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STOCHASTIC GEOMETRY AND DYNAMICS OF INFINITELY MANY PARTICLE SYSTEMS —RANDOM MATRICES AND INTERACTING BROWNIAN MOTIONS IN INFINITE DIMENSIONS

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Dedicated to the memory of Nobuyuki Ikeda

Abstract. We explain the general theories involved in solving an infinite-dimensional stochastic differential equation (ISDE) for interacting Brownian motions in infinite dimensions related to random matrices. Typical examples are the stochastic dynamics of infinite particle systems with logarithmic interaction potentials such as the sine, Airy, Bessel, and also for the Ginibre interacting Brownian motions. The first three are infinite-dimensional stochastic dynamics in one-dimensional space related to random matrices called Gaussian ensembles. They are the stationary distributions of interacting Brownian motions and given by the limit point processes of the distributions of eigenvalues of these random matrices.

The sine, Airy, and Bessel point processes and interacting Brownian motions are thought to be geometrically and dynamically universal as the limits of bulk, soft edge, and hard edge scaling. The Ginibre point process is a rotation- and translation-invariant point process on $\mathbb{R}^2$, and an equilibrium state of the Ginibre interacting Brownian motions. It is the bulk limit of the distributions of eigenvalues of non-Hermitian Gaussian random matrices.

When the interacting Brownian motions constitute a one-dimensional system interacting with each other through the logarithmic potential with inverse temperature $\beta = 2$, an algebraic construction is known in which the stochastic dynamics are defined by the space-time correlation function. The approach based on the stochastic analysis (called the analytic approach) can be applied to an extremely wide class. If we apply the analytic approach to this system, we see that these two constructions give the same stochastic dynamics. From the algebraic construction, despite being an infinite interacting particle system, it is possible to represent and calculate various quantities such as moments by the correlation functions. We can thus obtain quantitative information. From the analytic construction, it is possible to represent the dynamics as a solution of an ISDE. We can obtain qualitative information such as semi-martingale properties, continuity, and non-collision properties of each particle, and the strong Markov property of the infinite particle system as a whole.

Ginibre interacting Brownian motions constitute a two-dimensional infinite particle system related to non-Hermitian Gaussian random matrices. It has a logarithmic interaction potential with $\beta = 2$, but no algebraic configurations are known.The present result is the only construction.

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1. Prologue

We explain the general theories involved in solving an infinite-dimensional stochastic differential equation (ISDE) called the interacting Brownian motions in infinite dimensions. We have developed recently two general theories for constructing interacting Brownian motions. One offers a geometric approach for solving the ISDE (for convenience referred to as the first theory), and the other provides a new method to establish the existence of strong solutions and the pathwise uniqueness of solutions of the ISDE by introducing a new notion of solution of an ISDE (called the IFC solution) and examining various tail σ-fields (referred to as the second theory). The former is developed in a quartet of papers [40, 41, 42, 43], and the latter is developed in [49] together with [50, 52, 53, 51]. We shall explain the basic ideas from these papers.

Interacting Brownian motions arising from random matrices have logarithmic interaction potentials (two-dimensional Coulomb potentials). These potentials inherently have a very strong long-range effect, from which some interesting phenomena develop that we shall present.

Herbert Spohn at Minnesota in 1986:

The starting point of this research is a lecture presented in 1986 by Spohn attended by the author. The Institute for Mathematics and Its Applications (IMA) was established in Minnesota University in 1982. Between 1985–1986, the IMA had a program called *Stochastic Differential Equations and Their Applications*. George Papanicolaou organized the workshop "Hydrodynamic behavior and interacting particle systems", where the lecture was delivered. What the author would like to portray is the atmosphere at the workshop for Spohn's lecture.

Spohn was speaking quickly, writing equations on the whiteboard behind the platform of the big venue. The author could hardly understand his lecture on the stage, and the only thing that impressed most was the stochastic differential equation (SDE) he wrote in the corner of the whiteboard:

$$dX_t^i = dB_t^i + \sum_{j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{N}). \tag{1.1}$$

This is the ISDE called Dyson's model in infinite dimensions.

The reason why this equation impressed is that the ISDE (1.1) has a beautiful shape, and that there was a mysteriousness in solving this equation despite the effect of the interaction strongly remaining at infinity. In reading the proceedings [61], it turned out to be that Spohn obtained informally the ISDE (1.1) as a limit of the N-particle SDEs

$$dX_t^{N,i} = dB_t^i + \sum_{j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{1}{N} X_t^{N,i} dt \quad (i = 1, \dots, N) \tag{1.2}$$

and he did not solve the ISDE (1.1) itself.

Consider the stochastic dynamics X^N given by the finite-dimensional SDEs (FSDEs)

$$\mathsf{X}_t^N = \sum_{i=1}^{N} \delta_{X_t^{N,i}}.$$

Then the stationary distribution μ^N of solutions of (1.2) is

$$\frac{1}{Z}\{\prod_{i<j}^{N}|x_i-x_j|^2\}\exp\left\{-\frac{1}{2N}\sum_{k=1}^{N}|x_k|^2\right\}d\mathbf{x}_N.$$

The corresponding distribution of the configuration space S is denoted by the same symbol μ^N. This is the image measure $\mu^N\circ\mathfrak{u}^{-1}$ through the map $\mathfrak{u}((s_i))=\sum_i\delta_{s_i}$, and is the stationary distribution of the unlabeled dynamics X^N by construction. From the result of random matrix theory (orthogonal polynomial theory), its limit μ exists as the probability measure on the configuration space S and is called the Sine_2 point process. We recall here a probability measure on the configuration space is generally called a point process (also called a random point field).

Spohn proved that the natural positive bilinear form associated with μ is closable on $L^2(\mu)$. From this he constructed the $L^2(\mathsf{S},\mu)$-Markovian semi-group associated with the closure of the Dirichlet form. This was the meaning of the stochastic dynamics given by (1.1). His proof of the closability was via the free fermion theory. He used various tools such as random matrices, the matrix representation of correlation functions, and the fermion representation. Although a long time ago, the lecture left a long-lasting profound impression.

Spohn showed that the fluctuation of the particles is extremely small, so the infinite sum of the coefficients of the ISDE described above also has a significance as a conditional convergence. For long-range correlations such that the interaction potential is infinite range and polynomial decay, we could not solve the ISDE even with good potentials such as those of Ruelle's class. Therefore, it seemed impossible to solve the ISDE (1.1) with logarithmic interaction potentials.

Recently, Tsai [65] has solved a family of ISDEs including (1.1) such that for general $\beta\geq 1$

$$dX_t^i=dB_t^i+\frac{\beta}{2}\lim_{R\to\infty}\sum_{j\neq i;|X_t^i-X_t^j|<R}^{\infty}\frac{1}{X_t^i-X_t^j}dt\quad(i\in\mathbb{N}).$$

His method uses the special structure of one-dimensional system and a special monotonicity of the logarithmic interaction potential appearing only in this model.

2. Typical examples

In a sequence of papers [40, 41, 42, 43], we constructed a general theory to solve a class of ISDEs, called interacting Brownian motions. Interacting Brownian motions are usually described by SDEs such that

$$dX_t^i=dB_t^i-\frac{\beta}{2}\nabla\Phi(X_t^i)dt-\frac{\beta}{2}\sum_{j\neq i}^{\infty}\nabla\Psi(X_t^i,X_t^j)dt\quad(i\in\mathbb{N}),\tag{2.1}$$

where $\Phi:\mathbb{R}^d\to\mathbb{R}\cup\{\infty\}$ is a free potential and $\Psi:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}\cup\{\infty\}$ is an interaction potential, and β a non-negative constant called inverse temperature. If β is large, then the system is affected more strongly by the interaction potentials. In the following, all examples other than the Airy interacting Brownian motions are of the form given in (2.1). By definition, the solution $\mathbf{X}=(X^i)_{i\in\mathbb{N}}$ of (2.1) is an $(\mathbb{R}^d)^{\mathbb{N}}$-valued stochastic process.

What we were aiming at is solving the ISDE as related to random matrices, for which the interaction potential Ψ is

$$\Psi(x,y) = -\log|x-y|. \tag{2.2}$$

Below are a few typical examples of interacting Brownian motions. The space in which the particles move is denoted as S. Because $S = \mathbb{R}^d$, we identify S by the dimension d, the exception being the Bessel interacting Brownian motions for which $S = [0,\infty)$.

