Ginibre Random Field

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- Configuration spaces and Random Point Fields (rpf)
- Ginibre random point field
- Properties of Ginibre rpf
- Problem
- Main Theorems 1–3
- Proofs

Configuration space and Random Point Fields (rpf)

- Configuration space S over $\mathbb{C}$:

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\ s_i \in \mathbb{C},\ \ \mathsf{s}(|s| < r) < \infty\ \ (\forall r \in \mathbb{N})\}$$

$\mathsf{s} = \sum_i \delta_{s_i}$ denotes the set of unlabeled particles $\{s_i\}$ in $\mathbb{C}$

$\mathsf{s} = \sum_i \delta_{s_i} \in \mathsf{S}$ is called a cofiguration.

- A prob. meas. ν on S is called Random Point Field

Exam. Poisson rpf, Periodic rpf, Gibbs meas.

- ρ^n is called n correlation fun of ν w.r.t. a meas. m if

$$\int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(\mathbf{x}_n) \prod_{i=1}^{n} m(dx_i) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i) - k_i)!} d\nu$$

for any disjoint $A_i \in \mathcal{B}(S)$, $k_i \in \mathbb{N}$ s.t. $k_1 + \ldots + k_m = n$.

Ginibre Random Point Fields (rpf)

Let $K(x,y)$ be a kernel, m be a meas.

- ν is called a determinantal rpf generated by (K,m) if its n correaltion fun ρ^n w.r.t. m is given by

:gen1
$$\rho^n(\mathbf{x}_n) = \det[K(x_i,x_j)]_{1\le i,j\le n} \tag{1}$$

- Ginibre rpf μ is the det rpf generated by $(K_{\text{gin}}, \mathrm{g})$:

$$K_{\text{gin}}(x,y) = e^{x\bar{y}} \qquad \mathrm{g}(dx) = \pi^{-1} e^{-|x|^2} dx$$

- Ginibre rpf μ is det rpf generated by $(\widehat{K}_{\text{gin}}, dx)$

:gen2
$$\widehat{K}_{\text{gin}}(x,y) = \frac{1}{\pi} e^{-\frac{1}{2}|x|^2 + x\bar{y} - \frac{1}{2}|y|^2} \tag{2}$$

- The 1 correlation $\hat{\rho}^1$ w.r.t. dx of μ is $\hat{\rho}^1(x) = 1/\pi$.

Property of Ginibre rpf 1

(g1) μ is translation and rotation invariant

(g2) μ has small fluctuation: Let $D_r = \{|x| < r\}$. Then :small

$$\mathsf{Var}^{\mu}[\langle 1_{D_r}, \mathsf{s}\rangle] \sim r. \tag{3}$$

The order is r^2 in case of Poisson rpf whose intensity :small is 2D Lebesgue m. So (3) implies the fluctuation of Ginibre rpf is suppressed.

- The above property is similar to periodic rpf.
- On the other hand, μ has bounded correlation functions for all n by construction.

This quite different from periodic rpf, and is rather similar to Gibbs measures with Ruelle's class potentials.

(g3) μ is the thermodynamical limit of $\mu_{\mathsf{gin}}^{\mathsf{n}}$:

:muN

$$\mu_{\mathsf{gin}}^{\mathsf{n}} = \frac{1}{Z} \prod_{i<j}^{\mathsf{n}} |x_i - x_j|^2 \prod_{k=1}^{\mathsf{n}} \mathsf{g}(dx_k) \tag{4}$$

- $\mu_{\mathsf{gin}}^{\mathsf{n}}$ is the distribution of the eigen values of Gaussian random Matrices.
- $\mu_{\mathsf{gin}}^{\mathsf{n}}$ is det rpf generated by $(K_{\mathsf{gin}}^{\mathsf{n}}, \mathsf{g})$, where

$$K_{\mathsf{gin}}^{\mathsf{n}}(x,y) = \sum_{i=0}^{\mathsf{n}-1} \frac{(x\bar{y})^i}{i!}$$

