$ *q.e..*

Thm 14 (O.-Tanemura)**.** *Assume* (A1)–(A6). *Let* $\mathbf{X} = (X^i)$ *and* $\hat{\mathbf{X}} = (\hat{X}^i)$ *be strong sol of the ISDE*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s} = (s_i)_{i\in\mathbb{N}}$$

on the same Br m. Let $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ *and* $\hat{\mathsf{X}}_t = \sum_i \delta_{\hat{X}_t^i}$. *Suppose, for* μ*-a.s.* ξ,

$$\mathsf{X}_t \prec \mu_\xi \text{ and } \hat{\mathsf{X}}_t \prec \mu_\xi \ (\forall t)$$

Here $\mu_\xi = \mu(\cdot|\mathcal{T}(\mathsf{S}))(\xi)$ *(Thm 7). Then*

$$\mathbf{X} = \hat{\mathbf{X}} \text{ a.s.} \qquad \text{for } \mu\text{-a.s. } \mathsf{s} = \sum_{i=1}^{\infty} \delta_{s_i}$$

Thm 15 (O.-Tanemura). *Assume* (A1)–(A7). *Here*
(A7) μ *is tail trivial.*
Then the strong solution $\mathbf{X} = (X^i)$ *such that*

$$P_\mu \circ \mathsf{X}_t^{-1} \prec \mu \qquad \textit{for all } t$$

is unique for μ*-a.e.* $\mathsf{x} = \sum_i \delta_{x_i}$ *Here* X *is the unlabeled dynamics of* $\mathbf{X}$*:*

$$\mathsf{X}_t = \sum_i^\infty \delta_{X_t^i}$$

Cor If μ is a determinantal RPF, then the strong, solution of the ISDE that is reversible w.r.t. μ is unique.
• Tail σ-fields of Airy, Sine, Ginibre RPFs with $\beta = 2$ are trivial.

Let $\mathcal{D}^{\mu}_{\text{poly}}$ be the closure of the set of polynomials on S such that $\mathcal{E}^{\mu}_{1}(f,f) < \infty$. Then

$$\mathcal{D}^{\mu}_{\text{poly}} \subset \mathcal{D}^{\mu}$$

because polynomials are local and smooth.

Thm 16 (O.-Tanemura). *Assume* (A1)–(A7)*. Then the Dirichlet form that are extension of* $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}_{\text{poly}})$ *is unique.*

In particular, $\mathcal{D}^{\mu}_{\text{poly}} = \mathcal{D}^{\mu}$, and Lang's construction and O.'s construction are same.

Remark 2. If (A5) (non-explosion) does not hold. Then Thm 16 does not hold. This is very natural theorem that says the uniqueness of Dirichlet forms is related to the non-explosion problem of tagged problem.

All example below satisfy (A1)–(A6). Hence by Thm 14 we have a strong solution that preserves the tail σ field.

Gibbs measures :

- All Gibbs measures with Ruelle's class potentials (smooth outside the origin) satisfy the assumptions (A.1)–(A.6).

Non-collision (A4) does not hold in general. But it always holds for $d \geq 2$ and, for repulsive interaction Ψ in $d = 1$.

- In this case, the SDEs become

$$dX_t^i = dB_t^i - \frac{1}{2}\nabla\Phi(X_t^i)dt - \frac{1}{2}\sum_{j\neq i}\nabla\Psi(X_t^i - X_t^j)dt. \qquad (9)$$

Examples: Ruelle's class potentials

Lennard-Jones 6-12 potential

Let $\Phi_{6,12}(x) = c\{|x|^{-12} - |x|^{-6}\}$, where $d = 3$ and $c > 0$ is a constant. $\Phi_{6,12}$ is called the Lennard-Jones 6-12 potential. The corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{c}{2} \sum_{j=1, j\neq i}^{\infty} \{\frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}}\}dt \quad (i \in \mathbb{N})$$

Coulomb like potentials (not Coulomb!)

Let $a > d$ and set $\Phi_a(x) = (c/a)|x|^{-a}$, where $c > 0$. Then the corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{c}{2} \sum_{j=1, j\neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}} dt \quad (i \in \mathbb{N}). \qquad (10)$$

Examples: Ruelle's class potentials

Coulomb like potentials (not Coulomb!)

