Infinite-dimensional stochastic differential equations related to Airy random fields

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- Random matrices and log gasses
- Soft edge scaling limit and Airy random fields
- General theory for ISDEs:
  quasi-Gibbs property & log derivative
- Examples:
  Sine RPF, Bessel RPF,
  Airy RPF, Ginibre RPF,
  All canonical Gibbs measures

Gaussian ensembles & semi-circle law

- The dist of eigen values of the G(O/U/S)E Random Matrices are given by ($\beta = 1, 2, 4$)

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^{N}|x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \qquad (1)$$

- The distribution of

$$N^{-1}\sum_{i=1}^{N}\delta_{x_i} \quad \text{under } m_\beta^N$$

converge the semi-circle law

$$\varsigma(x)dx = \frac{1}{2\pi}\sqrt{4 - x^2}dx \qquad (2)$$

Sine rpf (Dyson's model)–Bulk scaling limit

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^{N}|x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \quad \varsigma(x)dx = \frac{1}{2\pi}\sqrt{4-x^2}dx$$

- Take $x_i = s_i/\sqrt{N}$ in (1) and set

$$\mu_{\mathrm{sin},\beta}^N(d\mathbf{s}_N) = \frac{1}{Z}\sum_{i<j}^{N}|s_i - s_j|^\beta \prod_{k=1}^{N} e^{-\beta|s_k|^2/4N} d\mathbf{s}_N \qquad (3)$$

- The associated N particle system is given by the SDE: 30p

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^{N}\frac{1}{X_t^i - X_t^j}dt - \frac{\beta}{4N}X_t^i dt \qquad (4)$$

- So the ass ∞ particle system is given by

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^{\infty}\frac{1}{X_t^i - X_t^j}dt$$

Airy rpf – Soft edge scaling limit

Airy rpf: $\mu_{\mathrm{Ai},\beta}$ ($S = \mathbb{R}$, $\beta = 1, 2, 4$)

Take the scaling $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$ in

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z} \prod_{i<j}^{N} |x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N$$

and set

$$\mu_{\mathrm{Ai},\beta}^N(d\mathbf{s}_N) = \frac{1}{Z} \prod_{i<j}^{N} |s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N} + N^{-1/6} s_i|^2} d\mathbf{s}_N.$$

Then $\mu_{\mathrm{Ai},\beta}$ is the TDL of $\mu_{\mathrm{Ai},\beta}^N$:

$$\lim_{N\to\infty} \mu_{\mathrm{Ai},\beta}^N = \mu_{\mathrm{Ai},\beta}$$

Airy rpf – Soft edge scaling limit

- $\beta = 2 \Rightarrow \mu_{\mathsf{Ai},\beta}$ is the det rpf gen by (K_{Ai}, dx):

$$K_{\mathsf{Ai}}(x,y) = \frac{\mathsf{Ai}(x)\mathsf{Ai}'(y) - \mathsf{Ai}'(x)\mathsf{Ai}(y)}{x-y}$$

Here $\mathsf{Ai}(\cdot)$ the Airy function such that

$$\mathsf{Ai}(z) = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{i(zk+k^3/3)}, \qquad z \in \mathbb{C}. \tag{5}$$

If $\beta = 1, 4$, the correlation func of $\mu_{\mathsf{Ai},\beta}$ are given by similar formula of quaternion determinant.

Airy rpf – Soft edge scaling limit

- From

$$\mu^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N$$

we deduce the SDE of the N particle system:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j=1,j\neq i}^N \frac{1}{X_t^i - X_t^j}dt - \frac{\beta}{2}\{N^{1/3} + \frac{1}{2N^{1/3}}X_t^i\}dt$$

- Problem: What is the limit SDE?

Does $\lim_{N\to\infty}\{\sum_{j=1,j\neq i}^N \frac{1}{X_t^i - X_t^j} - N^{1/3}\}$ converge ?

How to solve the limit SDE?

