Infinite-dimensional stochastic differential equations related to random matrices and a phase transition conjecture

- General theory for ISDEs:
  quasi-Gibbs property & log derivative
- Sine RPF, Bessel RPF, Airy RPF, Ginibre RPF,
- A phase transition conjecture for Ginibre Interacting Brownian motions
- Simulations by Ben Said & Otobe

General theorems for Infinite-dim SDE: set up

Let $S = \mathbb{R}^d$, $\mathbb{C}$, $[0, \infty)$.

S: Configuration space over S

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\ s_i \in S,\ \mathsf{s}(|s| < r) < \infty\ (\forall r \in \mathbb{N})\}$$

μ: RPF on S. i.e. prob meas. on S.

Prob: (1) To construct a *natural* stochastic dynamics

$$\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}} \quad \text{(labeled dynamics)}$$

related to μ, *i.e.*

$$\mathsf{X}_t = \sum_{i\in\mathbb{N}} \delta_{X_t^i} \quad \text{(unlabeled dynamics)}$$

is reversible w.r.t. μ.

(2) To find the ∞-dim. SDE that $\mathbf{X}_t$ satisfies.

- ρ^n is called the n-correlation function of μ w.r.t. Radon m. m if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(\mathbf{x}_n) \prod_{i=1}^{n} m(dx_i) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!} d\mu$$

for any disjoint $A_i \in \mathcal{B}(S)$, $k_i \in \mathbb{N}$ s.t. $k_1 + \ldots + k_m = n$.

- μ is called the determinantal RPF generated by (K, m) if its n-corraltion fun. ρ^n is given by

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{1\le i,j\le n}$$

- Ginibre RPF $S = \mathbb{C}$. μ_{gin} is generated by $(K_{\mathrm{gin},2}, \mathrm{g})$

$$K_{\mathrm{gin},2}(x,y) = e^{x\bar{y}} \qquad \mathrm{g}(dx) = \pi^{-1} e^{-|x|^2} dx$$

(g1) $\mu_{\mathsf{gin},2}$ is translation and rotation invariant
(g2) Singularity of Palm meas.
Palm meas. For $\mathbf{x} = \{x_1, \dots, x_{\mathsf{m}}\} \subset S^{\mathsf{m}}$ set

$$\mu_{\mathbf{x}} := \mu(\cdot - \sum_{l=1}^{\mathsf{m}} \delta_{x_l} \mid \mathsf{s}(\{x_l\}) \geq 1 (\forall l))$$

Thm 1 (with Shirai). *Let* $\mathsf{m}, \mathsf{n} \in \{0\} \cup \mathbb{N}$. *Then*
(1) *If* $\mathsf{m} = \mathsf{n}$, *then* $\mu_{\mathbf{x}} \sim \mu_{\mathbf{y}}$. *($\sim$ means ab. cont.)*
(2) *If* $\mathsf{m} \neq \mathsf{n}$, *then* $\mu_{\mathbf{x}}$ *and* $\mu_{\mathbf{y}}$ *are singular each other.*

Remark: • In case of Gibbs measures, it holds always

$$\mu_{\mathbf{x}} \prec \mu$$

• In this sense Ginibre RPF is similar to periodic RPF.

Thm 2 (with Shirai). *Suppose* $\mathsf{m} = \mathsf{n}$. *Then for* $\mu_{\mathbf{y}}$*-a.s.* $\mathsf{s} = \sum_i \delta_{s_i}$,

$$\frac{d\mu_{\mathbf{x}}}{d\mu_{\mathbf{y}}} = \frac{\Delta^{\mathsf{m}}(\mathbf{x}) \det[K_{\mathsf{gin},2}(x_i, x_j)]_{i,j=1}^{\mathsf{m}}}{\Delta^{\mathsf{m}}(\mathbf{y}) \det[K_{\mathsf{gin},2}(y_i, y_j)]_{i,j=1}^{\mathsf{m}}} \lim_{r\to\infty} \prod_{|s_i|<b_r} \frac{|\mathbf{x} - s_i|^2}{|\mathbf{y} - s_i|^2}$$

cpt uni in $\mathbf{x} \in \mathbb{C}^{\mathsf{m}}$.

- $\{b_r\}_{r\in\mathbb{N}}$: $\lim b_r = \infty$
- $|\mathbf{x} - s_i| = \prod_{m=1}^{\mathsf{m}} |x_m - s_i|$ *for* $\mathbf{x} = (x_1, \ldots, x_{\mathsf{m}})$
- $\Delta^{\mathsf{m}}(\mathbf{x}) = \prod_{i<j}^{\mathsf{m}} |x_i - x_j|^2$ *if* $\mathsf{m} \geq 2$, $\Delta^{\mathsf{m}}(\mathbf{x}) = 1$ *if* $\mathsf{m} = 1$.