Let $a > d$ and set $\Phi_a(x) = (c/a)|x|^{-a}$, where $c > 0$. Then the corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{c}{2} \sum_{j=1, j \neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}} dt \quad (i \in \mathbb{N}). \qquad (11)$$

At first glance the ISDE (11) resembles Ginibre IBMs, because these corresponds to the case $a = 0$ in (11). The sums in the drift terms, however, converge absolutely, unlike Coulomb (log) potentials. We emphasize that the structures of the dynamics given by the solutions of (11) and Ginibre IBMs are completely different from each other.

Ginibre rpf: $\Psi(x) = -\beta \log |x|$ $d = 2$, $\beta = 2$. If $\mu = \mu_{\mathrm{gin},2}$,

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \tag{12}$$

and also

$$dX_t^i = dB_t^i - X_t^i dt + \lim_{r\to\infty} \sum_{|X_t^j|<r \ \ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{13}$$

This comes from the plural expressions of $\mathsf{d}_{\mu_{\mathrm{gin},2}}$.
For finite N, these SDEs give different solution.
But in the limit $N \to \infty$ give the same solution if the initial distribution is closed to Ginibre rpf.

Examples: Bessel rpf–hard edge scaling limit

Bessel RPF (joint work with Honda):

$S = [0, \infty)$, $\beta = 2$, $a > 1$

$$dX_t^i = dB_t^i + \frac{a}{2X_t^i}dt + \lim_{r\to\infty} \frac{\beta}{2} \sum_{\substack{|X_t^j|<r \\ j\neq i}} \frac{1}{X_t^i - X_t^j}dt$$

$\beta = 1, 4$ are in progress.

Examples: sine rpf (Dyson's model)–bulk scaling limit

Sine$_\beta$ RPF: $S = R$, $\beta = 1, 2, 4$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r,\ j\neq i} \frac{1}{X_t^i - X_t^j} dt$$

Spohn (1987) considered the case $\beta = 2$:

$$dX_t^i = dB_t^i + \sum_{j\neq i} \frac{1}{X_t^i - X_t^j} dt$$

He constructed the dynamics as a Markov semigr by Dirichlet form.

The def of $\mu = \mu_{\mathrm{sin},\beta}$:

$\beta = 2 \Rightarrow \mu_{\mathrm{sin},\beta}$ is the det rpf generated by (K_{sin}, dx):

$$K_{\mathrm{sin}}(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}$$

$\beta = 1, 4 \Rightarrow$ the correlation funs are given by quaternion det.

Thm 17 (O.-Tanemura). *Let* $\beta = 1, 2, 4$. *Then:*

- *The log derivative* $\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}$ *is*

$$\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}(x, \mathsf{s}) = \beta \lim_{r\to\infty} \{ (\sum_{|x-s_i|<r} \frac{1}{x - s_i}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \}$$

Here

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x)$$

- *Airy rpf* $\mu_{\mathsf{Ai},\beta}$ *satisfy* (A1)–(A6) *and the limit ISDE is*

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

Examples: Airy rpf – Soft edge scaling limit

- The key idea is to take the rescaled semi-circle law ς, as the first approximation of the 1-correlation fun $\rho^{N,1}_{\mathsf{Ai},\beta}$.
- Our method can be applied to other soft edge scaling. Our result is the first time to clarify the SDE describing the limit infinite system for the soft edge.

Thm 18 (O.-Tanemura). *Assume $\beta = 2$.*

Let us label $X_t^i > X_t^{i+1}$ $(\forall i)$. Then :

(1) The top particle X_t^1 is the Airy process $\mathcal{A}(t)$ in the sense of Spohn.

(2) The infinite dim stochastic dynamics constructed by Spohn, Johansson & others by the space-time correlation fun is a solution of the prescribed SDE:

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

Examples: Airy rpf – Soft edge scaling limit

- The SDE gives a kind of Girsanov formula.
- These examples are the first time that the infinite dynamics are constructed for rpf appeared in random matrix theory with $\beta = 1, 4$ even if the bulk and the hard edge as well as the soft edge scaling

  In one dimensional system, the method of space-time correlation functions are available (Nagao, Katori-Tanemura, Spohn, and others), but this method is restricted to $\beta = 2$.
- By construction, if the total system start from the Airy$_\beta$ rpf $\mu_{\mathsf{Ai},\beta}$, then the distribution of the top particle X_t^1 equals $F_{\beta,edge}(x)$, the β Tracy-Widom distribution, where $\beta = 1, 2, 4$.

