# Strong solutions of infinite-dimensional stochastic differential equations

2013/12/4/Wed Courant/Probability and Mathematical Physics Seminar

## Motivation

- Soft edge scaling limit and Airy RPFs
- Coulomb stochastic dynamics in infinite dimensions

## General Theory

- Quasi-Gibbs measures & Log derivative
- Tail triviality
- Existence and uniqueness of strong solutions of ISDEs
- Uniqueness of quasi-regular Dirichlet forms

- Let $S = \mathbb{R}^d$. Consider infinte-many particles $\{X_t^i\}_{i\in\mathbb{N}}$ with potentials: $\Phi$ and $\Psi$

$$\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\} \quad \text{(free potential)}$$
$$\Psi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\} \quad \text{(interaction potential)}$$

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- The infinite-dimensional SDEs we study are (typically)

$$dX_t^i = dB_t^i - \frac{1}{2}\nabla\Phi(X_t^i)dt - \frac{1}{2}\sum_{j\neq i}^{\infty}\nabla\Psi(X_t^i - X_t^j)dt.$$

Here $(B^i)$ are indep copies of $d$-dim Brownian motions.

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# ISDEs

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Here $(B^i)$ are indep copies of $d$-dim Brownian motions.

- The purpose of this talk is to provide a general method to construct unique strong solutions of the above ISDEs, even if the interaction potentials are long range such as log potentials.

# Examples

All examples below have unique strong solutions :

- that preserves the tail $\sigma$ field of the configuration spaces.
- reversible (suitable) equilibrium states.

## Gibbs measures: Ruelle's class potentials

- All Gibbs measures with Ruelle's class potentials.

with marginal assumptions explained later.

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**Lennard-Jones 6-12 potential:** Let $d = 3$ and $\beta > 0$.

$$\Psi_{6,12}(x) = |x|^{-12} - |x|^{-6}$$

The corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^{\infty} \{ \frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}} \} dt \quad (i \in \mathbb{N})$$

# Examples: sine rpf (Dyson's model)–bulk scaling limit

Sine$_\beta$ RPF: $S = R,\ \beta = 1, 2, 4,\ \Phi = 0,\ \Psi(x) = -\log|x|$.

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r,\ j \neq i} \frac{1}{X_t^i - X_t^j} dt$$

# First motivation:

# Dynamical soft edge scaling limit

# of

# Gaussian ensembles

## Motivation 1

- The dist of eigen values of the G(O/U/S)E Random Matrices are given by ($\beta = 1, 2, 4$)

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^{N} |x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \qquad (1)$$

- The distribution of

$$N^{-1}\sum_{i=1}^{N}\delta_{x_i/\sqrt{N}} \quad \text{under } m_\beta^N$$

converges the semi-circle law

$$\varsigma(x)dx = \frac{1}{2\pi}\sqrt{4 - x^2}dx \qquad (2)$$

## Sine rpf (Dyson's model)–Bulk scaling limit

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^N |x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \quad \varsigma(x)dx = \frac{1}{2\pi}\sqrt{4-x^2}dx$$

- Take $x_i = s_i/\sqrt{N}$ in (1) and set

$$\mu_{\mathsf{sin},\beta}^N(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}^N |s_i - s_j|^\beta \prod_{k=1}^N e^{-\beta|s_k|^2/4N} d\mathbf{s}_N \tag{3}$$

- The associated $N$ particle system is given by the SDE: 30p

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^N \frac{1}{X_t^i - X_t^j}dt - \frac{\beta}{4N}X_t^i dt \tag{4}$$

- So the ass $\infty$ particle system is given by

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^\infty \frac{1}{X_t^i - X_t^j}dt$$

## Airy rpf – Soft edge scaling limit

Airy rpf: $\mu_{\mathsf{Ai},\beta}$ $(S = \mathbb{R},\ \beta = 1, 2, 4)$

Take the scaling $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$ in

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z} \prod_{i<j}^{N} |x_i - x_j|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^N |x_i|^2} d\mathbf{x}_N$$

and set

$$\mu_{\mathsf{Ai},\beta}^N(d\mathbf{s}_N) = \frac{1}{Z} \prod_{i<j}^{N} |s_i - s_j|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^N |2\sqrt{N} + N^{-1/6} s_i|^2} d\mathbf{s}_N.$$