Loosely, (g3) implies μ is a prob m. on $\mathbb{C}^{\mathbb{N}}$ given by

$$\bar{\mu} = \frac{1}{Z} \prod_{i<j}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} \frac{e^{-|x_k|^2}}{\pi} dx_k \tag{5}$$

:11z

Since μ_{gin} is translation inv, μ_{gin} can be written by

$$\bar{\mu} = \frac{1}{Z} \prod_{i<j}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} dx_k \tag{6}$$

:11zz

Both of them are not rigorous, but to some extent correct representations. Anyway, μ_{gin} is a infinite 2D Coulomb systems.

Ginibre rpf is not Gibbs meas in the sense that it does not satisfy the DLR eq. But its log derivative is related to 2D Coulomb pot.

Problem:

- Let ν be a translation invariant rpf on $\mathbb{C}$.
- Let $\mathsf{s} = \sum_i \delta_{s_i}$ be a sample point under ν.
- Take a finite number of particles from the sample points $\{s_i\}$.
- Can one detect the number of the removed particles?

Problem:

- Let ν be a translation invariant rpf on $\mathbb{C}$.
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- Take a finite number of particles from the sample points $\{s_i\}$.
- Can one detect the number of the removed particles?

If ν is a perioduc rpf, then "Yes".

If ν is a Poisson rpf, then "No".

The Ginibre rpf μ has a property between periodic and Poisson.

Palm meas. For a set of m-points $\mathbf{x} = \{x_1, \ldots, x_\mathsf{m}\}$ let

$$\mu_\mathbf{x} := \mu(\cdot - \sum_{l=1}^{\mathsf{m}} \delta_{x_l} \mid \mathsf{s}(\{x_l\}) \geq 1 \quad (l = 1, \ldots, \mathsf{m}))$$

1:21

Thm 1. *Let* $\mathsf{m}, \mathsf{n} \in \{0\} \cup \mathbb{N}$. *Then*

(1) *If* $\mathsf{m} = \mathsf{n}$, *then* $\mu_\mathbf{x}$ *and* $\mu_\mathbf{y}$ *are mutually ab. cont..*

(2) *If* $\mathsf{m} \neq \mathsf{n}$, *then* $\mu_\mathbf{x}$ *and* $\mu_\mathbf{y}$ *are singular each other.*

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Thm 1 Let $\mathsf{m}, \mathsf{n} \in \{0\} \cup \mathbb{N}$. Then

(1) If $\mathsf{m} = \mathsf{n}$, then $\mu_\mathbf{x}$ and $\mu_\mathbf{y}$ are mutually ab. cont..

(2) If $\mathsf{m} \neq \mathsf{n}$, then $\mu_\mathbf{x}$ and $\mu_\mathbf{y}$ are singular each other.

- (2) shows a special property of Ginibre rpf. Indeed,

Λ Poisson rpf $\Rightarrow \Lambda_\mathbf{x} = \Lambda$

ν Gibbs meas with Ruelle's class potentials $\Rightarrow \nu_\mathbf{x} \prec \nu$

- ν periodic rpf $\Rightarrow$ (2) holds

1:22

Thm 2. *Suppose* $\mathsf{m} = \mathsf{n}$. *Then for* $\mu_{\mathbf{y}}$*-a.s.* $\mathsf{s} = \sum_i \delta_{s_i}$

:21c

$$\frac{d\mu_{\mathbf{x}}}{d\mu_{\mathbf{y}}} = \frac{1}{Z_{\mathbf{xy}}} \lim_{r\to\infty} \prod_{|s_i|<b_r} \frac{|\mathbf{x} - s_i|^2}{|\mathbf{y} - s_i|^2} \tag{7}$$

compact uniformly in $\mathbf{x} \in \mathbb{C}^{\mathsf{m}}$, $\mathbf{y} \in \mathbb{C}^{\mathsf{m}} \backslash \{s_1, \ldots, s_{\mathsf{m}}\}$