Homogenization
and
Phase transition conj of Ginibre IBMs

Let $S=\mathbb{R}^2$. Let $\beta\in[0,\infty)$ and set

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\sum_{|X_t^i-X_t^j|<r,\, j\neq i}\frac{X_t^i-X_t^j}{|X_t^i-X_t^j|^2}dt.$$

When $\beta=2$, then the SDE has a solution, for general β we assume the existence of solution and the rpf $\mu_{\mathrm{gin},\beta}$.

- We tag $X_t^{i_0}$ and investgate the diffusive scaling:

$$\lim_{\epsilon\to 0}\epsilon X_{t/\epsilon^2}^{i_0} = \sqrt{2\alpha_{\mathrm{self}}[\mu_{\mathrm{gin},\beta}]}B_t$$

- Assume $X_0^{i_0}=0$ and $\sum_{i\neq i_0}\delta_{X_0^i}\sim\mu_{\mathrm{gin},\beta,\mathbf{o}}$.
- $\alpha_{\mathrm{self}}[\cdot]$ is called the self-diffusion matrix.

A phase transition conjecture for 2D Coulomb stochastic dynamics

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt$$

$$\lim_{\epsilon\to 0} X_{t/\epsilon^2}^{i_0} = \sqrt{2\alpha_{\mathsf{self}}[\mu_{\mathsf{gin},\beta}]} B_t, \qquad X_0^{i_0} = 0, \quad \sum_{i\neq i_0} \delta_{X_0^i} \sim \mu_{\mathsf{gin},\beta,\mathbf{o}}.$$

Conj: There exist constants $\beta_1 < \beta_2 < \beta_3$ such that

(C1) $\beta < \beta_1 \Rightarrow \alpha_{\mathsf{self}}[\mu_{\mathsf{gin},\beta}] > 0$ (diffusive)

(C2) $\beta_1 < \beta < \infty \Rightarrow \alpha_{\mathsf{self}}[\mu_{\mathsf{gin},\beta}] = 0$ (subdiffusive),

(C3) $\beta_2 < \beta < \infty \Rightarrow X_t^{i_0}$ has an inv prob measure

$X_t^{i_0} = O(\log t)$ (log behaivior)

(C4) $\beta_3 < \beta < \infty \Rightarrow (X_t^i)_{i\in\mathbb{N}}$ form a lattice like system.

Moreover,

$$\beta_1 \sim 1, \quad \beta_2 \sim 2.$$

Rigorous results: homogenization of diffusion in 2D Coulomb-periodic env.

Let $\mathsf{s} = \sum_i \delta_{s_i} \in \mathsf{S}$. Let $X_t^{\mathsf{s}} \in \mathbb{R}^2$ be the solution of

$$dX_t^{\mathsf{s}} = dB_t + \frac{\beta}{2} \lim_{q\to\infty} \sum_{|X_t^{\mathsf{s}} - s_i| < q} \frac{X_t^{\mathsf{s}} - s_i}{|X_t^{\mathsf{s}} - s_i|^2} dt$$

Let μ be a rpf, and set for a.s. s w.r.t. μ

$$\lim_{\varepsilon\to\infty} \varepsilon X^{\mathsf{s}}_{t/\varepsilon^2} = \sqrt{\alpha^{\beta}_{\text{eff}}[\mu]} B_t \tag{14}$$

Thm 19. μ_{per} *be a periodic rpf* $\Rightarrow$

(1) $\alpha^{\beta}_{\text{eff}}[\mu_{\text{per}}] > 0$.

(2) $\alpha^{\beta}_{\text{eff}}[\mu_{\text{per},0}] > 0$ *for* $\beta < 1$

$\alpha^{\beta}_{\text{eff}}[\mu_{\text{per},0}] = 0$, X_t^{s} *has a inv prob m for* $\beta > 2$

Rigorous results: homogenization of diffusion in Ginibre env.