In particular, if $\mathsf{m} = 1$, then

$$\frac{d\mu_x}{d\mu_y} = \frac{e^{-|x|^2}}{e^{-|y|^2}} \lim_{r\to\infty} \prod_{|s_i|<b_r} \frac{|x - s_i|^2}{|y - s_i|^2}$$

Index of the number of missing particles:
$D_q = \{z \in \mathbb{C}\,;\, |z| < \sqrt{q}\}$ $q \in \mathbb{N}$、

$$F_r(\mathsf{s}) = \frac{1}{r}\sum_{q=1}^{r}(\mathsf{s}(D_q) - q). \qquad (1)$$

Thm 3 (with Shirai). *Let* S *be the configuration space over* $\mathbb{C}$. *Let* $\mathsf{m} \in \mathbb{N}$. *Then for* $\mathbf{x} = (x_1, \ldots, x_{\mathsf{m}})$

$$\lim_{r\to\infty} F_r(\mathbf{s}) = -\mathsf{m} \qquad \textit{weakly in } L^2(\mathsf{S}, \mu_{\mathbf{x}}) \qquad (2)$$

Remark: m is the number of the removed particles.

$$\infty - \mathsf{m} \neq \infty$$

Property of Ginibre rpf: log gass

(g2) $\mu_{\text{gin},2}$ is the weak limit of $\mu_{\text{gin},2}^{N}$:

the labeled expression $\check{\mu}_{\text{gin}}^{N}$ of $\mu_{\text{gin},2}^{N}$ is

$$\check{\mu}_{\text{gin}}^{N} = \frac{1}{Z} \prod_{i<j}^{N} |x_i - x_j|^2 \prod_{k=1}^{N} \mathsf{g}(dx_k) \tag{3}$$

$\mu_{\text{gin},2}^{N}$ is the determinantal RPF gen. by $(K_{\text{gin},2}^{N}, \mathsf{g})$, where

$$K_{\text{gin},2}^{N}(x,y) = \sum_{i=0}^{N-1} \frac{(x\bar{y})^i}{i!}$$

Non rigorous expression of $\mu_{\mathrm{gin},2}$ as a meas $\check{\mu}_{\mathrm{gin},2}$ on $\mathbb{C}^{\mathbb{N}}$:

From (g3)

$$\check{\mu}_{\mathrm{gin},2} = \frac{1}{Z} \prod_{i<j}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} \frac{e^{-|x_k|^2}}{\pi} dx_k \tag{4}$$

From the trans inv we have another informal expression:

$$\check{\mu}_{\mathrm{gin},2} = \frac{1}{Z} \prod_{i<j}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} dx_k \tag{5}$$

Which representations are correct?

Non rigorous expression of $\mu_{\text{gin},2}$ as a meas $\breve{\mu}_{\text{gin},2}$ on $\mathbb{C}^{\mathbb{N}}$:

From (g3)

$$\breve{\mu}_{\text{gin},2} = \frac{1}{Z} \lim_{r\to\infty} \prod_{i<j, |x_i|, |x_j|<r}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} \frac{e^{-|x_k|^2}}{\pi} dx_k \quad (6)$$

From the translation invariance we have another informal expression:

$$\breve{\mu}_{\text{gin},2} = \frac{1}{Z} \lim_{r\to\infty} \prod_{i<j, |x_i - x_j|<r}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} dx_k \quad (7)$$

Which representations are correct?

Both

- Ψ: Ruelle class interaction potential,

$Q_r = \{|x| \le r\}$, $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap Q_r)$, $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap Q_r^c)$

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(Q_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi))$$

- μ is called (Φ, Ψ)-Gibbs m. if it satisfies DLR eq:

$$d\mu_{r,\xi}^m = \frac{1}{z_{r,\xi}} e^{-\mathcal{H}_r(\mathsf{s}) - \mathcal{W}_{r,\xi}(\mathsf{s})} \prod_{k=1}^m e^{-\Phi(s_k)} ds_k$$

$$\mathcal{H}_r = \sum_{s_i, s_j \in Q_r, i<j} \Psi(s_i - s_j), \ \mathcal{W}_{r,\xi} = \sum_{s_i \in Q_r, \xi_j \in Q_r^c} \Psi(s_i - \xi_j)$$

- Ginibre RPF: $\Phi = 0$ $\Psi(x) = -2\log|x|$

In Ginibre rpf, $\mathcal{W}_{r,\xi}$ diverge, so DLR does not make sense

(Φ,Ψ)-Quasi Gibbs measures

(Φ,Ψ)-Gibbs m. Let $\nu_r^m = \prod_{k=1}^m 1_{Q_r}(s_k)e^{-\Phi(s_k)}ds_k$

$$d\mu_{r,\xi}^m = \frac{1}{z_{r,\xi}^m} e^{-\mathcal{H}_r - \mathcal{W}_{r,\xi}} d\nu_r^m \qquad \text{(DLR eq)}$$

(Φ,Ψ)-quasi Gibbs m. $\exists\ c_{r,\xi}^m$

$$c_{r,\xi}^m{}^{-1} e^{-\mathcal{H}_r} d\nu_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\nu_r^m$$

- If μ is Ginibre RPF, $\mathcal{W}_{r,\xi}$ and $z_{r,\xi}^m$ diverge. But $e^{-\mathcal{W}_{r,\xi}}/z_{r,\xi}^m$ conv.

$$c_{r,\xi}^m{}^{-1} \le e^{-\mathcal{W}_{r,\xi}}/z_{r,\xi}^m \le c_{r,\xi}^m$$

- Quasi-Gibbs is very mild restriction. If μ is (Φ,Ψ)-quasi-Gibbs m, then μ is also $(\Phi+f,\Psi)$-quasi Gibbs m for any loc bdd m'able f.