Thm 1 (with Tanemura). *Let $\beta = 1, 2, 4$. Then:*

- *the limit ISDE is*

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x)$$

- *The above SDE has a unique, strong solution.*
- *So far the sto dyn related to Airy RPF was constructed only for $\beta = 2$ by Spohn, Johansson, and others by the method of space-time cor funs. This sto dyn is same as the above.*
- (X_t^i) *is a diffusion with state space* $\mathbb{R}^{\mathbb{N}}$*.*
- $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ *is reversible w.r.t.* $\mu_{\mathsf{airy},\beta}$*.*

Let $S = \mathbb{R}^d$, $\mathbb{C}$, $[0, \infty)$.

S: Configuration space over S

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\ s_i \in S,\ \mathsf{s}(|s| < r) < \infty\ (\forall r \in \mathbb{N})\}$$

μ: RPF on S. i.e. prob meas. on S.

Prob: (1) To construct a *natural* stochastic dynamics

$$\mathbf{X}_t = (X_t^i)_{i \in \mathbb{N}} \quad \text{(labeled dynamics)}$$

related to μ, *i.e.*

$$\mathsf{X}_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i} \quad \text{(unlabeled dynamics)}$$

is reversible w.r.t. μ.

(2) To find the ∞-dim. SDE that $\mathbf{X}_t$ satisfies.

- ρ^n is called the n-correlation function of μ w.r.t. Radon m. m if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(\mathbf{x}_n) \prod_{i=1}^{n} m(dx_i) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!} d\mu$$

for any disjoint $A_i \in \mathcal{B}(S)$, $k_i \in \mathbb{N}$ s.t. $k_1 + \ldots + k_m = n$.

- μ is called the determinantal RPF generated by (K, m) if its n-correlation fun. ρ^n is given by

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{1\le i,j\le n}$$

- Ginibre RPF $S = \mathbb{C}$. μ_{gin} is generated by $(K_{\mathsf{gin},2}, \mathrm{g})$

$$K_{\mathsf{gin},2}(x,y) = e^{x\bar{y}} \qquad \mathrm{g}(dx) = \pi^{-1} e^{-|x|^2} dx$$

- Ψ: Ruelle's class interaction potential,

$Q_r = \{|x| \le r\}$, $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap Q_r)$, $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap Q_r^c)$

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(Q_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi))$$

- μ is called (Φ, Ψ)-Gibbs m. if it satisfies DLR eq:

$$d\mu_{r,\xi}^m = \frac{1}{z_{r,\xi}} e^{-\mathcal{H}_r(\mathsf{s}) - \mathcal{W}_{r,\xi}(\mathsf{s})} \prod_{k=1}^m e^{-\Phi(s_k)} ds_k$$

$$\mathcal{H}_r = \sum_{s_i, s_j \in Q_r, i<j} \Psi(s_i - s_j), \ \ \mathcal{W}_{r,\xi} = \sum_{s_i \in Q_r, \xi_j \in Q_r^c} \Psi(s_i - \xi_j)$$

- Let $\Psi(x) = -2 \log |x|$.

Then, $\mathcal{W}_{r,\xi}$ diverge, so DLR does not make sense

(Φ, Ψ)-Gibbs m. Let $\nu_r^m = \prod_{k=1}^m 1_{Q_r}(s_k) e^{-\Phi(s_k)} ds_k$

$$d\mu_{r,\xi}^m = \frac{1}{z_{r,\xi}^m} e^{-\mathcal{H}_r - \mathcal{W}_{r,\xi}} d\nu_r^m \qquad \text{(DLR eq)}$$

(Φ, Ψ)-quasi Gibbs m. $\exists\ c_{r,\xi}^m$

$${c_{r,\xi}^m}^{-1} e^{-\mathcal{H}_r} d\nu_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\nu_r^m$$

- If μ is Airy RPF, $\mathcal{W}_{r,\xi}$ and $z_{r,\xi}^m$ diverge. But $e^{-\mathcal{W}_{r,\xi}}/z_{r,\xi}^m$ conv.

$${c_{r,\xi}^m}^{-1} \le e^{-\mathcal{W}_{r,\xi}}/z_{r,\xi}^m \le c_{r,\xi}^m$$

- Quasi-Gibbs is very mild restriction. If μ is (Φ, Ψ)-quasi-Gibbs m, then μ is also $(\Phi + f, \Psi)$-quasi Gibbs m for any loc bdd m'able f.