Let $\mathsf{s} = \sum_i \delta_{s_i} \in \mathsf{S}$. Let $X_t^{\mathsf{s}} \in \mathbb{R}^2$ be the solution of

$$dX_t^{\mathsf{s}} = dB_t + \lim_{q\to\infty} \sum_{|X_t^{\mathsf{s}}-s_i|<q} \frac{X_t^{\mathsf{s}} - s_i}{|X_t^{\mathsf{s}} - s_i|^2} dt$$

Thm 20. *Assume* $\mathsf{s} \sim \mu_{\mathsf{gin},2,\mathbf{o}}$ *and set*

$$\lim_{\varepsilon\to\infty} \varepsilon X^{\mathsf{s}}_{t/\varepsilon^2} = \sqrt{\alpha^2_{\mathsf{eff}}[\mu_{\mathsf{gin},2,\mathbf{o}}]} B_t$$

Then

$$\alpha^2_{\mathsf{eff}}[\mu_{\mathsf{gin},2,\mathbf{o}}] = 0$$

- We use (5) in Thm 10 to prove Thm 20

Conj: The positivity of $\alpha^2_{\mathsf{eff}}[\mu_{\mathsf{gin},2}]$ is an open problem. Since $\mu_{\mathsf{gin},2}$ is similar to μ_{per}, we should have

$$\alpha^2_{\mathsf{eff}}[\mu_{\mathsf{gin},2}] > 0$$

Observation: self-diffusion of 2D Coulomb system 1

Obs 0: $\mu_{\mathsf{gin},\beta}$ exists for general $\beta > 0$.

Obs 1: Since (by O.-Shirai [2012])

$$\mu_{\mathsf{gin},2} \perp \mu_{\mathsf{gin},2,\mathbf{o}} \tag{15}$$

we have for general $\beta > \beta_1$ $\qquad (\beta_1 \le 2)$

$$\mu_{\mathsf{gin},\beta} \perp \mu_{\mathsf{gin},\beta,\mathbf{o}}. \tag{16}$$

Let $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}}$ be the solution of

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \tag{17}$$

Note that

$$\mathsf{X}_t := \sum_{i\in\mathbb{N}} \delta_{X_t^i} \sim \mu_{\mathsf{gin},\beta} \tag{18}$$

Observation: self-diffusion of 2D Coulomb system 2

Let X_t^1 be the tag particle, and set $Y_t^i = X_t^{i+1} - X_t^1$.
$\mathsf{Y}_t = \sum_{i\in\mathbb{N}} \delta_{Y_t^i}$ is the env seen from the tagged particle.

Obs 2: By (16) and

$$\mathsf{Y}_t := \sum_{i \neq i_0} \delta_{Y_t^i} \sim \mu_{\mathsf{gin},\beta,\mathbf{o}}, \tag{19}$$

we have $X_t^* \in \mathbb{C}$ such that (by Goldman coupling if $\beta = 2$)

$$X_t^* \sim \text{prob m}, \qquad \mathsf{Y}_t + \delta_{X_t^*} \sim \mathsf{X}_t \sim \mu_{\mathsf{gin},\beta} \tag{20}$$

Obs 3:

$$dX_t^1 = dB_t^1 + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^1 - X_t^i|<r,\ i\geq 2} \frac{X_t^1 - X_t^i}{|X_t^1 - X_t^i|^2} dt \qquad (21)$$

$$dX_t^1 = dB_t^1 - \frac{\beta}{2} \lim_{r\to\infty} \sum_{|Y_t^i|<r,\ i\in\mathbb{N}} \frac{Y_t^i}{|Y_t^i|^2} dt$$

Observation: self-diffusion of 2D Coulomb system 4

Set $Y_t^* = X_t^* - X_t^1$. Then from

$$dX_t^1 = dB_t^1 - \frac{\beta}{2} \lim_{r\to\infty} \sum_{|Y_t^i|<r,\ i\in\mathbb{N}} \frac{Y_t^i}{|Y_t^i|^2} dt \tag{22}$$

we have

$$dX_t^1 = dB_t^1 - \frac{\beta}{2} \frac{X_t^1 - X_t^*}{|X_t^1 - X_t^*|^2} dt - \frac{\beta}{2} \lim_{r\to\infty} \sum_{|Y_t^i|<r,\ i\in\mathbb{N}\cup\{*\}} \frac{Y_t^i}{|Y_t^i|^2} dt$$