Then $\mu_{\mathsf{Ai},\beta}$ is the TDL of $\mu_{\mathsf{Ai},\beta}^N$:

$$\lim_{N\to\infty} \mu_{\mathsf{Ai},\beta}^N = \mu_{\mathsf{Ai},\beta}$$

- $\beta = 2 \Rightarrow \mu_{\mathsf{Ai},\beta}$ is the det rpf gen by $(K_{\mathsf{Ai}}, dx)$:

$$K_{\mathsf{Ai}}(x,y) = \frac{\mathsf{Ai}(x)\mathsf{Ai}'(y) - \mathsf{Ai}'(x)\mathsf{Ai}(y)}{x-y}$$

Here Ai(·) the Airy function such that

$$\mathsf{Ai}(z) = \frac{1}{2\pi}\int_{\gamma} dk\, e^{i(zk+k^3/3)}, \qquad z \in \mathbb{C}. \tag{5}$$

If $\beta = 1, 4$, the correlation func of $\mu_{\mathsf{Ai},\beta}$ are given by similar formula of quaternion determinant.

## Airy rpf – Dynamical soft edge scaling limit

- From

$$\mu^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N$$

we deduce the SDE of the $N$ particle system:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j=1, j\neq i}^N \frac{1}{X_t^i - X_t^j}dt - \frac{\beta}{2}\{N^{1/3} + \frac{1}{2N^{1/3}}X_t^i\}dt$$

- Problem: What is the limit SDE?

Does $\lim_{N\to\infty}\{\sum_{j=1, j\neq i}^N \frac{1}{X_t^i - X_t^j} - N^{1/3}\}$ converge ?

How to solve the limit SDE?

**Thm 1** (with Tanemura). *Let* $\beta = 1, 2, 4$. *Then:*

- *the limit ISDE is*

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \to \infty} \{ ( \sum_{j \neq i,\ |X_t^j| < r} \frac{1}{X_t^i - X_t^j} ) - \int_{|x| < r} \frac{\varrho(x)}{-x} dx \} dt$$

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty, 0]}(x)$$

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Airy rpf – Dynamical soft edge scaling limit

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- So far the sto dyn related to Airy RPF was constructed only for $\beta = 2$ by Spohn, Johansson, and others by the method of space-time cor funs. This sto dyn is same as the above.
- The labeled dyn $\mathbf{X}_t = (X_t^i)$ is a diffusion with state space $\mathbb{R}^{\mathbb{N}}$.
- The unlabeled dyn $\mathsf{X}_t = \sum_i^\infty \delta_{X_t^i}$ is reversible w.r.t. $\mu_{\mathrm{airy},\beta}$.

# Quasi-Gibbs measures,
# Log derivative,
# and
# (weak) solutions of ISDEs

## Ψ-Quasi-Gibbs meas.

- $S=\mathbb{R}^d$, $S_r=\{|x|\le r\}$, $\mathsf{S}=\{\mathsf{s}=\sum_i \delta_{s_i},\ \mathsf{s}(S_r)<\infty(\forall r)\}$
- $\pi_r,\pi_r^c:\mathsf{S}\to\mathsf{S}$, $\pi_r(\mathsf{s})=\mathsf{s}(\cdot\cap S_r)$, $\pi_r^c(\mathsf{s})=\mathsf{s}(\cdot\cap S_r^c)$

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$\Psi$-Quasi-Gibbs meas.

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- Let $\mu$ be a RPF over $S$.

$$\mu_{r,\xi}^m(\cdot)=\mu(\pi_r\in\cdot|\mathsf{s}(S_r)=m, \pi_r^c(\mathsf{s})=\pi_r^c(\xi))$$

- Let $\Psi : S\to\mathbb{R}\cup\{\infty\}$ (interaction).

$$\mathcal{H}_r=\sum_{s_i,s_j\in S_r, i<j}\Psi(s_i-s_j)$$

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## $\Psi$-Quasi-Gibbs meas.