The first four examples related to random matrices, we take (2.2). The last two examples are typical of interaction potentials in statistical physics; both are of Ruelle's class. All these potentials are long-range interactions, so the conventional theory cannot be applied in constructing the solutions of the ISDE.

When considering an ISDE, the equation cannot have meaning over the entire space $(\mathbb{R}^d)^{\mathbb{N}}$, and an appropriate subset has to be set. We first consider a stationary state μ naturally equipped with an ISDE and take the set as the support of μ. Here μ is not a probability measure on $(\mathbb{R}^d)^{\mathbb{N}}$ but the probability measure on the configuration space S on $\mathbb{R}^d$

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\, ; \mathsf{s}(S_r) < \infty \quad \text{for all } r \in \mathbb{N}\}, \tag{2.3}$$

where $S_r = \{|s| \le r\}$, δ_a is the Delta measure at a, and S is equipped with the vague topology. By construction S is a Polish space.

We call $\mathbf{s} = (s_i) \in (\mathbb{R}^d)^{\mathbb{N}}$ labeled particles and $\mathsf{s} = \sum_i \delta_{s_i} \in \mathsf{S}$ unlabeled particles, respectively. In the former, individual particles are numbered and distinguishable, whereas in the latter, individual particles are not distinguishable. Define the two stochastic dynamics $\mathbf{X} = \{\mathbf{X}_t\}$ and $\mathsf{X} = \{\mathsf{X}_t\}$ by

$$\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}} \quad \text{(labeled dynamics)},$$
$$\mathsf{X}_t = \sum_{i\in\mathbb{N}}^{\infty} \delta_{X_t^i} \quad \text{(unlabeled dynamics)}.$$

Below we set $\mathsf{X} = \sum_i \delta_{X^i}$ by $\mathsf{X}_t = \sum_i \delta_{X_t^i}$. The map $\mathfrak{u} : S^{\mathbb{N}} \to \mathsf{S}$, $\mathfrak{u}(\mathbf{s}) = \sum_i \delta_{s_i}$, is called the unlabeling map, and $\mathfrak{l} : \mathsf{S} \to S^{\mathbb{N}}$ is called a labeling map. The unlabeling map $\mathfrak{u}$ is unique, whereas we have infinitely many labeling maps $\mathfrak{l}$. One reason for introducing the unlabeled dynamics for interacting Brownian motions is labeled dynamics do not have any stationary distributions, whereas the unlabeled dynamics may have. This is similar to the role of unlabeled particles in the theory of infinite-volume Gibbs measures.

A solution space of an ISDE is the subset of $(\mathbb{R}^d)^{\mathbb{N}}$ in which the labeled particles move. The choice of solution space is an important issue in considering the ISDE (2.1). We consider the support S_μ of the probability measure μ on a configuration space and take a suitable subset of the inverse image $\mathfrak{u}^{-1}(\mathsf{S}_\mu)$ of S_μ as the solution space. Here, μ is suitably chosen for the ISDE. Hence, we have to clarify the meaning of the point process μ related to the ISDE. Indeed, choosing μ appropriately for each ISDE, as we shall show later, means solving "**differential equation** (7.5) **for** μ" determined from the ISDE. In this sense as well, our theory is geometric.

We recall some notation. A probability measure μ on the configuration space $(\mathsf{S}, \mathcal{B}(\mathsf{S}))$ is called a point process. For a point process μ on S, a symmetric function

$\rho^n : S^n \to [0,\infty)$ is called the n-point correlation function of μ with respect to the Radon measure m if ρ^n satisfies

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(x_1,\ldots,x_n)m(dx_1)\cdots m(dx_n) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!}d\mu,$$

where $A_1,\ldots,A_m \in \mathcal{B}(S),\ k_1,\ldots,k_m \in \mathbb{N},\ k_1+\cdots+k_m = n$. Here we set $\mathsf{s}(A_i)!/(\mathsf{s}(A_i)-k_i)! = 0$ for $\mathsf{s}(A_i)-k_i < 0$.

A point process μ is called a determinantal point process with kernel $K : S\times S \to \mathbb{C}$ and Radon measure m (a (K,m)-determinantal point process) if for each $n \in \mathbb{N}$ the n-correlation function of μ with respect to m is given by

$$\rho^n(x_1,\ldots,x_n) = \det[K(x_i,x_j)]_{i,j=1}^n.$$

If K is Hermite symmetric with spectrum contained in $[0,1]$, then the (K,m)-determinantal point process exists and is unique [27, 58, 60]. Here, we always take m to be the Lebesgue measure.

With this preparation, the rest of this section describes typical examples of interacting Brownian motions.

2.1. **Sine$_\beta$-interacting Brownian motion (Dyson's model in infinite dimensions)** [41, 65]**:**

Let $d = 1$, $\Phi(x) = 0$, $\Psi(x,y) = -\log|x-y|$, $\beta = 1,2,4$.

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{1}{X_t^i - X_t^j}dt \quad (i \in \mathbb{N}). \tag{2.4}$$

We note that the stationary distribution of the unlabeled dynamics associated with (2.4) has translation invariance in $\mathbb{R}$-action. Hence, the sum of the drift coefficient in (2.4) does not converge absolutely; it only enjoys conditional convergence. ISDE (2.4) is called Dyson's model in infinite dimensions, which corresponds to (1.1) mentioned before. The stationary distribution of the associated unlabeled dynamics is called the sine$_2$ point process, which is a determinantal point process for which the n-point correlation function with respect to the Lebesgue measure is given by

$$\rho^n_{\mathrm{sin},2}(x) = \det[K_{\mathrm{sin},2}(x_i - x_j)]_{i,j=1}^n,$$

where $K_{\mathrm{sin},2}$ is the sine kernel which is a continuous function such that

$$K_{\mathrm{sin},2}(x-y) = \frac{\sin 2(x-y)}{\pi(x-y)}. \tag{2.5}$$

If $\beta = 1,4$, then the analogous formula are given by the Pfaffian (or quaternion [29, 30]). The same holds for point processes in one dimension arising from random matrices. Sine$_\beta$-point processes are translation and rotation invariant ($\beta = 1,2,4$). These properties are inherited by sine$_\beta$-interacting Brownian motions.

2.2. **Airy$_\beta$-interacting Brownian motion** [50]**:**

Let $d = 1$, $\Phi(x) = 0$, $\Psi(x,y) = -\log|x-y|$, and $\beta = 1,2,4$.

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \Big\{ \Big(\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j} \Big) - \int_{|x|<r} \frac{\hat{\varrho}(x)}{-x} dx \Big\} dt \quad (i \in \mathbb{N}).$$

Here $\hat{\varrho}$ is given by

$$\hat{\varrho}(x) = \frac{1_{(-\infty,0)}(x)}{\pi}\sqrt{-x}.$$

The stationary distribution $\mu_{\mathrm{Ai},2}$ of the associated unlabeled dynamics is a determinantal point process called an Airy_{β}-point process. When $\beta = 2$, its n-point correlation function $\rho^n_{\mathrm{Ai},2}$ is such that

$$\rho^n_{\mathrm{Ai},2}(\mathbf{x}_n) = \det[K_{\mathrm{Ai},2}(x_i, x_j)]_{i,j=1}^n.$$

Here the kernel function is a continuous function given by

$$K_{\mathrm{Ai},2}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} \quad (x \neq y),$$

where $\mathrm{Ai}'(x) = d\mathrm{Ai}(x)/dx$ and $\mathrm{Ai}(\cdot)$ is the Airy function defined by

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{i(zk+k^3/3)}, \quad z \in \mathbb{R}.$$

2.3. **$\mathbf{Bessel}_{\alpha,\beta}$-interacting Brownian motion** [11]**:**

Let $d = 1$, $S = [0,\infty)$, and $1 \le \alpha < \infty$. The ISDE is given by

$$dX_t^i = dB_t^i + \{\frac{\alpha}{2X_t^i} + \frac{\beta}{2}\sum_{j\neq i}^{\infty}\frac{1}{X_t^i - X_t^j}\}dt \quad (i \in \mathbb{N}).$$

The stationary distribution $\mu_{\mathrm{Be},\alpha,\beta}$ of the associated unlabeled dynamics is a determinantal point process $\rho^n_{\mathrm{Ai},2}$ called a $\mathrm{Bessel}_{\alpha,\beta}$-point process, where $\beta = 1,2,4$. When $\beta = 2$, its n-point correlation function $\rho^n_{\mathrm{Be},\alpha,2}$ with respect to the Lebesgue measure on $[0,\infty)$ is given by

$$\rho^n_{\mathrm{Be},\alpha,2}(\mathbf{x}^n) = \det[\mathsf{K}_{\mathrm{Be},\alpha,2}(x_i,x_j)]_{i,j=1}^n.$$