Main theorems: Unlabeled level construction

Let $\mathbb{D}$ be the canonical square field on S: $\mathsf{s} = \sum_i \delta_{s_i}$, $\mathbf{s} = (s_i)$.

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_i \nabla_{s_i}\tilde{f}(\mathbf{s}) \cdot \nabla_{s_i}\tilde{g}(\mathbf{s})$$

Let $\mathcal{D}$ be the set of local smooth fun with $\mathcal{E}_1^{\mu}(f,f) < \infty$.

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g] d\mu$$

Thm 4. [O.95, 09, 10]
(1) *If μ is quasi-Gibbs with upper semi-cont potentials (Φ, Ψ), then $(\mathcal{E}^{\mu}, \mathcal{D}, L^2(\mathsf{S}, \mu))$ is closable.*
(2) *If $(\mathcal{E}^{\mu}, \mathcal{D}, L^2(\mathsf{S}, \mu))$ is closable & all correlation fun are loc bounded, then a diffusion X_t associated with the closure $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$ exists.*

If μ is Poisson rpf with Lebesgue intensity, then $\mathsf{X}_t = \sum_i \delta_{B_t^i}$.

Log derivative of μ

- Let μ_x be the (reduced) Palm m. of μ conditioned at x

$$\mu_x(\cdot) = \mu(\cdot - \delta_x | \mathsf{s}(x) \geq 1)$$

- Let μ^1 be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^1(A \times B) = \int_A \rho^1(x)\mu_x(B)dx$$

- $\mathsf{d}_\mu \in L^1(\mathbb{R}^d \times \mathsf{S}, \mu^1)$ is called the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^1 = -\int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}_\mu d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}$$

Here ∇_x is the nabla on $\mathbb{R}^d$, $\mathcal{D}$ is the space of local smooth functions on S with compact support.

- Very informally

$$\mathsf{d}_\mu = \nabla_x \log \mu^1$$

- Ginibre RPF: $\mathsf{d}_{\mu_{\mathrm{gin},2}}$ has plural representations

$$\mathsf{d}_{\mu_{\mathrm{gin},2}}(x,\mathsf{y}) = -2x + 2\lim_{r\to\infty}\sum_{|y_i|<r}\frac{x-y_i}{|x-y_i|^2} \quad \text{in } L^2_{\mathrm{loc}}(\mu^1)$$

$$\mathsf{d}_{\mu_{\mathrm{gin},2}}(x,\mathsf{y}) = 2\lim_{r\to\infty}\sum_{|x-y_i|<r}\frac{x-y_i}{|x-y_i|^2} \quad \text{in } L^2_{\mathrm{loc}}(\mu^1)$$

- Ginibre RPF: $\mathsf{d}_{\mu_{\mathrm{gin},2}}$ has plural representations

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- These correspond to the following:

$$\check{\mu}_{\mathrm{gin},2} = \frac{1}{Z}\lim_{r\to\infty}\prod_{i<j,|x_i|,|x_j|<r}^{\infty}|x_i-x_j|^2\prod_{k=1}^{\infty}\frac{e^{-|x_k|^2}}{\pi}dx_k \tag{13}$$

$$\check{\mu}_{\mathrm{gin},2} = \frac{1}{Z}\lim_{r\to\infty}\prod_{i<j,|x_i-x_j|<r}^{\infty}|x_i-x_j|^2\prod_{k=1}^{\infty}dx_k \tag{7}$$

Main theorems: Infinite-dim SDE

(A1) ρ^k of μ are locally bounded for all $k \in \mathbb{N}$
(A2) $(\mathcal{E}^{\mu}, \mathcal{D})$ is closable on $L^2(\mathsf{S}, \mu) \Leftarrow \mu$ is quasi-Gibbs
(A3) The log derivative $\mathsf{d}_{\mu} \in L^1_{\mathrm{loc}}(\mu^1)$ exists
(A4) $\{X_t^i\}$ do not collide each other (non-collision)
(A5) each tagged particle X_t^i never explode (non-explosion)

Let $\mathfrak{u} : S^{\mathbb{N}} \to \mathsf{S}$ such that $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$.

Thm 5. (O.10) (A1)–(A5) $\Rightarrow \exists \mathsf{S}_0 \subset \mathsf{S}$ *such that*

$$\mu(\mathsf{S}_0) = 1, \tag{8}$$

and that, for $\forall \mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$, $\exists \mathfrak{u}^{-1}(\mathsf{S}_0)$*-valued pr.* $(X_t^i)_{i\in\mathbb{N}}$ *and* $\exists S^{\mathbb{N}}$*-valued Brownian m.* $(B_t^i)_{i\in\mathbb{N}}$ *satisfying*

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}_{\mu}(X_t^i, \sum_{j\neq i}\delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s} \tag{9}$$

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}_\mu(X_t^i, \sum_{\mathrm{j}\neq\mathrm{i}} \delta_{X_t^j})dt, \quad (X_0^i)_{i\in\mathbb{N}} = \mathbf{s}$$

Thm 6 (O. (JMSJ 09)). *The family of processes* $\{(X_t^i)_{i\in\mathbb{N}}\}$ *is a diffusion with state space* $\mathfrak{u}^{-1}(\mathsf{S}_0) \subset S^{\mathbb{N}}$.