$$Z_{\mathbf{xy}} = \frac{\Delta(\mathbf{y}) \det[K_{\mathsf{gin}}(x_i, x_j)]_{i,j=1}^{\mathsf{m}}}{\Delta(\mathbf{x}) \det[K_{\mathsf{gin}}(y_i, y_j)]_{i,j=1}^{\mathsf{m}}}$$

$$\Delta(\mathbf{x}) = \prod_{i<j}^{\mathsf{m}} |x_i - x_j|^2, \qquad |\mathbf{x} - s_i| = \prod_{m=1}^{\mathsf{m}} |x_m - s_i|$$

$$\{b_r\}_{r\in\mathbb{N}} : \qquad b_r \uparrow \infty$$

Let $D_{\sqrt{q}} = \{z \in \mathbb{C}\,;\ |z| < \sqrt{q}\}$,

:61a

$$F_r(\mathsf{s}) = \frac{1}{r}\sum_{q=1}^{r}(\mathsf{s}(D_{\sqrt{q}}) - q). \tag{8}$$

By definition $\mathsf{s}(D_{\sqrt{q}})$ is the number of particles $\mathsf{s} = \sum_i \delta_{s_i}$ in the disk $D_{\sqrt{q}}$.

Thm 3. *Let* $\mathbf{x} = (x_1, \ldots, x_{\mathsf{m}})$.

:3

$$\lim_{r\to\infty} F_r(\mathbf{s}) = -\mathsf{m} \qquad \textit{weakly in } L^2(\mathsf{S}, \mu_{\mathbf{x}}) \tag{9}$$

1:3

- The 3 means we can determine the number of missing particles. So

$$\infty - \mathsf{m} \neq \infty$$

Proof of Thm 1

Thm 1[1:21] Let $m, n \in \{0\} \cup \mathbb{N}$. Then

(1) If $m = n$, then $\mu_{\mathbf{x}}$ and $\mu_{\mathbf{y}}$ are mutually ab. cont..

(2) If $m \neq n$, then $\mu_{\mathbf{x}}$ and $\mu_{\mathbf{y}}$ are singular each other.

Proof: Thm 1[1:21] follows from Thm 2[1:22] and Thm 3[1:3] immediately.

Proof of Thm 2

$$\frac{d\mu_{\mathbf{x}}}{d\mu_{\mathbf{y}}} = \frac{1}{Z_{\mathbf{xy}}} \lim_{r\to\infty} \prod_{|s_i|<b_r} \frac{|\mathbf{x}-s_i|^2}{|\mathbf{y}-s_i|^2} \quad \text{cpt uni in } \mathbf{x} \in \mathbb{C}^{\mathsf{m}}$$

where

$$Z_{\mathbf{xy}} = \frac{\Delta(\mathbf{y}) \det[K_{\mathsf{gin}}(x_i, x_j)]_{i,j=1}^{\mathsf{m}}}{\Delta(\mathbf{x}) \det[K_{\mathsf{gin}}(y_i, y_j)]_{i,j=1}^{\mathsf{m}}}$$

- Recall that

$$\mu = \lim_{\mathsf{n}\to\infty} \mu^{\mathsf{n}}_{\mathsf{gin}}$$

Hence

$$\mu_{\mathbf{x}} = \lim_{\mathsf{n}\to\infty} \mu^{\mathsf{n}}_{\mathsf{gin},\mathbf{x}}$$

- Since

$$\mu^{\mathsf{n}}_{\mathsf{gin}} = \frac{1}{Z} \prod_{i<j}^{\mathsf{n}} |s_i - s_j|^2 \prod_{k=1}^{\mathsf{n}} \mathsf{g}(ds_k),$$

we have

$$\mu^{\mathsf{n}}_{\mathsf{gin},\mathbf{x}} = \frac{1}{Z^{\mathsf{n}}_{\mathbf{x}}} \prod_{i=1}^{\mathsf{n}-\mathsf{m}} |\mathbf{x} - s_i|^2 \prod_{j<k}^{\mathsf{n}-\mathsf{m}} |s_j - s_k|^2 \prod_{l=1}^{\mathsf{n}-\mathsf{m}} \mathsf{g}(ds_l)$$