- $S = \mathbb{R}^d$, $S_r = \{|x| \le r\}$, $\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i},\ \mathsf{s}(S_r) < \infty (\forall r)\}$
- $\pi_r, \pi_r^c : \mathsf{S} \to \mathsf{S}$, $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r)$, $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$
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Def: $\mu$ is $\Psi$-quasi-Gibbs measure if $\exists\ c_{r,\xi}^m$ s.t.

$${c_{r,\xi}^m}^{-1} e^{-\mathcal{H}_r} d\Lambda_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\Lambda_r^m$$

Here $\Lambda_r^m = \Lambda(\cdot | \mathsf{s}(S_r) = m)$ and $\Lambda_r$ is the Poisson RPF with $1_{S_r} dx$.

- Gibbs measures $\Rightarrow$ Quasi-Gibbs measure .

## Application of quasi-Gibbs property to dynamics

(A1) $\mu$ is a $\Psi$-quasi-Gibbs m with upper-semicont $\Psi$.

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

Here $S_r = \{|x| < r\}$, $\mathsf{S}_r^k = \{\mathsf{s}(S_r) = k\}$, $\sigma_r^k$ is $k$-density fun on $S_r$.

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Let $\mathcal{D}_0$ be the set of local, smooth functions on $\mathsf{S}$.

Let $\tilde{f}(s_1, \ldots) = f(\mathsf{s})$, where $\tilde{f}$ is symmetric, $\mathsf{s} = \sum \delta_{s_i}$.

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g]\mu(d\mathsf{s}), \;\; \mathbb{D}[f,g] = \frac{1}{2}\sum_i \nabla_i \tilde{f} \cdot \nabla_i \tilde{g}$$

$$\mathcal{D}_0^{\mu} = \{f \in \mathcal{D}_0; \mathcal{E}^{\mu}(f,f) < \infty, f \in L^2(\mu)\}$$

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**Thm 2.** (1) (A1) $\Rightarrow$ $(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$ *is closable on* $L^2(\mu)$.

(2) (A1), (A2) $\Rightarrow$ $\exists$ *diffusion* $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ *associated with*

$(\mathcal{E}^{\mu}, \mathcal{D}_0^{\mu})$ *on* $L^2(\mu)$.

# General theorems on infinite-dim SDEs

(A1) $\mu$ is a $\Psi$-quasi-Gibbs m with upper-semicont $\Psi$.

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

(A3) $\{X_t^i\}$ do not collide each other (non-collision)

(A4) each tagged particle $X_t^i$ never explode (non-explosion)

By (A3) and (A4) the labeled dynamics

$$\mathbf{X}_t = (X_t^1, X_t^2, \dots)$$

can be constructed from the unlabeled dynamics

$$\mathsf{X}_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i}.$$

Indeed, the particles keep the initial label forever.

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To represent $\mathbf{X}_t$ by ISDEs, we introduce the log derivative of $\mu$.

## Log derivative of $\mu$: precise correspondence between RPFs & potentials

- Let $\mu_x$ be the (reduced) Palm m. of $\mu$ conditioned at $x$

$$\mu_x(\cdot) = \mu(\cdot - \delta_x | \mathsf{s}(x) \geq 1)$$

- Let $\mu^1$ be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^1(A \times B) = \int_A \rho^1(x)\mu_x(B)dx$$

- $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathsf{S}, \mu^1)$ is called the log derivative of $\mu$ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^1 = - \int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}^\mu d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}$$

Here $\nabla_x$ is the nabla on $\mathbb{R}^d$, $\mathcal{D}$ is the space of bounded, local smooth functions on S.