Here the kernel function $\mathsf{K}_{\mathrm{Be},\alpha,2}$ is a continuous function such that

$$\mathsf{K}_{\mathrm{Be},\alpha,2}(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)} \quad (x \neq y).$$

2.4. **Ginibre interacting Brownian motion** [41]**:**

Let $d = 2$ and $\Psi(x,y) = -\log|x-y|$. We consider the two ISDEs (2.6) and (2.7):

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\sum_{|X_t^i - X_t^j|<r,\ j\neq i}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt \quad (i \in \mathbb{N}), \tag{2.6}$$

$$dX_t^i = dB_t^i - X_t^i + \frac{\beta}{2}\lim_{r\to\infty}\sum_{|X_t^j|<r,\ j\neq i}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2}dt \quad (i \in \mathbb{N}). \tag{2.7}$$

If $\beta = 2$, the stationary distribution of the unlabeled dynamics associated with these ISDEs are the Ginibre point process μ_{Gin}, which is a determinantal point process with kernel function given by

$$K_{\mathrm{Gin}}(x,y) = \frac{1}{\pi}\exp\{-\frac{1}{2}|x|^2 + x\bar{y} - \frac{1}{2}|y|^2\},$$

where we naturally identify $\mathbb{R}^2$ with $\mathbb{C}$; $\bar{y}$ is the complex conjugate of $y \in \mathbb{C}$. Clearly, ISDEs (2.6) and (2.7) are different SDEs. These equations have the same solutions on the support of the Ginibre point process μ_{Gin} for μ_{Gin}-a.s. starting points. Both are strong solutions and enjoy pathwise uniqueness. Thus, the different ISDEs have the same pathwise-unique strong solutions. This is the first example of the dynamical rigidity of interacting Brownian motions with logarithmic interaction potentials.

All the examples given above are related to random matrices. We next present examples of Gibbs measures associated with Ruelle's class interaction potentials. Here, we mean Gibbs measures are point processes for which conditional distributions are given by the Dobrushin–Lanford–Ruelle (DLR) equation. The precise definition of Gibbs measure is given by (5.5). Our general theory can be applied to essentially all Gibbs measures.

2.5. **Lennard-Jones 6-12 potential:**
Let $d = 3$, $\beta > 0$, and $\Psi_{6,12}(x) = \{|x|^{-12} - |x|^{-6}\}$. The interacting potential $\Psi_{6,12}$ is called the Lennard–Jones 6-12 potential. The associated ISDE is

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j=1,j\neq i}^{\infty}\{\frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}}\}dt \quad (i \in \mathbb{N}).$$

2.6. **Riesz potentials of Ruelle's class:**
Let $d < a \in \mathbb{N}$, $0 < \beta$, and $\Psi_a(x) = (\beta/a)|x|^{-a}$. The associated ISDE is

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j=1,j\neq i}^{\infty}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}}dt \quad (i \in \mathbb{N}). \tag{2.8}$$

At first glance this ISDE resembles (2.4) and (2.6). Indeed, (2.8) corresponds to (2.4) and (2.6) with $a = 0$. The sum of the drift term converges *absolutely* unlike (2.4) and (2.6).

3. Random matrices and interacting Brownian motions

In this section, we explain the relationship between random matrices and the interacting Brownian motions. We assume $\beta = 1, 2, 4$ throughout this section. We refer to [29, 1, 5] for the general theory of random matrices.

Gaussian random matrices of order N are square matrices $M^N = [m_{ij}]_{i,j=1}^N$ for which each elements are either real, Hermitian, or quaternionic—that is, the distributions are invariant under orthogonal, unitary, or symplectic transformations (denoted by capital letters O/U/S)— and are independent except for these symmetries. These random matrices are referred to as Gaussian ensembles labeled G(O/U/S)E. Let $\mathbb{F}$ denote one of the real/complex/quaternion fields; they correspond respectively to G(O/U/S)E. We assume M^N is $\mathbb{F}$-symmetric, and its elements are mean free, $\mathbb{F}$-valued Gaussian random variables. Moreover, their covariances are one for $i < j$. On the diagonal, that is $i = j$, they are real Gaussian random variable with variance one. Then the eigenvalue distribution of M^N is given by

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\{\prod_{i<j}^{N}|x_i - x_j|^\beta\}\exp\left\{-\frac{\beta}{4}\sum_{k=1}^{N}|x_k|^2\right\}d\mathbf{x}_N, \tag{3.1}$$

where $\mathbf{x}_N = (x_1, \ldots, x_N)$, $d\mathbf{x}_N = dx_1 \cdots dx_N$. Here GOE, GUE, and GSE correspond to $\beta = 1, 2$, and 4, respectively. Equation (3.1) makes sense for all $0 < \beta < \infty$ and correspond to typical log gases [5].

Let $\mathcal{P}$ denote the set of all probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Under $m_\beta^N(d\mathbf{x}_N)$, we consider $\mathcal{P}$-valued random variable

$$\mathbb{X}_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i/\sqrt{N}}$$

and denote by μ_β^N its distribution. By definition, μ_β^N is a probability measure on $\mathcal{P}$. Let $\sigma_{\mathrm{semi}}(x)dx \in \mathcal{P}$ such that

$$\sigma_{\mathrm{semi}}(x) = \frac{1}{\pi}\sqrt{4-x^2}\mathbf{1}_{(-2,2)}(x). \tag{3.2}$$

The probability measure $\sigma_{\mathrm{semi}}(x)dx$ is called the semi-circle distribution. The celebrated Wigner semicircle law asserts that $\{\mu_\beta^N\}$ converges weakly to $\delta_{\sigma_{\mathrm{semi}}(x)dx}$:

$$\lim_{N\to\infty} \mu_\beta^N = \delta_{\sigma_{\mathrm{semi}}(x)dx} \quad \text{weakly.}$$

Because the limit distribution is non random, we can regard this as a law of large numbers in random matrix theory. Then what is the counterpart of the central limit theorem in random matrix theory? Furthermore, will it lead to invariance principles?

We call a point θ in the support of the semi-circle distribution a macro-position. We rescale (3.2) at $\theta \in [-2,2]$ to obtain meaningful limits. We can divide the support $[-2,2]$ into two parts $|\theta| < 2$ and $\theta = \pm 2$. The former is called the bulk and the latter is called the soft edge.

3.1. **Bulk limits and universality.** The scaling at a bulk position $\theta \in \{|\theta| < 2\}$ is called the bulk scaling. We now take this scaling:

$$x_i \mapsto \frac{s_i + \theta N}{\sqrt{N}}.$$

Then the distribution of $m_\beta^N(d\mathbf{s}_N)$ is

$$\widetilde{m}_\beta^N(d\mathbf{s}_N) = \frac{1}{Z}\{\prod_{i<j}^N |s_i - s_j|^\beta\} \exp\left\{-\frac{\beta}{4}\sum_{k=1}^N \Big|\frac{s_k + \theta N}{\sqrt{N}}\Big|^2\right\} d\mathbf{s}_N. \tag{3.3}$$

Let us denote by $\mu_{\beta,\theta}^N$ the corresponding distribution in the configuration space S. Then the limit of $\mu_{\beta,\theta}^N$ becomes the $\mathrm{sine}_{\beta,\theta}$ point process $\mu_{\beta,\theta}$:

$$\lim_{N\to\infty} \mu_{\beta,\theta}^N = \mu_{\beta,\theta} \quad \text{weakly.}$$

Here $\mu_{\beta,\theta}$ is the determinantal point process for which correlation functions with respect to the Lebesgue measure is given by the kernel function

$$K_\theta(x,y) = \frac{\sin\{\sqrt{4-\theta^2}(x-y)\}}{\pi(x-y)}.$$

The bulk scaling limit has a universality in the sense that the limits are always the sine_2 point process with different constant density. The case $\theta = 0$ has already appeared in (2.5). We next consider its dynamical counterpart.