Remark 1. (1) (A1)–(A5) can be checked for Ginibre RPF ($\beta = 2$), Sine RPFs, Airy RPFs and Bessel RPFs ($\beta = 1, 2, 4$).
(2) We can calcurate the log derivatives of these measures.
(3) We have general theorems for quasi-Gibbs property and the log derivatives (O. PTRF10, AOP). The statements are too messy to be omited here.

Thm 7 (with Tanemura). *If μ is a (Φ, Ψ) quasi-Gibbs measure with smooth potential (on $x \neq 0$). Assume*
(H.1) The tail sigma field of μ is trivial.
Then
(1) The SDE has a unique strong solution for q.e. initial staring points (s_i).
(2) The associated martingale problem is unique.
(3) The Dirichlet form that are extention of $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ is unique.

Remark: We do not know the tail triviality hold for Ginibre rpf and others. It is known that all discrete determinantal rpf have trivial tail sigma fields by Russel Lyons. So it is quite likely the same also hold for the determinantal rpfs in continuous spaces.

Gibbs measures :

- All Gibbs measures with Ruelle's class upper semi-cont potentials satisfy the assumptions (A.1)–(A.5).
- In this case, the SDEs become

$$dX_t^i = dB_t^i - \frac{1}{2}\nabla\Phi(X_t^i)dt - \frac{1}{2}\sum_{j\neq i}\nabla\Psi(X_t^i - X_t^j)dt. \qquad (10)$$

Examples: Ginibre rpf

Ginibre rpf: $\Psi(x) = -\beta \log |x|$ $d = 2$, $\beta = 2$. If $\mu = \mu_{\text{gin},2}$,

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j| < r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \tag{11}$$

and also

$$dX_t^i = dB_t^i - X_t^i dt + \lim_{r\to\infty} \sum_{|X_t^j| < r \ \ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{12}$$

This comes from the pulral expressions of $\mathsf{d}_{\mu_{\text{gin},2}}$.
For finite N, these SDEs give different solution.
But in the limit $N \to \infty$ give the same solution if the initial distribution is closed to Ginibre rpf.

Examples: Bessel rpf–hard edge scaling limit

Bessel RPF (joint work with Honda):

$S = [0,\infty)$, $\beta = 2$, $a > 1$

$$dX_t^i = dB_t^i + \frac{a}{2X_t^i}dt + \lim_{r\to\infty}\frac{\beta}{2}\sum_{\substack{|X_t^j|<r\\ j\neq i}}\frac{1}{X_t^i - X_t^j}dt$$

$\beta = 1, 4$ are in progress.

Examples: sine rpf (Dyson's model)–bulk scaling limit

Sine$_\beta$ RPF: $S = R$, $\beta = 1, 2, 4$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \to \infty} \sum_{|X_t^i - X_t^j| < r,\ j \neq i} \frac{1}{X_t^i - X_t^j} dt$$

Spohn (1987) considered the case $\beta = 2$:

$$dX_t^i = dB_t^i + \sum_{j \neq i} \frac{1}{X_t^i - X_t^j} dt$$

He constructed the dynamics as a Markov semigr by Dirichlet form.

The def of $\mu = \mu_{\mathrm{sin},\beta}$:

$\beta = 2 \Rightarrow \mu_{\mathrm{sin},\beta}$ is the det rpf generated by (K_{sin}, dx):

$$K_{\mathrm{sin}}(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}$$

$\beta = 1, 4 \Rightarrow$ the correlation funs are given by quaternion det.

Examples: sine rpf (Dyson's model)–bulk scaling limit

- The dist of eigen values of the G(O/U/S)E RMs are given by

$$m_{\beta}^{N}(d\mathbf{x}_N) = \frac{1}{Z}\prod_{i<j}^{N}|x_i - x_j|^{\beta} e^{-\frac{\beta}{4}\sum_{i=1}^{N}|x_i|^2} d\mathbf{x}_N, \tag{13}$$

- m_{β}^{N} converge the semi-circle law $\varsigma(x)dx = \frac{1}{\pi}\sqrt{2\pi - x^2}dx$

- Take $x_i = s_i/\sqrt{N}$ in (13) and set

$$\mu_{\mathsf{sin},\beta}^{N}(d\mathbf{s}_N) = \frac{1}{Z}\sum_{i<j}^{N}|s_i - s_j|^{\beta}\prod_{k=1}^{N} e^{-\beta|s_k|^2/4N} d\mathbf{s}_N \tag{14}$$

- The stationary m. $\mu = \mu_{\mathsf{sin},\beta}$ is TDL of $\mu_{\mathsf{sin},\beta}^{N}$
- The associated N particle system is given by the SDE:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^{N}\frac{1}{X_t^i - X_t^j}dt - \frac{\beta}{2N}X_t^i dt$$

Examples: Airy rpf – Soft edge scaling limit

Airy rpf: $\mu_{\mathsf{Ai},\beta}$ $(S = \mathbb{R},\ \beta = 1, 2, 4)$ joint work with Tanemura.