- Very informally

$$\mathsf{d}^\mu = \nabla_x \log \mu^1$$

## Log derivative

- If $\mu^1(dxd\mathsf{s}) = m(x, s_1, \ldots)dx \prod_i ds_i$, then

$$
\begin{aligned}
& -\int \nabla_x f(x, s_1, \ldots)\mu^1(dx ds_1 \cdots) \\
= & -\int \nabla_x f(x, s_1, \ldots) m(x, s_1, \ldots) dx \prod_i ds_i \\
= & \int f(x, s_1, \ldots) \nabla_x m(x, s_1, \ldots) dx \prod_i ds_i \\
= & \int f(x, s_1, \ldots) \frac{\nabla_x m(x, s_1, \ldots)}{m(x, s_1, \ldots)} m(x, s_1, \ldots) dx \prod_i ds_i .
\end{aligned}
$$

Hence

$$
\mathsf{d}^{\mu} = \frac{\nabla_x m(x, s_1, \ldots)}{m(x, s_1, \ldots)} = \nabla_x \log m(x, s_1, \ldots).
$$

This is very informal calculation.

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**Thm 3.** (O.12(PTRF)) (A1)–(A5) $\Rightarrow \exists \mathsf{S}_0 \subset \mathsf{S}$ *such that*

$$\mu(\mathsf{S}_0) = 1,$$

*and that, for* $\forall \mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, $\exists \mathfrak{u}^{-1}(\mathsf{S}_0)$*-valued pr.* $(X_t^i)_{i\in\mathbb{N}}$ *and* $\exists S^{\mathbb{N}}$*-valued Brownian m.* $(B_t^i)_{i\in\mathbb{N}}$ *satisfying*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^\mu(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

Here $\mathfrak{u}: S^{\mathbb{N}} \to \mathsf{S}$ such that $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$.

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The solution $(\mathbf{X}, \mathbf{B})$ is not a strong solution.

We next construct a strong solution from a weak solution.

# Strong solutions

- To construct strong solutions

we have two important geometric properties of RPFs.

Tail triviality & Tail decomposition

Let $\mathcal{T} = \mathcal{T}(\mathsf{S})$ be the tail $\sigma$ field of $\mathsf{S}$:

$$\mathcal{T}(\mathsf{S}) = \cap_{r=1}^{\infty} \sigma[\pi_r^c] \quad (\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)).$$

**Thm 5.** *Let $\mu$ be a det RPF. Then $\mathcal{T}(\mathsf{S})$ is $\mu$-trivial.*

- Thm 5 is a generalization of that for the discrete determinantal RPFs due to Russel Lyons, Shirai-Takahashi.

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Let $\mathcal{T} = \mathcal{T}(\mathsf{S})$ be the tail $\sigma$ field of S:

$$\mathcal{T}(\mathsf{S}) = \cap_{r=1}^{\infty} \sigma[\pi_r^c] \quad (\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)).$$

**Thm** 5 Let $\mu$ be a det RPF. Then $\mathcal{T}(\mathsf{S})$ is $\mu$-trivial.

**Thm 6.** *Let $\mu$ be a quasi-Gibbs measure. Let $\mu(\cdot|\mathcal{T})$ be the regular conditional probability. Then*

$$\mu(\cdot) = \int_{\mathsf{S}} \mu(\cdot|\mathcal{T})(\xi)\mu(d\xi)$$

*and, for $\mu$-a.s. $\xi$,*

$$\mu(A|\mathcal{T})(\xi) = 1_A(\xi) \qquad \textit{for any } A \in \mathcal{T}.$$

• Thm 6 is a generalization of that for the discrete Gibbs m due to Georgii.

existence of strong solution

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{\mathsf{j}\neq\mathsf{i}} \delta_{X_t^j}) dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

We consider a condition such that the drifts $\mathsf{d}^{\mu}(x, \mathsf{s})$ are locally Lipschitz continuous in $x$.

Let $S_r = \{|x| < r\}$ and

$$\mathsf{H}(r,n) = \{\mathsf{s} = \sum_i \delta_{s_i}; |\nabla_x \mathsf{d}^{\mu}(s_i, \mathsf{s} - \delta_{s_i})| < n \text{ for } \forall i \text{ s.t. } s_i \in S_r\},$$

$$\mathsf{H} = \bigcap_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \mathsf{H}(r,n).$$

(A6) $\mathrm{Cap}^{\mu}(\mathsf{H}^c) = 0$.