Hence

$$\begin{aligned} dX_t^1 &= dB_t^1 - \frac{\beta}{2} \frac{X_t^1}{|X_t^1|^2} dt \\ &+ \frac{\beta}{2} \{ \frac{X_t^1}{|X_t^1|^2} - \frac{X_t^1 - X_t^*}{|X_t^1 - X_t^*|^2} \} dt - \frac{\beta}{2} \lim_{r\to\infty} \sum_{\substack{|Y_t^i|<r, \\ i\in\mathbb{N}\cup\{*\}}} \frac{Y_t^i}{|Y_t^i|^2} dt \end{aligned} \tag{23}$$

$$dX_t^1 = dB_t^1 - \frac{\beta}{2}\frac{X_t^1}{|X_t^1|^2}dt + \frac{\beta}{2}\{\frac{X_t^1}{|X_t^1|^2} - \frac{X_t^1 - X_t^*}{|X_t^1 - X_t^*|^2}\}dt - \frac{\beta}{2}\lim_{r\to\infty}\sum_{|Y_t^i|<r,\ i\in\mathbb{N}\cup\{*\}}\frac{Y_t^i}{|Y_t^i|^2}dt \tag{24}$$

Obs 4: (1) By homogenization, $\exists\sqrt{2a[\beta]} \le E$

$$\epsilon\{B^1_{u/\epsilon^2} - \frac{\beta}{2}\lim_{r\to\infty}\int_0^{u/\epsilon^2}\sum_{|Y_t^i|<r,\ i\in\mathbb{N}\cup\{*\}}\frac{Y_t^i}{|Y_t^i|^2}dt\} = \sqrt{2a[\beta]}\hat{B}_u \tag{25}$$

Since X_t^* has inv prob

$$\epsilon\int_0^{u/\epsilon^2}\frac{\beta}{2}\{\frac{X_t^1}{|X_t^1|^2} - \frac{X_t^1 - X_t^*}{|X_t^1 - X_t^*|^2}\}dt \sim o(\epsilon) \tag{26}$$

Hence we have (approximately)

$$dX_t^1 = \sqrt{2a[\beta]}dB_t^1 - \frac{\beta}{2}\frac{X_t^1}{|X_t^1|^2}dt \tag{27}$$

By the simple calculation ($\beta > \beta_{00}$, $\tilde{B}_t$ is 1D Br m)

$$d|X_t^1| = \sqrt{2a[\beta]}d\tilde{B}_t + (a[\beta] - \frac{\beta}{2})\frac{1}{|X_t^1|}dt \tag{28}$$

So the phase transition follows from the one of Bessel processes.

Simulation of Coulomb interacting Brownian motions

Simulation of Ginibre IBM (2D Coulomb system) and phase transition

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r \ \ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \qquad \text{(T)}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{-\alpha X_t^i + \lim_{r\to\infty} \sum_{|X_t^j|<r \ \ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\} dt. \qquad \text{(OU)}$$

Here, since $\rho^1 = 1/\pi$, $\alpha = |\{|x| \le 1\}|\rho^1 = 1$.

- Taking (OU) & (T)into account we take the model:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{-X_t^i + \sum_{j=1 \ \ j\neq i}^{N} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\} dt \qquad \text{(OUN)}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{ \sum_{j=1 \ \ j\neq i}^{N} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\} dt \qquad \text{(TN)}$$

Simulation: 3D Coulomb system

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r \ \ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \tag{T}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{-\alpha X_t^i + \lim_{r\to\infty} \sum_{|X_t^j|<r \ \ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\} dt. \tag{OU}$$

We take $\rho^1 = 1$. So $\alpha = |\{|x| \leq 1\}|\rho^1 = 4\pi/3$.

- Taking (OU) & (T)into account we take the model:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{-\frac{4\pi}{3} X_t^i + \sum_{j=1 \ \ j\neq i}^{N} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\} dt \tag{OUN}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{\sum_{j=1 \ \ j\neq i}^{N} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\} dt \tag{TN}$$