Take the scaling $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$ in

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z} \prod_{i<j}^{N} |x_i - x_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |x_i|^2} d\mathbf{x}_N$$

and set

$$\mu_{\mathsf{Ai},\beta}^N(d\mathbf{s}_N) = \frac{1}{Z} \prod_{i<j} |s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N} + N^{-1/6} s_i|^2} d\mathbf{s}_N.$$

Then $\mu_{\mathsf{Ai},\beta}$ is the TDL of $\mu_{\mathsf{Ai},\beta}^N$:

$$\lim_{N\to\infty} \mu_{\mathsf{Ai},\beta}^N = \mu_{\mathsf{Ai},\beta}$$

- $\beta = 2 \Rightarrow \mu^N_{\mathsf{Ai},\beta}$ is the det rpf gen by (K_{Ai}, dx):

$$K_{\mathsf{Ai}}(x,y) = \frac{\mathsf{Ai}(x)\mathsf{Ai}'(y) - \mathsf{Ai}'(x)\mathsf{Ai}(y)}{x-y}$$

Here Ai(·) the Airy function such that

$$\mathsf{Ai}(z) = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{i(zk+k^3/3)}, \qquad z \in \mathbb{C}. \tag{15}$$

Examples: Airy rpf – Soft edge scaling limit

- From

$$\mu^N_{\mathsf{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\prod_{i<j}|s_i - s_j|^\beta e^{-\frac{\beta}{4}\sum_{i=1}^N |2\sqrt{N}+N^{-1/6}s_i|^2} d\mathbf{s}_N$$

we deduce the SDE of the N particle system:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j=1,j\neq i}^N \frac{1}{X_t^i - X_t^j}dt - \frac{\beta}{2}\{N^{1/3} + \frac{1}{2N^{1/3}}X_t^i\}dt$$

- The point is

$$\lim_{N\to\infty}\{\sum_{j=1,j\neq i}^N \frac{1}{X_t^i - X_t^j} - N^{1/3}\} \quad \text{converge}$$

Examples: Airy rpf – Soft edge scaling limit

Thm 8 (with Tanemura). *Let* $\beta = 1, 2, 4$. *Then:*

- *The log derivative* $\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}$ *is*

$$\mathsf{d}^{\mu_{\mathsf{Ai},\beta}}(x, \mathsf{s}) = \beta \lim_{r\to\infty} \{ (\sum_{|x-s_i|<r} \frac{1}{x - s_i}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \}$$

Here

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} 1_{(-\infty,0]}(x)$$

- *Airy rpf* $\mu_{\mathsf{Ai},\beta}$ *satisfy* (A1)–(A5) *and the limit ISDE is*

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

Examples: Airy rpf – Soft edge scaling limit

- The key idea is to take the rescaled semi-circle law ς, as the first approximation of the 1-correlation fun $\rho^{N,1}_{\mathrm{Ai},\beta}$.
- Our method can be applied to other soft edge scaling.
- Let us label $X^i_t > X^{i+1}_t$ $(\forall i)$.

If $\beta = 2$, then the top particle X^1_t is the Airy process $\mathcal{A}(t)$ in the sense of Spohn.

In fact, if the tail sigma field of $\mu_{\mathrm{Ai},\beta}$ is trivial, then the SDE has a unique strong solution.

Even if this is not the case, we proved that the infinite dim stochastic dynamics constructed by Spohn, Johansson & others by the space-time correlation fun is a solution of the prescribed SDE:

$$dX^i_t = dB^i_t + \frac{\beta}{2} \lim_{r\to\infty} \{ (\sum_{j\neq i,\ |X^j_t|<r} \frac{1}{X^i_t - X^j_t}) - \int_{|x|<r} \frac{\varrho(x)}{-x} dx \} dt$$

- Recently the Airy process has been extensively studued by Spohn, Johansson, and many others. Our result is the first time to clarify the SDE describing the limit infinite system for the soft edge.

Examples: Airy rpf – Soft edge scaling limit

- The SDE gives a kind of Girsanov formula.
- These examples are the first time that the infinite dynamics are constructed for rpf appeared in random matrix theory with $\beta = 1, 4$ even if the bulk and the hard edge as well as the soft edge scaling

  In one dimensional system, the method of space-time correaltion functions are avialable (Nagao, Katori-Tanemura, Spohn, and others), but this method is restricted to $\beta = 2$.
- By construction, if the total system start from the Airy$_\beta$ rpf $\mu_{\mathsf{Ai},\beta}$, then the distribution of the top particle X_t^1 equals $F_{\beta,edge}(x)$, the β Tracy-Widom distribution, where $\beta = 1, 2, 4$.