## existence of strong solution

(A1) $\mu$ is a $\Psi$-quasi-Gibbs m with upper-semicont $\Psi$.

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

(A3) $\{X_t^i\}$ do not collide each other

(A4) each tagged particle $X_t^i$ never explode

(A5) The log derivative $\mathsf{d}^{\mu} \in L^1_{\mathrm{loc}}(\mu^1)$ exists

(A6) $\mathrm{Cap}^{\mu}(\mathsf{H}^c) = 0$.

**Thm 7** (O.-Tanemura)**.** (A1)–(A6). $\Rightarrow$ (1) *The ISDE*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \ (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

*has a strong solution for* $\mathbf{s} = (s_i) \in S^{\mathbb{N}}$ *s.t.* $\sum_i \delta_{s_i} \in \mathsf{H}$.

• • •

existence of strong solution

(A1) $\mu$ is a $\Psi$-quasi-Gibbs m with upper-semicont $\Psi$.
(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$
(A3) $\{X_t^i\}$ do not collide each other
(A4) each tagged particle $X_t^i$ never explode
(A5) The log derivative $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mu^1)$ exists
(A6) $\mathrm{Cap}^\mu(\mathsf{H}^c) = 0$.

**Thm 7**[O.-Tanemura] (A1)–(A6). $\Rightarrow$ (1) The ISDE

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^\mu(X_t^i, \sum_{\mathrm{j}\neq\mathrm{i}} \delta_{X_t^j})dt, \ \ (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

has a strong solution for $\mathbf{s} = (s_i) \in S^{\mathbb{N}}$ s.t. $\sum_i \delta_{s_i} \in \mathsf{H}$.
(2) The ass unlabeled diffusion $\mathsf{X} = \sum_i \delta_{X^i}$ satisfies

$$P_{\mu_\xi} \circ \mathsf{X}_t^{-1} \prec \mu_\xi \quad (\forall t) \quad \text{for } \mu\text{-a.s. } \xi$$

Here $\mu_\xi = \mu(\cdot|\mathcal{T}(\mathsf{S}))(\xi)$ in Thm 6.

**Thm 8** (O.-Tanemura). *Assume* (A1)–(A6).
*Let* $\mathbf{X} = (X^i)$ *and* $\hat{\mathbf{X}} = (\hat{X}^i)$ *be strong sol of the ISDE*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s} = (s_i)_{i\in\mathbb{N}}$$

*on the same Br m. Let* $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ *and* $\hat{\mathsf{X}}_t = \sum_i \delta_{\hat{X}_t^i}$.
*Suppose, for* $\mu$*-a.s.* $\xi$,

$$P_{\mu_\xi} \circ \mathsf{X}_t^{-1} \prec \mu_\xi \text{ and } P_{\mu_\xi} \circ \hat{\mathsf{X}}_t^{-1} \prec \mu_\xi \ (\forall t)$$

*Then*

$$\mathbf{X} = \hat{\mathbf{X}} \text{ a.s.} \qquad \text{for } \mu\text{-a.s. } \mathsf{s} = \sum_{i=1}^{\infty} \delta_{s_i}$$

**Thm 9** (O.-Tanemura). *Assume* (A1)–(A7). *Here*
*(A7) $\mu$ is tail trivial.*
*Then the strong solution* $\mathbf{X} = (X^i)$ *such that*

$$P_\mu \circ \mathsf{X}_t^{-1} \prec \mu \qquad \textit{for all } t$$

*is unique for* $\mu$*-a.e.* $\mathsf{x} = \sum_i \delta_{x_i}$ *Here* $\mathsf{X}$ *is the unlabeled dynamics of* $\mathbf{X}$*:*

$$\mathsf{X}_t = \sum_i^\infty \delta_{X_t^i}$$

Cor If $\mu$ is a determinantal RPF, then the strong, solution of the ISDE that is reversible w.r.t. $\mu$ is unique.

- Tail $\sigma$-fields of Airy, Sine, Ginibre RPFs with $\beta = 2$ are trivial.