Main theorems: Unlabeled level construction

Let $\mathbb{D}$ be the canonical square field on S: $\mathsf{s} = \sum_i \delta_{s_i}$, $\mathbf{s} = (s_i)$.

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_i \nabla_{s_i}\tilde{f}(\mathbf{s}) \cdot \nabla_{s_i}\tilde{g}(\mathbf{s})$$

Let $\mathcal{D}$ be the set of local smooth fun with $\mathcal{E}_1^{\mu}(f,f) < \infty$.

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g] d\mu$$

Thm 2. [O.96[CMP], 10[JMSJ], 12?[AOP]]
(1) *If μ is quasi-Gibbs with upper semi-cont potentials (Φ, Ψ), then $(\mathcal{E}^{\mu}, \mathcal{D}, L^2(\mathsf{S}, \mu))$ is closable.*
(2) *If $(\mathcal{E}^{\mu}, \mathcal{D}, L^2(\mathsf{S}, \mu))$ is closable & all correlation fun are loc bounded, then a diffusion X_t associated with the closure $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$ exists.*

If μ is Poisson rpf with Lebesgue intensity, then $\mathsf{X}_t = \sum_i \delta_{B_t^i}$.

- Let μ_x be the (reduced) Palm m. of μ conditioned at x

$$\mu_x(\cdot) = \mu(\cdot - \delta_x | \mathsf{s}(x) \geq 1)$$

- Let μ^1 be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^1(A \times B) = \int_A \rho^1(x)\mu_x(B)dx$$

- $\mathsf{d}_\mu \in L^1(\mathbb{R}^d \times \mathsf{S}, \mu^1)$ is called the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^1 = -\int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}_\mu d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}$$

Here ∇_x is the nabla on $\mathbb{R}^d$, $\mathcal{D}$ is the space of bounded, local smooth functions on S.

- Very informally

$$\mathsf{d}_\mu = \nabla_x \log \mu^1$$

Main theorems: Infinite-dim SDE

(A1) μ is a quasi-Gibbs measure. (closability)

(A2) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$ (quasi-regular)

Here $S_r = \{|x| < r\}$, $\mathsf{S}_r^k = \{\mathsf{s}(S_r) = k\}$, σ_r^k is k-density fun on S_r.

(A3) The log derivative $\mathsf{d}_\mu \in L^1_{\text{loc}}(\mu^1)$ exists (SDE rep)

(A4) $\{X_t^i\}$ do not collide each other (non-collision)

(A5) each tagged particle X_t^i never explode (non-explosion)

Let $\mathfrak{u}: S^{\mathbb{N}} \to \mathsf{S}$ such that $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$.

Main theorems: labeled diffusions

Thm 3. (O.12(PTRF)) (A1)–(A5) $\Rightarrow \exists \mathsf{S}_0 \subset \mathsf{S}$ *such that*

$$\mu(\mathsf{S}_0) = 1, \tag{6}$$

and that, for $\forall \mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, $\exists \mathfrak{u}^{-1}(\mathsf{S}_0)$*-valued pr.* $(X_t^i)_{i\in\mathbb{N}}$ *and* $\exists S^{\mathbb{N}}$*-valued Brownian m.* $(B_t^i)_{i\in\mathbb{N}}$ *satisfying*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}_\mu(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s} \tag{7}$$

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}_\mu(X_t^i, \sum_{\mathsf{j}\neq\mathsf{i}} \delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