To sum up

Thm 9. *Ginibre RPF ($\beta = 2$), Sine RPFs, Airy RPFs and Bessel RPFs ($\beta = 1, 2, 4$) are qausi-Gibbs m. for $\Psi(x) = -\beta \log |x|$, and the log derivative can be calculated. The asociated ISDE has a solution. If the tail σ field is trivial, then unique strong solution exists.*

- The key point of the proof is to use the small fluctuation property (SFP) of linear statistics for these measures.
- SFP was established by Soshnikov (Sine, Airy, Bessel RPFs), Shirai (Ginibre RPF).
- Proof consists of several parts:

(1) To find a good finite particle approximation $\{\mu^N\}$

(2) To prove uniform *small fluctuation* of $\{\mu^N\}$

(3) To prove uni bounds of 1 & 2 cor funs of $\{\mu^N\}$

(4) To carry out the limiting procedure of d_{μ^N} & quasi-Gibbs property by using general theorems. (O. 11,12)

Let $\beta \in [0, \infty)$ and set

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \to \infty} \sum_{|X_t^i - X_t^j| < r, j \neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt.$$

When $\beta = 2$, then the SDE has a solution, for general β we assume the existence of solution and the rpf $\mu_{\mathsf{gin},\beta}$.

- We tag $X_t^{i_0}$ and investgate the diffusive scaling:

$$\lim_{\epsilon \to 0} \epsilon X^{i_0}_{t/\epsilon^2} = \sqrt{2\alpha_{\mathsf{self}}[\mu_{\mathsf{gin},\beta}]} B_t$$

- Assume $X_0^{i_0} = 0$ and $\sum_{i \neq i_0} \delta_{X_0^i} \sim \mu_{\mathsf{gin},\beta,\mathbf{o}}$.
- $\alpha_{\mathsf{self}}[\cdot]$ is called the self-diffusion matrix.

A phase transition conjecture for 2D Coulomb stochastic dynamics

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt$$

$$\lim_{\epsilon\to 0} X_{t/\epsilon^2}^{i_0} = \sqrt{2\alpha_{\mathsf{self}}[\mu_{\mathsf{gin},\beta}]} B_t, \qquad X_0^{i_0} = 0, \quad \sum_{i\neq i_0} \delta_{X_0^i} \sim \mu_{\mathsf{gin},\beta,\mathbf{o}}.$$

Conj: There exist constants $\beta_1 < \beta_2 < \beta_3$ such that
(C1) $\beta < \beta_1 \Rightarrow \alpha_{\mathsf{self}}[\mu_{\mathsf{gin},\beta}] > 0$ (diffusive)
(C2) $\beta_1 < \beta < \infty \Rightarrow \alpha_{\mathsf{self}}[\mu_{\mathsf{gin},\beta}] = 0$ (subdiffusive),
(C3) $\beta_2 < \beta < \infty \Rightarrow X_t^{i_0}$ has an inv prob measure
$X_t^{i_0} = O(\log t)$ (log behaivior)
(C4) $\beta_3 < \beta < \infty \Rightarrow (X_t^i)_{i\in\mathbb{N}}$ form a lattice like system.
Moreover,

$$\beta_1 \sim 1, \quad \beta_2 \sim 2.$$

Rigorous results: homogenization of diffusion in 2D Coulomb-periodic env.

Let $\mathsf{s} = \sum_i \delta_{s_i} \in \mathsf{S}$. Let $X_t^{\mathsf{s}} \in \mathbb{R}^2$ be the solution of

$$dX_t^{\mathsf{s}} = dB_t + \frac{\beta}{2} \lim_{q\to\infty} \sum_{|X_t^{\mathsf{s}}-s_i|<q} \frac{X_t^{\mathsf{s}} - s_i}{|X_t^{\mathsf{s}} - s_i|^2} dt$$

Let μ be a rpf, and set for a.s. s w.r.t. μ

$$\lim_{\varepsilon\to\infty} \varepsilon X^{\mathsf{s}}_{t/\varepsilon^2} = \sqrt{\alpha^{\beta}_{\mathrm{eff}}[\mu]} B_t \tag{16}$$

Thm 10. μ_{per} *be a periodic rpf* $\Rightarrow$

(1) $\alpha^{\beta}_{\mathrm{eff}}[\mu_{\mathrm{per}}] > 0$.

(2) $\alpha^{\beta}_{\mathrm{eff}}[\mu_{\mathrm{per},0}] > 0$ *for* $\beta < 1$

$\alpha^{\beta}_{\mathrm{eff}}[\mu_{\mathrm{per},0}] = 0$, X_t^{s} *has a inv prob m for* $\beta > 2$

Rigorous results: homogenization of diffusion in Ginibre env.

Let $\mathsf{s} = \sum_i \delta_{s_i} \in \mathsf{S}$. Let $X_t^{\mathsf{s}} \in \mathbb{R}^2$ be the solution of

$$dX_t^{\mathsf{s}} = dB_t + \lim_{q\to\infty} \sum_{|X_t^{\mathsf{s}}-s_i|<q} \frac{X_t^{\mathsf{s}} - s_i}{|X_t^{\mathsf{s}} - s_i|^2} dt$$

Thm 11. *Assume* $\mathsf{s} \sim \mu_{\mathrm{gin},2,\mathbf{o}}$ *and set*

$$\lim_{\varepsilon\to\infty} \varepsilon X_{t/\varepsilon^2}^{\mathsf{s}} = \sqrt{\alpha_{\mathrm{eff}}^2[\mu_{\mathrm{gin},2,\mathbf{o}}]} B_t$$

Then

$$\alpha_{\mathrm{eff}}^2[\mu_{\mathrm{gin},2,\mathbf{o}}] = 0$$

Conj: The positivity of $\alpha_{\mathrm{eff}}^2[\mu_{\mathrm{gin},2}]$ is an open problem. Since $\mu_{\mathrm{gin},2}$ is similar to μ_{per}, we should have

$$\alpha_{\mathrm{eff}}^2[\mu_{\mathrm{gin},2}] > 0$$

Observation: self-diffusion of 2D Coulomb system 1

Obs 0: $\mu_{\mathsf{gin},\beta}$ exists for general $\beta > 0$.