Let $\mathcal{D}^{\mu}_{\text{poly}}$ be the closure of the set of polynomials on S such that $\mathcal{E}^{\mu}_{1}(f,f) < \infty$. Then

$$\mathcal{D}^{\mu}_{\text{poly}} \subset \mathcal{D}^{\mu}$$

because polynomials are local and smooth.

**Thm 10** (O.-Tanemura)**.** *Assume* (A1)–(A7)*. Then quasi-regular Dirichlet forms that are extension of* $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}_{\text{poly}})$ *are unique.*

In particular, $\mathcal{D}^{\mu}_{\text{poly}} = \mathcal{D}^{\mu}$, and Lang's construction and O.'s construction are same.

*Remark* 1. If (A5) (non-explosion) does not hold. Then Thm 10 does not hold. This is very natural theorem that says the uniqueness of Dirichlet forms is related to the non-explosion problem of tagged problem.

# Application to soft-edge scaling Airy RPFs

## Examples: Airy rpf – Soft edge scaling limit

**Thm 11** (O.-Tanemura). *Let* $\beta = 1, 2, 4$. *Then:*

- *The log derivative* $\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}$ *is*

$$\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}(x, \mathsf{s}) = \beta \lim_{r\to\infty} \{ ( \sum_{|x - s_i| < r} \frac{1}{x - s_i} ) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \}$$

*Here*

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty, 0]}(x)$$

- *Airy rpf* $\mu_{\mathsf{Ai},\beta}$ *satisfy* (A1)–(A6) *and the limit ISDE is*

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ ( \sum_{j \neq i,\ |X_t^j| < r} \frac{1}{X_t^i - X_t^j} ) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

- The key idea is to take the rescaled semi-circle law $\varsigma$, as the first approximation of the 1-correlation fun $\rho^{N,1}_{\mathsf{Ai},\beta}$.
- Our method can be applied to other soft edge scaling. Our result is the first time to clarify the SDE describing the limit infinite system for the soft edge.

**Thm 12** (O.-Tanemura). *Assume* $\beta = 2$.

*Let us label* $X_t^i > X_t^{i+1}$ $(\forall i)$. *Then :*

(1) *The top particle* $X_t^1$ *is the Airy process* $\mathcal{A}(t)$ *in the sense of Spohn.*

(2) *The infinite dim stochastic dynamics constructed by Spohn, Johansson & others by the space-time correlation fun is a solution of the prescribed SDE:*

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ ( \sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j} ) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

Examples: Airy rpf – Soft edge scaling limit

- The SDE gives a kind of Girsanov formula.
- These examples are the first time that the infinite dynamics are constructed for rpf appeared in random matrix theory with $\beta = 1, 4$ even if the bulk and the hard edge as well as the soft edge scaling

  In one dimensional system, the method of space-time correlation functions are available (Nagao, Katori-Tanemura, Spohn, and others), but this method is restricted to $\beta = 2$.
- By construction, if the total system start from the $\mathrm{Airy}_2$ rpf $\mu_{\mathrm{Ai},2}$, then the distribution of the top particle $X_t^1$ equals $F_{2,edge}(x)$, the 2 Tracy-Widom distribution.

# Idea of "strong sol of ISDEs"

2013/12/4 Courant

- General theory to construct unique, strong solutions of infinite-dimensional stochastic differential equations

## Strong solutions of ISDE: Non Markov type

$S = \mathbb{R}^d, [0,\infty), \mathbb{C}$

$W(S^{\mathbb{N}}) = C([0,T); S^{\mathbb{N}})$, $(0 < T < \infty)$ labeled path sp.