Thm 4 (O. (JMSJ 10)). *The family of processes* $\{(X_t^i)_{i\in\mathbb{N}}\}$ *is a diffusion with state space* $\mathfrak{u}^{-1}(\mathsf{S}_0) \subset S^{\mathbb{N}}$*.*

Remark 1. (1) (A1)–(A5) can be checked for Ginibre RPF ($\beta = 2$), Sine RPFs, Airy RPFs and Bessel RPFs ($\beta = 1, 2, 4$).
(2) We can calculate the log derivatives of these measures.
(3) We have general theorems for quasi-Gibbs property and the log derivatives (O. PTRF12, to appear in AOP, preprint). The statements are too messy to be omitted here.

$$\mathsf{H}^1 = \{(x, \mathsf{s}) \in S \times \mathsf{S}\,;\, \mathsf{d}_\mu(x, \mathsf{s}) \text{ is locally Lips cont. }\}$$

Here "locally" means we regard $\mathsf{d}_\mu(x, \mathsf{s})$ as symmetric fun on S_r with fixed particles outside S_r^c for $\forall r$ except a capacity zero set (non-single points, say).

Let $\mathsf{H} = \{\delta_x + \mathsf{s}\,;\, (x, \mathsf{s}) \in \mathsf{H}^1\}$ Assume

(A6) $\mathrm{Cap}^\mu(\mathsf{H}^c) = 0$.

Thm 5 (with Tanemura). *Assume (A1)–(A6). Then the SDE has a unique, strong solution for initial starting points $(s_i) \in S^{\mathbb{N}}$ such that $\sum_i \delta_{s_i} \in \mathsf{H}$ q.e..*

Remark: (1) It is quite likely that all determinantal rpfs in continuous spaces satisfy (A1)–(A6).

(2) It is likely that the conclusion of Thm 5 holds for all initial points $\mathbf{s} = (s_i)$ such that $\sum_i \delta_{s_i} \in \mathsf{H}$. (in progress)

Let $\mathcal{D}^{\mu}_{\text{poly}}$ be the closure of the set of polynomials on S such that $\mathcal{E}^{\mu}_{1}(f,f) < \infty$. Then

$$\mathcal{D}^{\mu}_{\text{poly}} \subset \mathcal{D}^{\mu}$$

because polynomials are local and smooth.

Thm 6 (with Tanemura)**.** *Assume* (A1)–(A6)*. Then the Dirichlet form that are extension of* $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}_{\text{poly}})$ *is unique.*

In particular, $\mathcal{D}^{\mu}_{\text{poly}} = \mathcal{D}^{\mu}$, and Lang's construction and Osada's construction are same.

Remark 2. If (A5) (non-explosion) does not hold. Then Thm 6 does not hold. This is very natural theorem that says the uniqueness of Dirichlet forms is related to the non-explosion problem of tagged problem.

All example below satisfy (A1)–(A6). Hence by Thm 5 we have a unique, strong solution.

Gibbs measures :

- All Gibbs measures with Ruelle's class potentials (smooth outside the origin) satisfy the assumptions (A.1)–(A.6).

Non-collision (A4) does not hold in general. But it always holds for $d \ge 2$ and, for repulsive interaction Ψ in $d = 1$.

- In this case, the SDEs become

$$dX_t^i = dB_t^i - \frac{1}{2}\nabla\Phi(X_t^i)dt - \frac{1}{2}\sum_{j\neq i}\nabla\Psi(X_t^i - X_t^j)dt. \qquad (8)$$

Examples: Ruelle's class potentials

Lennard-Jones 6-12 potential

Let $\Phi_{6,12}(x) = c\{|x|^{-12} - |x|^{-6}\}$, where $d = 3$ and $c > 0$ is a constant. $\Phi_{6,12}$ is called the Lennard-Jones 6-12 potential. The corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{c}{2} \sum_{j=1, j\neq i}^{\infty} \{\frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}}\}dt \quad (i \in \mathbb{N})$$

Coulomb like potentials (not Coulomb!)