We deduce from (3.3) the SDE describing a N-particle systems as follows: For each $i = 1, \ldots, N$, $\mathbf{X}^N = (X^{N,i})_{i=1}^N$ の is given by

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j \neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2N} X_t^{N,i} dt - \frac{\beta}{2} \theta dt \tag{3.4}$$

Here $\beta > 0$ is taken to be general. As $N \to \infty$, (3.4) becomes

$$dX_t^{\infty,i} = dB_t^i + \frac{\beta}{2} \sum_{j \neq i}^{\infty} \frac{1}{X_t^{\infty,i} - X_t^{\infty,j}} dt - \frac{\beta}{2} \theta dt.$$

This ISDE does not give a correct answer other than $\theta = 0$. Indeed, the limit ISDE is independent of θ and we *always* have

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \to \infty} \sum_{|X_t^i - X_t^j| < r,\ j \neq i} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{N}). \tag{2.4}$$

Recently, we proved an SDE gap phenomena:

Theorem 3.1 (Kawamoto-O. [22]). *Let $\beta = 2$. Let the initial distribution of the unlabeled dynamics be $\mu_{2,\theta}^N$ and take the label $\mathfrak{l}^N$ such that $\mu_{2,\theta}^N \circ (\mathfrak{l}^N)^{-1}$ converge weakly to $\mu_{2,\theta} \circ \mathfrak{l}^{-1}$. Then, for each $m \in \mathbb{N}$, we have the first m-particles of $\mathbf{X}^N = (X^{N,i})_{i=1}^N$ converge weakly in $C([0,\infty); \mathbb{R}^m)$ to $(X^i)_{i=1}^m$; that is,*

$$\lim_{N \to \infty} (X^{N,i})_{i=1}^m = (X^i)_{i=1}^m \qquad \textit{(weakly)}.$$

Here $(X^i)_{i=1}^m$ is the first m-components of the solution of (2.4).

We mentioned the "SDE gap" above because the form of the SDEs is different for finite-particle systems and the limit ISDE. We thus have a gap of SDEs. This phenomenon has dynamical universality corresponding to the above (geometric) universality in the sense that the limit stochastic dynamics is always described by the same ISDE independent of θ.

In addition, θ is included only in the initial condition. The solution of this ISDE is highly nonergodic, and it will stay in the strata (infinite-dimensional submanifold) evolving by itself as determined by θ [21]. However, the ISDE describing it is the same. This result can be interpreted as proving that the drift coefficient of the ISDE (2.4) is oriented in the direction tangential to the submanifold.

We expect that this result holds for general β-ensembles. We construct a general result regarding the convergence of SDE solutions of N-particle systems to that of ISDE [24]; as a corollary, Theorem 3.1 obtains. If the interaction potentials are of Ruelle's class, then we can apply [24] straightforwardly without any calculation. For logarithmic potentials, as we see in Theorem 3.1, we need a fine calculation. In one-dimensional systems with a logarithmic potential and $\beta = 2$, we can prove the same result by an algebraic method based on a calculation of space-time correlation functions [53, 51]. We therefore see that for such a class there exist two completely different methods for constructing stochastic dynamics.

3.2. **Soft edge limit and Airy interacting Brownian motions.**

At the positions $\theta = \pm 2$, the scaling is called the soft-edge scaling. We then consider the correspondence such that

$$x \longmapsto 2\sqrt{N} + \frac{s}{N^{1/6}}.$$

The distributions $m^N_{\mathrm{Ai},\beta}(d\mathbf{s})$ of the labeled N-particles are given by

$$m^N_{\mathrm{Ai},\beta}(d\mathbf{s}_N) = \frac{1}{Z}\{\prod_{i<j}^N |s_i - s_j|^\beta\} \exp\Big\{ -\frac{\beta}{4}\sum_{k=1}^N |2\sqrt{N} + \frac{s_k}{N^{1/6}}|^2 \Big\} d\mathbf{s}_N.$$

From this we deduce that the SDE describing the reversible N-particle systems $\mathbf{X}^N = (X_t^{N,1}, \dots, X_t^{N,N})$ is given by

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\sum_{j=1,\, j\neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2}\{N^{1/3} + \frac{1}{2N^{1/3}} X_t^{N,i}\} dt. \tag{3.5}$$

We now want to take N to infinity. The difficulty is that the coefficient in (3.5) contains the divergent term

$$-\frac{\beta}{2} N^{1/3} dt.$$

In [50], we solved the ISDE

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\Big\{\big(\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}\big) - \int_{|x|<r} \frac{\hat{\varrho}(x)}{-x} dx\Big\} dt, \tag{3.6}$$

and in [24] it was proved that the solutions of (3.5) converge weakly to that of (3.6) under suitable assumptions regarding the initial distributions.

In the following, we clarify the reason why (3.6) appears in the limit. We first consider the inverse transformation of the soft-edge scaling and rescale the limit semicircle distribution according to this:

$$\hat{\varrho}^N(x) = N^{1/3}\sigma_{\mathrm{semi}}(xN^{-2/3} + 2). \tag{3.7}$$

We regard $\hat{\varrho}^N$ in (3.7) as a first approximation of the one-point correlation function of the reduced Palm measure of N-particle systems conditioned at x. Then clearly

$$\int_{\mathbb{R}} \hat{\varrho}^N(x) dx = N.$$

A simple calculation shows that

$$\hat{\varrho}^N(x) = \frac{1_{(-4N^{2/3},0)}(x)}{\pi}\sqrt{-x\left(1 + \frac{x}{4N^{2/3}}\right)}, \tag{3.8}$$

$$\lim_{N\to\infty} \hat{\varrho}^N(x) = \hat{\varrho}(x) \quad \text{compact uniformly.} \tag{3.9}$$

The key point is the following identity.

$$N^{1/3} = \int_{\mathbb{R}} \frac{\hat{\varrho}^N(x)}{-x} dx. \tag{3.10}$$

The appearance $\hat{\varrho}(x)$ in (3.6) stems from (3.8) and (3.9). Indeed, as $N \to \infty$,

$$\begin{aligned} dX_t^i &\sim dB_t^i + \frac{\beta}{2}\Big\{\Big(\sum_{j\neq i,\ j=1}^N \frac{1}{X_t^i - X_t^j}\Big) - N^{1/3}\Big\} dt \\ &\sim dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\Big\{\big(\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}\big) - \int_{|x|<r} \frac{\hat{\varrho}^N(x)}{-x} dx\Big\} dt \quad \text{by (3.10)} \\ &\sim dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\Big\{\big(\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j}\big) - \int_{|x|<r} \frac{\hat{\varrho}(x)}{-x} dx\Big\} dt \quad \text{by (3.9).} \end{aligned}$$

We thus obtain ISDE (3.6). We expect that this procedure is common in the soft-edge scaling limits. We note that the solutions of the ISDE in the limit satisfy the non-collision property [37]; that is, $\mathbf{X} = (X_t^i)_{i\in\mathbb{N}}$ satisfies the following:

$$P(X_t^i \neq X_t^j \ 0 \le \forall t < \infty, i \neq j) = 1.$$

Hence, we label the particles as $X_t^i > X_t^j$ $(\forall\, i < j \in \mathbb{N})$. Then the stochastic dynamics $(X_t^i)_{i\in\mathbb{N}}$ is an $\mathbb{R}_>^{\mathbb{N}}$–valued process, where $\mathbb{R}_>^{\mathbb{N}} = \{(x_i) \in \mathbb{R}^{\mathbb{N}}; x_i > x_j \ (i < j)\}$. If $\beta = 2$, then the stochastic dynamics can be constructed by the algebraic method (see Section 4). This construction has been studied by Johannson, Spohn, Ferrari, Katori-Tanemura, and others. The right-most particle X^1 is called the Airy process and has been extensively studied. These two stochastic dynamics constructed by completely different methods coincide with each other [49, 50, 52, 53, 51, 22].

4. The algebraic construction: method of space-time correlation functions

When $d = 1$ and $\beta = 2$, we can construct the stochastic dynamics using explicit expressions for the space-time correlation functions in terms of extended kernel functions. In this section, we present these explicit expressions for examples given in Section 1 such as the sine, Airy, and Bessel point processes with $\beta = 2$.