Obs 1: Since (by O.-Shirai [2012])

$$\mu_{\mathsf{gin},2} \perp \mu_{\mathsf{gin},2,\mathbf{o}} \tag{17}$$

we have for general $\beta > \beta_1$ $\qquad (\beta_1 \le 2)$

$$\mu_{\mathsf{gin},\beta} \perp \mu_{\mathsf{gin},\beta,\mathbf{o}}. \tag{18}$$

Let $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}}$ be the solution of

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt \tag{19}$$

Note that

$$\mathsf{X}_t := \sum_{i\in\mathbb{N}} \delta_{X_t^i} \sim \mu_{\mathsf{gin},\beta} \tag{20}$$

Observation: self-diffusion of 2D Coulomb system 2

Let X_t^1 be the tag particle, and set $Y_t^i = X_t^{i+1} - X_t^1$.

Obs 2: By (18) and

$$\mathsf{Y}_t := \sum_{i \neq i_0} \delta_{Y_t^i} \sim \mu_{\mathsf{gin},\beta,\mathbf{o}}, \tag{21}$$

we have $X_t^* \in \mathbb{C}$ such that

$$X_t^* \sim \text{prob m}, \qquad \mathsf{Y}_t + \delta_{X_t^*} \sim \mathsf{X}_t \sim \mu_{\mathsf{gin},\beta} \tag{22}$$

Obs 3:

$$dX_t^1 = dB_t^1 + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^1 - X_t^i| < r,\ i \geq 2} \frac{X_t^1 - X_t^i}{|X_t^1 - X_t^i|^2} dt \tag{23}$$

$$dX_t^1 = dB_t^1 - \frac{\beta}{2} \lim_{r\to\infty} \sum_{|Y_t^i| < r,\ i \in \mathbb{N}} \frac{Y_t^i}{|Y_t^i|^2} dt$$

Observation: self-diffusion of 2D Coulomb system 3

Set $Y_t^* = X_t^* - X_t^1$. Then from

$$dX_t^1 = dB_t^1 - \frac{\beta}{2} \lim_{r\to\infty} \sum_{|Y_t^i|<r,\ i\in\mathbb{N}} \frac{Y_t^i}{|Y_t^i|^2} dt \tag{24}$$

we have

$$dX_t^1 = dB_t^1 - \frac{\beta}{2} \frac{X_t^1 - X_t^*}{|X_t^1 - X_t^*|^2} dt - \frac{\beta}{2} \lim_{r\to\infty} \sum_{|Y_t^i|<r,\ i\in\mathbb{N}\cup\{*\}} \frac{Y_t^i}{|Y_t^i|^2} dt$$

Hence

$$\begin{aligned} dX_t^1 = dB_t^1 &- \frac{\beta}{2} \frac{X_t^1}{|X_t^1|^2} dt \\ &+ \frac{\beta}{2} \left\{ \frac{X_t^1}{|X_t^1|^2} - \frac{X_t^1 - X_t^*}{|X_t^1 - X_t^*|^2} \right\} dt - \frac{\beta}{2} \lim_{r\to\infty} \sum_{\substack{|Y_t^i|<r, \\ i\in\mathbb{N}\cup\{*\}}} \frac{Y_t^i}{|Y_t^i|^2} dt \end{aligned} \tag{25}$$

$$dX_t^1 = dB_t^1 - \frac{\beta}{2}\frac{X_t^1}{|X_t^1|^2}dt + \frac{\beta}{2}\{\frac{X_t^1}{|X_t^1|^2} - \frac{X_t^1 - X_t^*}{|X_t^1 - X_t^*|^2}\}dt - \frac{\beta}{2}\lim_{r\to\infty}\sum_{|Y_t^i|<r,\ i\in\mathbb{N}\cup\{*\}}\frac{Y_t^i}{|Y_t^i|^2}dt \tag{26}$$

Obs 4: (1) By homogenization, $\exists\sqrt{2a[\beta]} \le E$

$$\epsilon\{B_{u/\epsilon^2}^1 - \frac{\beta}{2}\lim_{r\to\infty}\int_0^{u/\epsilon^2}\sum_{|Y_t^i|<r,\ i\in\mathbb{N}\cup\{*\}}\frac{Y_t^i}{|Y_t^i|^2}dt\} = \sqrt{2a[\beta]}\hat{B}_u \tag{27}$$