Let $a > d$ and set $\Phi_a(x) = (c/a)|x|^{-a}$, where $c > 0$. Then the corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{c}{2} \sum_{j=1, j\neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}} dt \quad (i \in \mathbb{N}). \qquad (9)$$

Examples: Ruelle's class potentials

Coulomb like potentials (not Coulomb!)

Let $a > d$ and set $\Phi_a(x) = (c/a)|x|^{-a}$, where $c > 0$. Then the corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{c}{2} \sum_{j=1, j\neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}} dt \quad (i \in \mathbb{N}). \tag{10}$$

At first glance the ISDE (10) resembles Ginibre IBMs, because these corresponds to the case $a = 0$ in (10). The sums in the drift terms, however, converge absolutely, unlike Coulomb (log) potentials. We emphasize that the structures of the dynamics given by the solutions of (10) and Ginibre IBMs are completely different from each other.

Examples: Ginibre rpf

Ginibre rpf: $\Psi(x) = -\beta \log |x|$ $d = 2$, $\beta = 2$. If $\mu = \mu_{\text{gin},2}$,

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \tag{11}$$

and also

$$dX_t^i = dB_t^i - X_t^i dt + \lim_{r\to\infty} \sum_{|X_t^j|<r \ \ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{12}$$

This comes from the plural expressions of $\mathsf{d}_{\mu_{\text{gin},2}}$.
For finite N, these SDEs give different solution.
But in the limit $N \to \infty$ give the same solution if the initial distribution is closed to Ginibre rpf.

Examples: Bessel rpf–hard edge scaling limit

Bessel RPF (joint work with Honda):

$S = [0, \infty)$, $\beta = 2$, $a > 1$

$$dX_t^i = dB_t^i + \frac{a}{2X_t^i}dt + \lim_{r\to\infty} \frac{\beta}{2} \sum_{\substack{|X_t^j|<r \\ j\neq i}} \frac{1}{X_t^i - X_t^j}dt$$

$\beta = 1, 4$ are in progress.

Examples: sine rpf (Dyson's model)–bulk scaling limit

Sine$_\beta$ RPF: $S = R$, $\beta = 1, 2, 4$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r,\ j \neq i} \frac{1}{X_t^i - X_t^j} dt$$

Spohn (1987) considered the case $\beta = 2$:

$$dX_t^i = dB_t^i + \sum_{j \neq i} \frac{1}{X_t^i - X_t^j} dt$$

He constructed the dynamics as a Markov semigr by Dirichlet form.

The def of $\mu = \mu_{\sin,\beta}$:

$\beta = 2 \Rightarrow \mu_{\sin,\beta}$ is the det rpf generated by $(K_{\sin}, dx)$:

$$K_{\sin}(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}$$

$\beta = 1, 4 \Rightarrow$ the correlation funs are given by quaternion det.

Thm 7 (with Tanemura). *Let* $\beta = 1, 2, 4$. *Then:*

- *The log derivative* $\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}$ *is*

$$\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}(x, \mathsf{s}) = \beta \lim_{r\to\infty}\{(\sum_{|x-s_i|<r} \frac{1}{x - s_i}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx\}$$

Here

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x)$$

- *Airy rpf* $\mu_{\mathsf{Ai},\beta}$ *satisfy* (A1)–(A6) *and the limit ISDE is*

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty}\{(\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx\} dt$$

Examples: Airy rpf – Soft edge scaling limit

- The key idea is to take the rescaled semi-circle law ς, as the first approximation of the 1-correlation fun $\rho_{\mathsf{Ai},\beta}^{N,1}$.
- Our method can be applied to other soft edge scaling. Our result is the first time to clarify the SDE describing the limit infinite system for the soft edge.

Thm 8 (with Tanemura). *Assume* $\beta = 2$.

Let us label $X_t^i > X_t^{i+1}$ $(\forall i)$.