We define multi-time moment generating functions of S-valued process X_t as follows:

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \mathbb{E}\left[\exp\left\{\sum_{m=1}^{M}\int_{\mathbb{R}} f_m d\mathsf{X}_{t_m}\right\}\right].$$

Let $\mathbb{K}(s,x;t,y)$ be an extended kernel [16, 19]. For the one-dimensional examples in Section 2, we can represent $\Psi^{\mathbf{t}}[\mathbf{f}]$ by using the Fredholm determinant of $\mathbb{K}$:

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \underset{\substack{(s,t)\in\{t_1,t_2,\dots,t_M\}^2,\\ (x,y)\in\mathbb{R}^2}}{\mathrm{Det}} \Big[\delta_{st}\delta(x-y) + \mathbb{K}(s,x;t,y)\chi_t(y)\Big].$$

Here $M \in \mathbb{N} = \{1, 2, \dots\}$, $\mathbf{f} = (f_1, f_2, \dots, f_M) \in C_0(\mathbb{R})^M$, $\mathbf{t} = (t_1, t_2, \dots, t_M)$ $(0 < t_1 < \cdots < t_M < \infty)$, and $\chi_{t_m} = e^{f_m} - 1$, $1 \le m \le M$.

(i)**Extended sine kernel:** $\mathbb{K}_{\sin}(s,x;t,y), s,t \in \mathbb{R}^+ = \{x \in \mathbb{R} : x \ge 0\}, x,y \in \mathbb{R}$:

$$\mathbb{K}_{\sin}(s,x;t,y) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^1 du\, e^{u^2(t-s)/2}\cos\{u(y-x)\} & \text{if } s<t, \\ K_{\sin}(x,y) & \text{if } s=t, \\ -\dfrac{1}{\pi}\displaystyle\int_1^\infty du\, e^{u^2(t-s)/2}\cos\{u(y-x)\} & \text{if } s>t. \end{cases}$$

(ii) **Extended Airy kernel:** $\mathbb{K}_{\mathrm{Ai}}(s,x;t,y), s,t \in \mathbb{R}^+, x,y \in \mathbb{R}$:

$$\mathbb{K}_{\mathrm{Ai}}(s,x;t,y) = \begin{cases} \displaystyle\int_0^\infty du\, e^{-u(t-s)/2}\mathrm{Ai}(u+x)\mathrm{Ai}(u+y) & \text{if } s<t, \\ K_{\mathrm{Ai}}(x,y) & \text{if } s=t, \\ -\displaystyle\int_{-\infty}^0 du\, e^{-u(t-s)/2}\mathrm{Ai}(u+x)\mathrm{Ai}(u+y) & \text{if } s>t. \end{cases}$$

(iii) **Extended Bessel kernel:** $\mathbb{K}_{J_\nu}(s,x;t,y), s,t \in \mathbb{R}^+, x,y \in \mathbb{R}^+$:

$$\mathbb{K}_{J_\nu}(s,x;t,y) = \begin{cases} \displaystyle\int_0^1 du\, e^{-2u(s-t)} J_\nu(2\sqrt{ux}) J_\nu(2\sqrt{uy}) & \text{if} \quad s<t, \\ K_{J_\nu}(x,y) & \text{if} \quad s=t, \\ \displaystyle -\int_1^\infty du\, e^{-2u(s-t)} J_\nu(2\sqrt{ux}) J_\nu(2\sqrt{uy}) & \text{if} \quad s>t. \end{cases}$$

It is known that one can construct S-valued stochastic dynamics through these kernels. For finite particle systems, there exists a representation given by these kernels, and the infinite unlabeled dynamics are their limits as $N \to \infty$. The Markov property of the limit dynamics thus constructed is not clear. Indeed, the underlying measures are singular each other if the numbers of particles are different, hence it is not clear such a property is inherited by the infinite volume stochastic dynamics. The Markov property was proved in [19] and the strong Markov property was proved in [53]. Combining these results with [49, 50, 52, 53, 51], we see that the stochastic dynamics constructed by the space-time correlation functions and by the stochastic analysis in [49] are the same.

Originally, the stochastic dynamics associated with the Airy kernel was constructed by Prähofer and Spohn and by Johansson as a limit of the finite systems of the space-time correlation functions [54, 15]. At the very beginning, even the continuity of trajectories of particles was an issue. In particular, proving the non-collision property of particles is difficult with this method. A semi-martingale property of each particle was proposed as an open problem in [15] and solved in Hägg [10] and Corwin-Hammond [4]. We also remark that in [50] the ISDE describing the Airy interacting Brownian motions was solved, and from this and [53], the semi-martingale property follows immediately.

Katori–Tanemura studied the algebraic method based on the space-time correlation functions to construct the infinite-volume dynamics of infinite particle systems. The extended kernels are the analogy of the "transition probability density" in infinite dimensions. They used this to represent the transition probability of the dynamics explicitly [18]. In this sense, this research can be regard as solvable models in probability theory [16, 17, 18, 19, 20].

5. Beginning of the general theory of interacting Brownian motions

A system of an infinite number of Brownian particles moving in $\mathbb{R}^d$ interacting via the potential $\Psi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is described by the following ISDE,

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j \neq i}^{\infty} \nabla \Psi(X_t^i - X_t^j) dt \quad (i \in \mathbb{N}), \tag{5.1}$$

which is a special case of (2.1) with $\Phi = 0$ and $\Psi(x,y) = \Psi(x-y)$. Here $\beta \geq 0$ is a constant called the inverse temperature, and $\{B^i\}_{i \in \mathbb{N}}$ is a system of infinite, independent d-dimensional standard Brownian motions.

The stochastic dynamics $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ are $(\mathbb{R}^d)^{\mathbb{N}}$-valued. Intuitively, the invariant measure $\check{\mu}$ of the solutions of (5.1) is given by

$$\check{\mu}(d\mathbf{x}) = \frac{1}{\mathcal{Z}} e^{-\beta \sum_{i<j \in \mathbb{N}} \Psi(x_i - x_j)} \prod_{k=1}^{\infty} dx_k. \tag{5.2}$$

Because this representation contains the infinite product of Lebesgue measures

$$d\mathbf{x}^{\infty} = \prod_{k=1}^{\infty} dx_k, \tag{5.3}$$

we cannot justify (5.2) as it is. The traditional method to solve this is to introduce the Gibbs measures based on the DLR equation [55].

We consider the configuration space S over $\mathbb{R}^d$; S is the space consisting of unlabeled particles. We denote by $\mu_{r,\xi}^n$ the regular conditional probability of a point process μ:

$$\mu_{r,\xi}^n(d\mathsf{x}) = \mu(\pi_{S_r}(\cdot) \in \cdot | \pi_{S_r^c}(\mathsf{x}) = \pi_{S_r^c}(\xi),\, \mathsf{x}(S_r) = n), \tag{5.4}$$

where $\pi_A(\mathsf{s}) = \mathsf{s}(\cdot \cap A)$, $S_r = \{s \in S;\, |s| < r\}$, $\xi \in \mathsf{S}$. By defitition $\mu_{r,\xi}^n(d\mathsf{x})$ is a conditional probability such that there exists n particles in S_r. Here we regard $\mu_{r,\xi}^n(d\mathsf{x})$ as a probability on S, and we often identify $\mu_{r,\xi}^n(d\mathsf{x})$ as a symmetric probability on S^n.

Let Λ be the Poisson point process for which the intensity is the Lebesgue measure. We set $\Lambda_r^n = \Lambda(\cdot \cap \mathsf{S}_r^n)$, where $\mathsf{S}_r^n = \{\mathsf{s} \in \mathsf{S}; \mathsf{s}(S_r) = n\}$. We call μ a (Φ, Ψ)-canonical Gibbs measure if μ satisfies the DLR equation. For each $n, r \in \mathbb{N}$ and μ-a.s. $\xi \in \mathsf{S}$, we set

$$\mu_{r,\xi}^n(d\mathsf{x}) = \frac{1}{\mathcal{Z}} e^{-\mathcal{H}_{r,\xi}} \Lambda_r^n(d\mathsf{x}), \tag{5.5}$$

where $\mathcal{H}_{r,\xi} = \mathcal{H}_r + \mathcal{I}_{r,\xi}$ and

$$\begin{aligned} \mathcal{H}_r(\mathsf{s}) &= \beta\{\sum_{i=1}^{n} \Phi(s_i) + \sum_{i<j,\, s_i, s_j \in S_r} \Psi(s_i, s_j)\}, \\ \mathcal{I}_{r,\xi} &= \beta \sum_{s_i \in S_r,\, \xi_k \in S_r^c} \Psi(s_i, \xi_k). \end{aligned} \tag{5.6}$$

Here $\mathcal{H}_r$ is a Hamiltonian in S_r and $\mathcal{I}_{r,\xi}$ denotes the interaction term between inside and outside particles. We thus consider (Φ, Ψ)-canonical Gibbs measures instead of (5.2) through DLR equation.