Since X_t^* has inv prob

$$\epsilon\int_0^{u/\epsilon^2}\frac{\beta}{2}\{\frac{X_t^1}{|X_t^1|^2} - \frac{X_t^1 - X_t^*}{|X_t^1 - X_t^*|^2}\}dt \sim o(\epsilon) \tag{28}$$

Hence we have (approximately)

$$dX_t^1 = \sqrt{2a[\beta]}dB_t^1 - \frac{\beta}{2}\frac{X_t^1}{|X_t^1|^2}dt \tag{29}$$

By the simple calculation ($\beta > \beta_{00}$, $\tilde{B}_t$ is 1D Br m)

$$d|X_t^1| = \sqrt{2a[\beta]}d\tilde{B}_t + (a[\beta] - \frac{\beta}{2})\frac{1}{|X_t^1|}dt \tag{30}$$

So the phase transition follows from the one of Bessel processes.

Simulation of Ginibre IBM (2D Coulomb system) and phase transition

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r \ \ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \tag{T}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{-\alpha X_t^i + \lim_{r\to\infty} \sum_{|X_t^j|<r \ \ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\} dt. \tag{OU}$$

Here, since $\rho^1 = 1/\pi$, $\alpha = |\{|x| \le 1\}|\rho^1 = 1$.

- Taking (OU) & (T)into account we take the model:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{-X_t^i + \sum_{j=1 \ \ j\neq i}^{N} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\} dt \tag{OUN}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{\sum_{j=1 \ \ j\neq i}^{N} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\} dt \tag{TN}$$

Simulation: 3D Coulomb system

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\sum_{|X_t^i - X_t^j|<r \ \ j\neq i}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt \tag{T}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{-\alpha X_t^i + \lim_{r\to\infty}\sum_{|X_t^j|<r \ \ j\neq i}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\}dt. \tag{OU}$$

We take $\rho^1 = 1$. So $\alpha = |\{|x| \leq 1\}|\rho^1 = 4\pi/3$.

- Taking (OU) & (T)into account we take the model:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{-\frac{4\pi}{3}X_t^i + \sum_{j=1 \ \ j\neq i}^{N}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\}dt \tag{OUN}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\{\sum_{j=1 \ \ j\neq i}^{N}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}\}dt \tag{TN}$$

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Representation of α Let $\tilde{\mathcal{E}} : L^2(\mu_{\mathsf{gin},,2,,2,\mathbf{o}}) \otimes L^2(\mu_{\mathsf{gin},,2,\mathbf{o}}) \to \mathbb{R}$:

$$\tilde{\mathcal{E}}(\mathbf{f}, \mathbf{g}) = \int_{\mathsf{S}} \frac{1}{2} \sum_{i=1}^{2} f_i g_i d\mu_{\mathsf{gin},,2,\mathbf{o}} \qquad \text{for } \mathbf{f} = (f_1, f_2)$$

$$\tilde{\mathcal{D}} = \overline{\{(D_1 f, D_2 f); f \in \mathcal{D}_0\}}$$

Here D_i is the generator of the translation, $\mathbf{e}_i$ unit vector. There's a unique $\mathbf{u}_i \in \tilde{\mathcal{D}}$ s.t.

$$\tilde{\mathcal{E}}(\mathbf{u}_i, \mathbf{g}) = \tilde{\mathcal{E}}(\mathbf{e}_i, \mathbf{g}) \text{ for all } \mathbf{g} \in \tilde{\mathcal{D}} \tag{31}$$

Thm 12 (O.98). *The effective-diffusion matrix α is given by*

$$\alpha_{ij} = \tilde{\mathcal{E}}(\mathbf{e}_i - \mathbf{u}_i, \mathbf{e}_j - \mathbf{u}_j) \tag{32}$$

Moreover, $\alpha = 0$ if and only if $\mathbf{e}_i \in \tilde{\mathcal{D}}$ for all i.

- We need to prove $\mathbf{e}_1, \mathbf{e}_2 \in \tilde{\mathcal{D}}$
- One can check $(F_r, 0) \in \tilde{\mathcal{D}}$. Hence

$$\mathbf{e}_1 = \lim_{r \to \infty} (-F_r, 0) \quad \text{weakly in } \tilde{\mathcal{D}}$$

This completes the proof of Thm 8.

Conj: If we replace $\mu_{\mathrm{gin},,2,\mathbf{o}}$ by $\mu_{\mathrm{gin},2}$, then $\alpha > 0$. Indeed, in case of periodic μ, this is the case.

Related problems:

- Yoo proved that Determinantal RPF with

$$\mathrm{Spec}(K) \subset [0,1)$$

are *Gibbs* measures. So it is likely all Determinantal RPF are quasi Gibbs measures, i.e., under the condition

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To strength Yoo's result like this is important because RPFs in infinite volume appeared in RMT usually satisfy that

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- To calculate the log derivative of Determinantal RPFs.

- β ensemble of Sine, Bessel, Airy for general $\beta > 0$:

(Valkó, B.-Virág, B., Ramírez, J.-Rider, B.-Virág, B.)

Good finite approximations are clear: Log gasses.

The problem is to control correlation functions and to prove small fluctuations.

- The spectrum of Gaussian Analitic functions

(Some progress done by Shirai)

- In particular, GAF with Bergmann Kernel

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