- Since

$$\mu^{\mathsf{n}}_{\mathsf{gin},\mathbf{x}} = \frac{1}{Z^{\mathsf{n}}_{\mathbf{x}}} \prod_{i=1}^{\mathsf{n}-\mathsf{m}} |\mathbf{x} - s_i|^2 \prod_{j<k}^{\mathsf{n}-\mathsf{m}} |s_j - s_k|^2 \prod_{l=1}^{\mathsf{n}-\mathsf{m}} \mathsf{g}(ds_l),$$

we have

$$\frac{d\mu^{\mathsf{n}}_{\mathsf{gin},\mathbf{x}}}{d\mu^{\mathsf{n}}_{\mathsf{gin},\mathbf{y}}}(\mathsf{s}) = \frac{Z^{\mathsf{n}}_{\mathbf{y}}}{Z^{\mathsf{n}}_{\mathbf{x}}} \prod_{i=1}^{\mathsf{n}-\mathsf{m}} \frac{|\mathbf{x} - s_i|^2}{|\mathbf{y} - s_i|^2} \qquad (\mathsf{s} = \sum_i \delta_{s_i}).$$

- Therefore we obtain

$$\frac{d\mu_{\mathbf{x}}}{d\mu_{\mathbf{y}}}(\mathsf{s}) = \lim_{r\to\infty} \{ \lim_{\mathsf{n}\to\infty} \{ E[\frac{Z^{\mathsf{n}}_{\mathbf{y}}}{Z^{\mathsf{n}}_{\mathbf{x}}} \prod_{i=1}^{\mathsf{n}-\mathsf{m}} \frac{|\mathbf{x} - s_i|^2}{|\mathbf{y} - s_i|^2} | \mathcal{B}(\pi_r)]\}\}.$$

- Let $\pi_r : \mathsf{S} \to \mathsf{S}$ such that $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap D_r)$

To prove

$$\frac{d\mu_{\mathbf{x}}}{d\mu_{\mathbf{y}}}(\mathsf{s}) = \lim_{r\to\infty}\{\lim_{\mathsf{n}\to\infty}\{E[\frac{Z_{\mathbf{y}}^{\mathsf{n}}}{Z_{\mathbf{x}}^{\mathsf{n}}}\prod_{i=1}^{\mathsf{n}-\mathsf{m}}\frac{|\mathbf{x}-s_i|^2}{|\mathbf{y}-s_i|^2}|\mathcal{B}(\pi_r)]\}\},$$

we use the small fluctuation property of Ginibre rpf:

$$\sup_{\mathsf{n}} E^{\mu^{\mathsf{n}}_{\mathsf{gin},\mathbf{x}}}[\langle 1_{Q_r}, \mathsf{s}\rangle] = O(r) \quad \text{(small fluctuation)}$$

Note that, in case of Poisson rpf, $O(r^2)$.

Proof of Thm 3

(1): Let $\mathbf{0}_{\mathsf{m}} = (0, \ldots, 0) \in \mathbb{C}^{\mathsf{m}}$ and $\mathbf{x} \in \mathbb{C}^{\mathsf{m}}$.

- By Th 1 (1), $\mu_{\mathbf{x}} \sim \mu_{\mathbf{0}_{\mathsf{m}}}$.
- Hence it enugh to show Thm 3 only for $\mu_{\mathbf{0}_{\mathsf{m}}}$ and $\mu_{\mathbf{0}_{\mathsf{n}}}$.

(2): $\mu_{\mathbf{0}_{\mathsf{m}}}$ is rotation invariant.