Except for the special case in $\mathbb{R}$ due to Lippner and Rost, ISDE (5.1) was first solved by Lang [25, 26] in a general frame work (see also [56]). Fritz [6] constructed non-equilibrium solutions in $\mathbb{R}^d$ for $d \leq 4$. Tanemura [62] solved the ISDE of hard-core Brownian balls. Usually, we use the Itô scheme to solve the SDE. Indeed, we use a kind of the Picard approximation as similarly used in ordinary differential equations. We then need, at least locally, Lipschitz continuity for the coefficients. The difficulty of performing this method in infinite dimensions is that the Lipschitz continuity for the coefficients cannot be expected at all in infinite dimensions, and that localization is also very complicated. Furthermore, the coefficients are defined on only a small region of the space. Lang studied the case whereby $\Phi = 0$ and that

$$\Psi \in C_0^3(\mathbb{R}^d).$$

He carried out the analysis by combining estimates of Gibbs measures. Even in the manageable category of Ruelle's class, for Ψ of polynomial decay, Lang's method

— using the traditional method of Itô — was impossible to prove. For the infinite-dimensional Dyson model (1.1), the interacting potential is the logarithmic potential:

$$\Psi(x,y) = -\beta \log |x-y|$$

Instead of an attenuation, this logarithmic potential diverges to infinity at the point of infinity. Normally, the DLR equation has meaning, and various means such as uniform evaluation of local density can be used, but they become impossible for a logarithmic potential. Conversely, the appearance of particles moving under the potential to generate strong interactions in such distant places should be vividly different from that of the ordinary Ruelle's class. Pursuing this aspect appears to be an interesting problem.

In general, an ISDE has an infinite number of different coefficients σ_i and b_i and is given in the form

$$\begin{aligned} dX_t^1 &= \sigma_1(\mathbf{X}_t)dB_t^1 + b_1(\mathbf{X}_t)dt \\ dX_t^2 &= \sigma_2(\mathbf{X}_t)dB_t^2 + b_2(\mathbf{X}_t)dt \\ dX_t^3 &= \sigma_3(\mathbf{X}_t)dB_t^3 + b_3(\mathbf{X}_t)dt \\ &\cdots . \end{aligned} \tag{5.7}$$

If (σ_i, b_i) converges fast enough to $(1,0)$ as $i \to \infty$, (5.7) can be solved similarly to an ordinary SDE (depending on the convergence speed). The problem is that, if (σ_i, b_i) has "symmetry", it does not converge to $(1,0)$ as $i \to \infty$. That is, the coefficients of the ISDE are given by a single function $(\sigma, b) : \mathbb{R}^d \times \mathsf{R}^d \to (\mathbb{R}^{d^2} \times \mathbb{R}^d) \cup \{\infty\}$ as follows:

$$\begin{aligned} dX_t^1 &= \sigma(X_t^1, \mathsf{X}_t^{1\diamondsuit})dB_t^1 + b(X_t^1, \mathsf{X}_t^{1\diamondsuit})dt \\ dX_t^2 &= \sigma(X_t^2, \mathsf{X}_t^{2\diamondsuit})dB_t^2 + b(X_t^2, \mathsf{X}_t^{2\diamondsuit})dt \\ dX_t^3 &= \sigma(X_t^3, \mathsf{X}_t^{3\diamondsuit})dB_t^3 + b(X_t^3, \mathsf{X}_t^{3\diamondsuit})dt \\ &\cdots , \end{aligned} \tag{5.8}$$

where $\mathsf{X}^{i\diamondsuit} = \sum_{j\neq i} \delta_{X^j}$. We emphasize again that the function $(\sigma, b)(x, \mathsf{s})$ is independent of particle label i. Symmetry became an obstacle to using conventional techniques. Conversely, by considering the entire system as an object to take values in the configuration space, we can use the notion of unlabeled stochastic dynamics

$$\mathsf{X} = \sum_{i=1}^{\infty} \delta_{X^i}.$$

Because X can have an invariant probability measure, we can use the geometric stochastic analytic method; that is, the Dirichlet form theory becomes effective.

6. Dirichlet form approach: In case of Brownian motions

The Dirichlet form $(\mathcal{E}, \mathcal{D})$ is a nonnegative closed form defined on domain $\mathcal{D}$ and has the Markov property. The combination with the underlying L^2 space is called the Dirichlet space [7].

For Brownian motions on $\mathbb{R}^d$, the associated Dirichlet form on $L^2(\mathbb{R}^d, dx)$ is given by

$$\mathcal{E}^{dx}(f,g) = \int_{\mathbb{R}^d} \mathbb{D}[f,g]dx, \quad \mathcal{D} = H^1(\mathbb{R}^d),$$

where $\mathbb{D}$ is the standard carré du champ operator on $\mathbb{R}^d$; that is,

$$\mathbb{D}[f,g] = \frac{1}{2}(\nabla f, \nabla g)_{\mathbb{R}^d}.$$

The reason why we call $\mathbb{D}$ standard is that when combined with the Lebesgue measure it defines the standard Brownian motion. The naturalness of the caré du champ $\mathbb{D}$ should present no objection. Normally, the Dirichlet form $(\mathcal{E}, \mathcal{D})$ is often defined as the closure of a closable form $(\mathcal{E}, \mathcal{D}_0)$. We can take $\mathcal{D}_0 = C_0^\infty(\mathbb{R}^d)$ for the standard Brownian motion on $\mathbb{R}^d$.

The important thing is that through $\mathbb{D}$ we have a correspondence such that

$$dx \Longleftrightarrow (\mathcal{E}^{dx}, C_0^\infty(\mathbb{R}^d), L^2(\mathbb{R}^d, dx)) \Longleftrightarrow (\mathcal{E}^{dx}, H^1(\mathbb{R}^d), L^2(\mathbb{R}^d, dx)) \Longleftrightarrow B. \tag{6.1}$$

Even if dx is replaced by a general Radon measure μ, this relation is still valid under a mild condition. We then have

$$\mu \Longleftrightarrow (\mathcal{E}^\mu, C_0^\infty(\mathbb{R}^d), L^2(\mathbb{R}^d, \mu)) \Longleftrightarrow (\mathcal{E}^\mu, \overline{C_0^\infty(\mathbb{R}^d)}^\mu, L^2(\mathbb{R}^d, \mu)) \Longleftrightarrow X \tag{6.2}$$

Here, the rightmost side is the μ-reversible diffusion process. A sufficient condition for the validity of this correspondence is that μ has an upper semi-continuous density to the Lebesgue measure. In this case, the corresponding diffusion process (or Dirichlet form) is called a distorted Brownian motion.

Taking (6.2) in mind, we introduce the map $F_{\mathbb{D}} = F_{\mathbb{D}}(\mu)$ from the space of Radon measures to the space of "positive bilinear forms", and then to the space of "diffusion processes"

$$\mu \xrightarrow{[0]} (\mathcal{E}^\mu, C_0^\infty(\mathbb{R}^d), L^2(\mathbb{R}^d, \mu)) \xrightarrow{[1]} (\mathcal{E}^\mu, \overline{C_0^\infty(\mathbb{R}^d)}^\mu, L^2(\mathbb{R}^d, \mu)) \xrightarrow{[2]} X. \tag{6.3}$$

One issue is to what extend this mapping really has meaning. We shall find a sufficient condition under which the space of diffusion processes is reachable. The correspondence of [0] is self-explanatory. From the general theory, [1] is reduced to the closability of $(\mathcal{E}^\mu, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \mu)$ and [2] is reduced to the regularity of the Dirichlet form $(\mathcal{E}^\mu, \overline{C_0^\infty(\mathbb{R}^d)}^\mu)$ on $L^2(\mathbb{R}^d, \mu)$. See [7] for the regularity of Dirichlet forms.

We remark that this strategy is not limited to $\mathbb{R}^d$, being still valid even in infinite dimensions as long as the space has a good carré du champ operator $\mathbb{D}$. Then the problem becomes "What is a good carré du champ operator?". It is $\mathbb{D}$ appearing from the best diffusion (Brownian motion) with the best measure (Lebesgue measure) that appears in the correspondence in (6.1). In other words, if there are two Brownian motions, a Lebesgue measure, and a $\mathbb{D}$, we can expect that one can be constructed from the other naturally.