- a quadruplet $(\{\sigma^i\}, \{b^i\}, W_{\mathsf{sol}}, \mathbf{S_0})$

$W_{\mathsf{sol}}$ : a Borel subset of $W(S^{\mathbb{N}})$ sp of solutions of ISDE

$\sigma^i, b^i : W_{\mathsf{sol}} \to W(S^{\mathbb{N}})$ coefficients of ISDE

$\mathbf{S_0}$ be a Borel subset of $S^{\mathbb{N}}$ initial starting points of ISDE

- the ISDE on $S^{\mathbb{N}}$ of the form

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N}) \tag{6}$$

$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i \in \mathbb{N}} \in \mathbf{S_0} \tag{7}$$

$$\mathbf{X} \in W_{\mathsf{sol}}. \tag{8}$$

- $\mathbf{X} = \{(X_t^i)_{i\in\mathbb{N}}\}_{t\in[0,T)} \in W_{\mathsf{sol}}$
- $\mathbf{B} = (B^i)$ $(i \in \mathbb{N})$ is the $S^{\mathbb{N}}$-valued standard Br motion.

## Strong solutions of ISDE: Assump (P1)

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N})$$
$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathbf{S_0}$$
$$\mathbf{X} \in W_{\mathrm{sol}}.$$

(P1) ISDE (6) has a solution $(\mathbf{X}, \mathbf{B})$. (not a strong sol! )

Here $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ is the Brownian motion on $S^{\mathbb{N}}$

Problem: Prove that $\mathbf{X}$ is a functional of the Br $\mathbf{B}$

Idea:
Strong solutions of Infinite-dimensional SDE
$\Leftrightarrow$
Infinite-many, finite-dimensional SDEs with consistency
$+$
Triviality of Tail $\sigma$-field of label pathes

Assump (P2) infinite-many, finite-dimensional SDEs with consistency

- $\bar{P}_{\mathbf{s}}$: a prob meas on $W(S^{\mathbb{N}}) \times W^{\mathbf{0}}(S^{\mathbb{N}})$
- $\bar{P}_{\mathbf{s},\mathbf{B}} = \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot|\mathbf{B})$: the regular conditional prob
- $\mathbf{P}_{\mathbf{s}} = \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot)$, $\quad P^{\infty}_{\mathsf{Br}} = \bar{P}_{\mathbf{s}}(\mathbf{B} \in \cdot)$

For $\mathbf{X} \in W_{\mathsf{sol}}$, $\mathsf{s} \in \mathbf{S_0}$, and $m \in \mathbb{N}$,
we introduce a new SDE (11) on $\mathbf{Y}^m = (Y_t^1, \ldots, Y_t^m)$.

$$dY_t^i = \sigma^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dt \tag{9}$$

$$\mathbf{Y}_0^m = (s_1, \ldots, s_m) \in \mathsf{S}^m, \quad \text{where } \mathsf{s} = (s_i)_{i=1}^{\infty},$$

$$\mathbf{Y}^m + \mathbf{X}^{m*} \in W_{\mathsf{sol}}.$$

Here $\mathbf{X}^{m*} = (0, \ldots, 0, X_t^{m+1}, X_t^{m+2}, \ldots)$ and we set

$$\mathbf{Y}^m + \mathbf{X}^{m*} = (Y_t^1, \ldots, Y_t^m, X_t^{m+1}, X_t^{m+2}, \ldots). \tag{10}$$

$\mathbf{X}^{m*}$ is interpreted as a part of the coefficients of the SDE (11).

$$dY_t^i = \sigma^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m + \mathbf{X}^{m*})_t dt \tag{11}$$
$$\mathbf{Y}_0^m = (s_1, \ldots, s_m) \in \mathsf{S}^m,$$
$$\mathbf{Y}^m + \mathbf{X}^{m*} \in W_{\mathsf{sol}}.$$

(P2) The SDE (11) has a unique, strong solution for each $\mathbf{s} \in \mathbf{S_0}$, $\mathbf{X} \in W_{\mathsf{sol}}^{\mathbf{s}}$, and $m \in \mathbb{N}$.

## Strong solutions of ISDE: (P3) Tail triviality

Let $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ be the tail $\sigma$-field of $W(S^{\mathbb{N}})$; we set

$$\mathit{Tail}\,(W(S^{\mathbb{N}})) = \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}] \tag{12}$$

$$\mathit{Tail}^{\,[1]}(\mathbf{P}) = \{A \in \mathit{Tail}\,(W(S^{\mathbb{N}}))\,;\, \mathbf{P}(A) = 1\}.$$

Here $\mathbf{P}$ is a probability measure on $W(S^{\mathbb{N}})$.