(3): $\mu_{\mathbf{0}_{\mathsf{m}}}$ is a det rpf generated by $(K_{\mathsf{m}}^*, \mathrm{g})$, where

$$K_{\mathsf{m}}^*(x, y) = \sum_{k=\mathsf{m}}^{\infty} \frac{1}{k!} x^k \bar{y}^k$$

1.3a (4): $\xi_i \geq 0$, $\xi_i^2 \sim \Gamma_i$, where $\Gamma_i = \Gamma(i)^{-1} t^{i-1} e^{-t} dt$

Prop 1. *Let* $\Xi = \prod_{i=1}^{\infty} P^{\xi_i}$, $\iota(\mathsf{s}) = \sum_i \delta_{|s_i|}$,

:qq
$$\mu_{0_\mathrm{m}} \circ \iota^{-1} \sim \Big(\sum_{i=\mathrm{m}+1}^{\infty} \delta_{\xi_i}, \Xi \Big) \tag{10}$$

(5): Let $D_{\sqrt{q}} = \{z \in \mathbb{C}\,;\, |z| < \sqrt{q}\}$ $q \in \mathbb{N}$、

$$F_r(\mathsf{s}) = \frac{1}{r} \sum_{q=1}^{r} (\mathsf{s}(D_{\sqrt{q}}) - q), \quad G_r(\mathsf{s}) = \frac{1}{r} \sum_{q=1}^{r} (\mathsf{s}([0, \sqrt{q})) - q)$$

- Then $F_r \circ \iota^{-1} = G_r$.
- Let $\kappa_\mathrm{m}((x_i)_{i\in\mathbb{N}}) = \sum_{i=\mathrm{m}+1} \delta_{x_i}$. Then

:pf5a
$$(F_r, \mu_{0_\mathrm{m}}) \sim (G_r \circ \kappa_\mathrm{m}, \Xi) \tag{11}$$

(6): Key estimate:

:6a

$$\lim_{r\to\infty} E^{\mu_{0\mathsf{m}}}[F_r] = -\mathsf{m} \tag{12}$$

l:6b

$$\overline{\lim_{r\to\infty}} \mathsf{Var}^{\mu_{0\mathsf{m}}}[F_r] < \infty \tag{13}$$

- By these and (11), we see :pf5a

$$\{G_r \circ \kappa_{\mathsf{m}}\} \text{ is bounded in } L^2([0,\infty)^{\mathbb{N}}, \Xi)$$

- Hence we can take a subsequene such that

:6c

$$\lim_{r'\to\infty} G_{r'} \circ \kappa_{\mathsf{m}} = H_{\mathsf{m}} \quad \text{weakly in } L^2([0,\infty)^{\mathbb{N}}, \Xi) \tag{14}$$

(7): We next investigate the property of H_{m}:

:6c

$$\lim_{r'\to\infty} G_{r'}\circ\kappa_{\mathsf{m}} = H_{\mathsf{m}} \quad \text{weakly in } L^2([0,\infty)^{\mathbb{N}},\Xi) \quad (15)$$

- $G_r(\mathsf{s}) = \frac{1}{r}\sum_{q=1}^{r}(\mathsf{s}([0,\sqrt{q})) - q) \Rightarrow H_{\mathsf{m}}$ is tail m'able, $\kappa_{\mathsf{m}}((x_i)_{i\in\mathbb{N}}) = \sum_{i=\mathsf{m}+1}\delta_{x_i}$
- (11) :pf5a $(F_r,\mu_{\mathbf{0}_{\mathsf{m}}}) \sim (G_r\circ\kappa_{\mathsf{m}},\Xi) \Rightarrow H_{\mathsf{m}} - H_{\mathsf{n}} = -\mathsf{m}+\mathsf{n}$
- H_{m} is const by Kolmogorov's 0-1 law.
- $E[H_{\mathsf{m}}] = -\mathsf{m}$ by (12) :6a.

Threfore

$$H_{\mathsf{m}} = -\mathsf{m}$$

:pf5a
(11) implies

$$\lim_{r\to\infty}(F_r,\mu_{\mathbf{0}_{\mathsf{m}}}) \sim H_{\mathsf{m}} = -\mathsf{m}$$

Thank You !