Generally, in each regular symmetric Dirichlet space, it is proved that there is always a carré du champ operator. The point is that a "good" $\mathbb{D}$ is robust, and it is commonly chosen in the Dirichlet space accompanied with a very wide range of Radon measures μ, beyond being a carré du champ operator of one specific regular symmetric Dirichlet space. We expect that the carré du champ operator of the best diffusion process on each space has this property. In the general theory, we can take μ in the Dirichlet form and μ in L^2-space differently. Taking a common μ

affords an advantage in that we can use a geometric method. For example, in finite-dimensional spaces, we can estimate heat kernels using Nash's inequality through isoperimetric inequalities associated with the carré du champ operator.

In (6.1) and (6.2), we took $\overline{C_0^\infty(\mathbb{R}^d)}^\mu$. Generally, the choice of domain $\mathcal{D}$ is an important issue because the behavior of the particles changes markedly depending on the choice of the domain $\mathcal{D}$ in the Dirichlet form $(\mathcal{E}, \mathcal{D})$. For example, even if μ has no density for the Lebesgue measure, we can construct good diffusions by choosing a good new domain $\mathcal{D}$ other than $\overline{C_0^\infty(\mathbb{R}^d)}^\mu$. Indeed, a family of good diffusions on the Sierpinski carpets and other fractals were constructed by this method [36, 38, 39].

6.1. **A scheme for constructing infinite-dimensional Brownian motion: Dirichlet forms without underlying measures.**

The space to solve the ISDE is $(\mathbb{R}^d)^{\mathbb{N}}$. The ISDE includes $(\mathbb{R}^d)^{\mathbb{N}}$-Brownian motion $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$. Fortunately, in $(\mathbb{R}^d)^{\mathbb{N}}$, we have $(\mathbb{R}^d)^{\mathbb{N}}$-Brownian motion $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ and a natural caré du champ $\mathbb{D}^\infty[f,g]$. Indeed, the construction of $(\mathbb{R}^d)^{\mathbb{N}}$-valued Brownian motion $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ is easy because we simply prepare infinite copies of independent d-dimensional Brownian motions. Its generator $\mathbb{L}^\infty$ is given by

$$\mathbb{L}^\infty = \frac{1}{2}\sum_{i=1}^{\infty} \Delta_i,$$

where each Δ_i denotes the Laplacian on $\mathbb{R}^d$. The caré du champ $\mathbb{D}^\infty$ on $(\mathbb{R}^d)^{\mathbb{N}}$ is

$$\mathbb{D}^\infty[f,g] = \frac{1}{2}\sum_{i=1}^{\infty}(\nabla_i f, \nabla_i g)_{\mathbb{R}^d}.$$

Therefore, considering the Dirichlet form from the above correspondence in the Dirichlet form, we have intuitively

$$\mathcal{E}^{d\mathbf{x}^\infty}(f,g) = \int_{(\mathbb{R}^d)^\infty} \mathbb{D}^\infty[f,g] d\mathbf{x}^\infty, \quad L^2((\mathbb{R}^d)^\infty, d\mathbf{x}^\infty).$$

However, this cannot be justified because this contains the infinite product of the Lebesgue measures $d\mathbf{x}^\infty$. Hence, it breaks from the first stage of the correspondence of (6.3). We thus fall into a troubling situation whereby "there is no Dirichlet form despite the Brownian motion." Therefore, we have to consider Dirichlet forms on the space without measures. This leads us to introducing a sequence of Dirichlet forms on the spaces with measures.

6.2. **Approximate sequence of** $(\mathbb{R}^d)^{\mathbb{N}}$ **—a tiny infinite dimensions and a huge infinite dimensions—.**

We consider the configuration space S on $\mathbb{R}^d$ defined by (2.3) instead of $(\mathbb{R}^d)^{\mathbb{N}}$. The Poisson point process Λ for which the intensity is the Lebesgue measure is normally used as a substitute for the infinite product of the Lebesgue measure $d\mathbf{x}^\infty$. By definition, the Poisson point process with the Lebesgue intensity is the probability measure on S such that for $A, B \in \mathcal{B}(\mathbb{R}^d)$

(1) If $A \cap B = \emptyset$, then $\Lambda \circ \pi_A^{-1}$ and $\Lambda \circ \pi_B^{-1}$ are independent,
(2) $\Lambda(\mathsf{s}(A) = n) = e^{-\lambda(A)}\lambda(A)^n/n!$,

where $\pi_A : \mathsf{S} \to \mathsf{S}$ is $\pi_A(\mathsf{s}) = \mathsf{s}(\,\cdot\, \cap A)$. The function f on S can be represented uniquely by the symmetric function $\check{f}$ defined on a subset of $\bigcup_{i\in\{0,\mathbb{N},\infty\}}(\mathbb{R}^d)^i$ as follows.

$$f(\mathsf{s}) = \check{f}(s_1, s_2, \ldots) \quad (\mathsf{s} = \sum_i \delta_{s_i}).$$

See [32, 3] for more rigorous definition of $\check{f}$. We say a function f on S is local if f is $\sigma[\pi_{S_r}]$-measurable for some $r \in \mathbb{N}$, and smooth if $\check{f}$ is smooth. We denote by $\mathcal{D}_\circ$ the set consisting of all local and smooth functions on S.

We set a caré du champ $\mathbb{D}$ on S by

$$\mathbb{D}[f,g](\mathsf{s}) = \mathbb{D}^\infty[\check{f}, \check{g}](s_1, s_2, \ldots).$$

The right-hand side is symmetric in (s_i) and can be regarded as a function of $\mathsf{s} = \sum_i \delta_{s_i}$. Consider the Dirichlet form

$$\mathcal{E}^\Lambda(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g]d\Lambda, \quad L^2(\mathsf{S}, \Lambda).$$

Let $\mathcal{D}_\circ^\Lambda = \{f \in \mathcal{D}_\circ\,;\, \mathcal{E}^\Lambda(f,f) < \infty,\ f \in L^2(\mathsf{S},\Lambda)\}$. Then $(\mathcal{E}^\Lambda, \mathcal{D}_\circ^\Lambda)$ is closable on $L^2(\mathsf{S},\Lambda)$. Let $(\mathcal{E}^\Lambda, \mathcal{D}^\Lambda)$ be the closure of $(\mathcal{E}^\Lambda, \mathcal{D}_\circ^\Lambda)$. Then the S-valued Brownian motion

$$\mathsf{B} = \sum_{i\in\mathbb{N}} \delta_{B^i} \tag{6.4}$$

is associated with the Dirichlet form $(\mathcal{E}^\Lambda, \mathcal{D}^\Lambda)$ on $L^2(\mathsf{S},\Lambda)$.

We therefore see that the infinite-dimensional space S being very tiny compared with $(\mathbb{R}^d)^\mathbb{N}$, has *Lebesgue measure* Λ, the caré du champ $\mathbb{D}$, and *Brownian motion* B. This triplet satisfies the relation (6.1). In addition, various choices exist for B^i satisfying (6.4) at each time t. With a suitably choice, $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ becomes the $(\mathbb{R}^d)^\mathbb{N}$-valued Brownian motions. As shall be described later, if each B^i is continuous and does not collide with each other, then it is uniquely determined only by the initial label's arbitrariness.

There is a big difference between the two infinite-dimensional spaces S and $(\mathbb{R}^d)^\mathbb{N}$. Hence, we connect the tiny infinite dimension S with the huge infinite dimension $(\mathbb{R}^d)^\mathbb{N}$ by considering a sequence of infinite-dimensional spaces such that

$$\mathsf{S},\quad \mathbb{R}^d\times\mathsf{S},\quad (\mathbb{R}^d)^2\times\mathsf{S},\quad (\mathbb{R}^d)^3\times\mathsf{S},\quad (\mathbb{R}^d)^4\times\mathsf{S},\quad (\mathbb{R}^d)^5\times\mathsf{S},\quad \cdots$$

The sequence of *Lebesgue measures* on each spaces is given by

$$\Lambda,\quad dx\times\Lambda,\quad dx^2\times\Lambda,\quad dx^3\times\Lambda,\quad dx^4\times\Lambda,\quad dx^5\times\Lambda,\quad \cdots.$$

Furthermore, the sequence of Brownian motions is given by

$$\mathsf{B},\quad (\mathbf{B}^1, \mathsf{B}^1),\quad (\mathbf{B}^2, \mathsf{B}^2),\quad (\mathbf{B}^3, \mathsf{B}^3),\quad (\mathbf{B}^4, \mathsf{B}^4),\quad (\mathbf{B}^5, \mathsf{B}^5),\quad \cdots,$$

where $\mathbf{B}^n = (B^1, \ldots, B^n)$ and $\mathsf{B}^n = \sum_{i=n+1}^\infty \delta_{B^i}$. The Dirichlet form

$$(\mathcal{E}^{dx^n\times\Lambda}, L^2((\mathbb{R}^d)^n\times\mathsf{S}, dx^n\times\Lambda))$$

is associated with the *Brownian motion* $(\mathbf{B}^n, \mathsf{B}^n)$ at the nth element of these columns. Let $\Lambda^{[n]} = dx^n\times\Lambda$ be the Campbell measure of Λ. Put $\mathsf{S}^{[n]} = (\mathbb{R}^d)^n\times\mathsf{S}$. Define $\Xi^{[n]}$ by

$$\Xi^{[n]}(\Lambda) = (\mathcal{E}^{\Lambda^{[n]}}, L^2(\mathsf{S}^{[n]}, \Lambda^{[n]})). \tag{6.5}$$

The representation $\Xi^{[n]}(\Lambda)$ makes sense for general μ and thus we write $\Xi^{[n]}(\mu)$ in Theorem 7.3.