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P_s}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

(P1) ISDE (6) has a solution $(\mathbf{X}, \mathbf{B})$.

(P2) SDE (11) has a unique, strong solution for all $\mathbf{s}, \mathbf{X}, m$.

(P3) $\mathcal{T}ail\,(W(S^{\mathbb{N}}))$ is $\mathbf{P}_{\mathbf{s}}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

**Thm 13.** *Assume* (P1)–(P3). *Then*

(1) *ISDE* (6)–(8) *has a strong solution for each* $\mathbf{s} \in \mathbf{S}_0$.

(2) *Let* $\mathbf{Y}_{\mathbf{s}}$ *and* $\mathbf{Y}'_{\mathbf{s}}$ *be strong solutions of ISDE* (6)–(8) *starting at* $\mathbf{s} \in \mathbf{S}_0$ *defined on the same space of Brownian motions* $\mathbf{B}$. *Then* $\mathbf{Y}_{\mathbf{s}} = \mathbf{Y}'_{\mathbf{s}}$ *a.s. if and only if*

$$\mathcal{T}ail^{[1]}(\mathrm{Law}(\mathbf{Y}_{\mathbf{s}})) = \mathcal{T}ail^{[1]}(\mathrm{Law}(\mathbf{Y}'_{\mathbf{s}})). \tag{13}$$

## Strong solutions of ISDE: Idea of Main Theorem 1 (1)

(P1) ISDE (6) has a solution $(\mathbf{X}, \mathbf{B})$.

(P2) SDE (11) has a unique, strong solution for all $\mathsf{s}, \mathbf{X}, m$.

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P}_{\mathsf{s}}$-trivial for each $\mathsf{s} \in \mathbf{S_0}$.

- $(\mathbf{X}, \mathbf{B})$: sol of ISDE by (P1). Let $(\mathbf{X}, \mathbf{B})$ be fixed.
- $\mathbf{Y}^m$ is a unique strong sol of SDE(10) by (P2)
- $\mathbf{Y}^m$ is $\sigma[\mathbf{B}] \bigvee \sigma[\mathbf{X}^{m*}]$-m'ble. $\mathbf{X}^{m*} = (X^n)_{m<n<\infty}$.
- $\mathbf{Y}^m = (X^1, \ldots, X^m)$. by (P2)
- $\mathbf{X}$ is $\sigma[\mathbf{B}] \bigvee \mathit{Tail}\,(W(S^{\mathbb{N}}))$-m'ble by $m \to \infty$.
- $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is trivial by (P3) $\Rightarrow$ $\mathbf{X}$ is a strong solution.

## Strong solutions of ISDE: How to prove (P1)–(P3)

(P1) ISDE (6) has a solution $(\mathbf{X}, \mathbf{B})$.

(P2) SDE (11) has a unique, strong solution for all $\mathbf{s}, \mathbf{X}, m$.

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ is $\mathbf{P_s}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$.

- (P1) follows from a general theory of O..
- (P2) is classical.
- How to prove (P3)? $\Rightarrow$ Tail Theorems.

(Q1) $\mu$ is tail trivial.

(Q2) $P_\mu \circ \mathsf{X}_t^{-1} \prec \mu$ for all $t$.

Let $S_r = \{|x| < r\}$, $\mathsf{X}_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i}$, $X^i = \{X_t^i\}$.

$\mathsf{m}_r = \inf\{m \in \mathbb{N}; X^i \in C([0,T]; S_r^c) \text{ for } m < \forall i \in \mathbb{N}\}$.

(Q3) $P_\mu(\cap_{r=1}^{\infty}\{\mathsf{m}_r(\mathsf{X}) < \infty\}) = 1$.

**Thm 14.** *Assume* (Q1)–(Q3). *Then* (P3) *holds.*

(P3) $\mathit{Tail}\,(W(S^{\mathbb{N}}))$ *is* $\mathbf{P_s}$*-trivial for each* $\mathbf{s} \in \mathbf{S_0}$.

# End