- *The top particle* X_t^1 *is the Airy process* $\mathcal{A}(t)$ *in the sense of Spohn.*
- *The infinite dim stochastic dynamics constructed by Spohn, Johansson & others by the space-time correlation fun is a solution of the prescribed SDE:*

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

- The SDE gives a kind of Girsanov formula.
- These examples are the first time that the infinite dynamics are constructed for rpf appeared in random matrix theory with $\beta = 1, 4$ even if the bulk and the hard edge as well as the soft edge scaling

  In one dimensional system, the method of space-time correlation functions are available (Nagao, Katori-Tanemura, Spohn, and others), but this method is restricted to $\beta = 2$.
- By construction, if the total system start from the Airy_β rpf $\mu_{\mathrm{Ai},\beta}$, then the distribution of the top particle X_t^1 equals $F_{\beta,edge}(x)$, the β Tracy-Widom distribution, where $\beta = 1, 2, 4$.

Thm 9. *Ginibre RPF ($\beta = 2$), Sine RPFs, Airy RPFs ($\beta = 1, 2, 4$) and Bessel RPFs ($\beta = 2$) are quasi-Gibbs m. for $\Psi(x) = -\beta \log |x|$, and the log derivative can be calculated. The associated ISDE has a unique, strong solution.*

Remark 3. Virág et all have been constructed the RPF for all β on Dyson, Airy and Bessel RPFs (called β ensemble). It is quite likely that these RPFs satisfy our assumptions (A1)–(A6). But unfortunately, they have not yet prove the existence of correlation functions for these models. Only an existence of TDL has been established!

It is important to prove these are quasi-Gibbs measures and to calculate the log derivative.

Thank You !

To sum up

- The key point of the proof is to use the small fluctuation property (SFP) of linear statistics for these measures.
- SFP was established by Soshnikov (Sine, Airy, Bessel RPFs), Shirai (Ginibre RPF).
- Proof consists of several parts:

(1) To find a good finite particle approximation $\{\mu^N\}$

(2) To prove uniform *small fluctuation* of $\{\mu^N\}$

(3) To prove uni bounds of 1 & 2 cor funs of $\{\mu^N\}$

(4) To carry out the limiting procedure of $\mathtt{d}_{\mu^N}$ & quasi-Gibbs property by using general theorems. (O. 11,12)

Derivation of (4): $(\mathcal{E}^{\mu^N_{\sin,\beta}}, L^2(\mu^N_{\sin,\beta}))$

$$\mathcal{E}^{\mu^N_{\sin,\beta}}(f,g) = \int \mathbb{D}[f,g]\mu^N_{\sin,\beta}(d\mathbf{x}), \quad \mathbb{D}[f,g] = \frac{1}{2}\sum_{i=1}^{N} \frac{\partial f}{\partial s_i}\frac{\partial g}{\partial s_i}$$

$$\mathcal{E}^{\mu^N_{\sin,\beta}}(f,g) = \int \mathbb{D}[f,g]\frac{1}{Z}\sum_{i<j}^{N} |s_i - s_j|^\beta \prod_{k=1}^{N} e^{-\beta|s_k|^2/4N} d\mathbf{s}_N$$

$$= -\frac{1}{2}\int \{\Delta f + \sum_{i\neq j}^{N} \frac{\beta}{s_i - s_j}\frac{\partial f}{\partial s_i} - \sum_{k=1}^{N} \frac{\beta s_k}{2N}\frac{\partial f}{\partial s_k}\} g$$

$$\frac{1}{Z}\sum_{i<j}^{N} |s_i - s_j|^\beta \prod_{k=1}^{N} e^{-\beta|s_k|^2/4N} d\mathbf{s}_N$$

$$= -\int \{\frac{1}{2}\Delta f + \frac{\beta}{2}\sum_{i=1}^{N}[(\sum_{j\neq i}^{N}\frac{1}{s_i - s_j}) - \frac{s_i}{2N}]\frac{\partial f}{\partial s_i}\} g \ \mu^N_{\sin,\beta}(d\mathbf{s})$$