Suppose $d \geq 2$ and take an initial label $\mathfrak{l}$. Each particle then can move without changing its label. Hence we can construct a natural map $\mathfrak{l}_{\mathrm{path}}$ from the S-valued path space $C([0,\infty);\mathsf{S})$ to the $(\mathbb{R}^d)^{\mathbb{N}}$-valued path space $C([0,\infty);(\mathbb{R}^d)^{\mathbb{N}})$ through the label $\mathfrak{l}$. Then we have

$$\mathfrak{l}_{\mathrm{path}}(\mathsf{B}) = \mathbf{B}.$$

Importantly, this correspondence gives the couplings among Dirichlet forms. Indeed, we can represent the $(\mathbb{R}^d)^n \times \mathsf{S}$-valued diffusion process given associated with the Dirichlet space $\Xi^{[n]}$ by this mapping $\mathfrak{l}_{\mathrm{path}}$ and the diffusion process (unlabeled Brownian motion B) originally given by the Dirichlet space $\Xi^{[0]}$. In other words, each element $\Xi^{[n]}$ of the countably infinite number of Dirichlet forms defines the diffusion process. Originally, these diffusions are unrelated to each other. The map $\mathfrak{l}_{\mathrm{path}}$ gives the couplings among these $(\mathbb{R}^d)^n \times \mathsf{S}$-valued diffusions $(\mathbf{B}^n, \mathsf{B})$. All these diffusions can be represented as a functional of the Brownian motion B on the smallest space. We thus see that the smallest Dirichlet space plays a role giving the structure of the mutual relationship of the infinitely many Dirichlet spaces. Since we have certainly constructed the Dirichlet space $\Xi^{[n]}$

$$\mathbf{B}^n = (B^1, \dots, B^n)$$

for each $n \in \mathbb{N}$ and found the afore-mentioned relation, we can say that we have created a Dirichlet space on $(\mathbb{R}^d)^{\mathbb{N}}$ that handles $\mathbf{B} = (B^n)_{n\in\mathbb{N}}$. Although this viewpoint cannot cover all of the ISDEs on $(\mathbb{R}^d)^{\mathbb{N}}$ as in (5.7), it is sufficient to solve the target ISDE, that is, ISDE with symmetry given by (5.8).

7. Dirichlet form approach to interacting Brownian motions: (the first theory)

In this section, we apply the idea in the preceding section to non-trivial examples and state the main theorem. The purpose of this section is to develop a general theory to solve ISDEs of the form

$$dX_t^i = \sigma(X_t^i, \mathsf{X}_t^{i\diamond})dB_t^i + b(X_t^i, \mathsf{X}_t^{i\diamond})dt \tag{7.1}$$

$$\mathbf{X} \in \mathbf{W}^{\mathrm{sol}}. \tag{7.2}$$

We solve the ISDE on the interval $[0,T]$ for each $T \in \mathbb{N}$ and set $W((\mathbb{R}^d)^{\mathbb{N}}) = C([0,T];(\mathbb{R}^d)^{\mathbb{N}})$. $\mathbf{W}^{\mathrm{sol}}$ is a symmetric subset of $W((\mathbb{R}^d)^{\mathbb{N}})$ that includes the solutions of ISDE. We regard the coefficients of the ISDE as functions defined on a subset of $W((\mathbb{R}^d)^{\mathbb{N}})$ naturally, and we then suppose $\mathbf{W}^{\mathrm{sol}}$ is contained by the subset.

This class of ISDE includes (2.1), (5.1), (5.8), and all concrete examples Section 2. The key notions are **quasi-Gibbs property** and **logarithmic derivative** of point process μ. The former plays an important role in the construction of the unlabeled diffusion and the latter for the representation as a solution of ISDE (7.1).

7.1. Construction of unlabeled diffusions.

Taking the correspondence (6.3) into account, we introduce the general method by which to generate the unlabeled diffusion from a given point process μ. We begin with the notion of **quasi-Gibbs measures**.

Definition 7.1. Let Φ and Ψ be the free and interaction potentials, respectively. A point process μ is called a (Φ, Ψ)-quasi Gibbs measure, if the regular conditional probability $\mu^n_{r,\xi}$ of μ defined by (5.4) satisfies the inequality such that

$$C(r,\xi,n)^{-1}e^{-\mathcal{H}_r(\mathsf{s})}d\Lambda^n_r \le \mu^n_{r,\xi}(d\mathsf{s}) \le C(r,\xi,n)e^{-\mathcal{H}_r(\mathsf{s})}d\Lambda^n_r$$

for μ-a.s. ξ and all $r, n \in \mathbb{N}$ with a positive constant $C = C(r,\xi,n)$ depending on (r,ξ,n). Here $\mu \le \nu$ means that two measures μ and ν satisfy $\mu(A) \le \nu(A)$ for all A, and $\mathcal{H}_r(\mathsf{s})$ denotes the Hamiltonian given by (5.6) defined only inside S_r.

Remark 7.1. (1) Clearly, Gibbs measures are quasi-Gibbs measures.
(2) Note that $C(r,\xi,n)$ depends on ξ. Then this notion is robust for perturbation of free potentials. If μ is a (Φ,Ψ)-quasi Gibbs measure with locally bounded Φ_0, then μ is also a $(\Phi+\Phi_0,\Psi)$-quasi Gibbs measure. Point processes with logarithmic interaction potentials in typical examples are all $(0, -\beta\log|x-y|)$-quasi Gibbs measures.

We set $a(x,\mathsf{s}) = \sigma(x,\mathsf{s})\,{}^t\sigma(x,\mathsf{s})$, where σ the coefficient in (7.1). We assume:
(**A1**) a is uniformly elliptic and bounded. μ is a (Φ,Ψ)-quasi Gibbs measure. There exists an upper semi-continuous potential (Φ_0,Ψ_0) and two positive constants c_1 and c_2 such that

$$c_1(\Phi_0,\Psi_0) \le (\Phi,\Psi) \le c_2(\Phi_0,\Psi_0).$$

(**A2**) There exists a constant $p > 1$ such that each n-point correlation function ρ^n is L^p locally bounded.

For μ and a satisfying these assumptions we set the Dirichlet form as follows.

$$\mathcal{E}^{a,\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}^a[f,g]d\mu, \quad \mathbb{D}^a[f,g] = \frac{1}{2}\sum_i a(s_i,\mathsf{s}^{i\diamondsuit})\frac{\partial \check{f}}{\partial s_i}\cdot\frac{\partial \check{g}}{\partial s_i}.$$

Here we set $\mathsf{s}^{i\diamondsuit} = \sum_{j\ne i}\delta_{s_j}$ for $\mathsf{s} = \sum_i \delta_{s_i}$. Let $\mathcal{D}_\circ$ be the function space consisting of local and smooth functions S as before, and put

$$\mathcal{D}^{a,\mu}_\circ = \{f \in \mathcal{D}_\circ \cap L^2(\mathsf{S},\mu); \mathcal{E}^{a,\mu}(f,f) < \infty\}.$$

Theorem 7.1 ([32, 33, 42])**.** *Assume* (**A1**) *and* (**A2**). *Then* $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}_\circ)$ *is closable on* $L^2(\mathsf{S},\mu)$. *The diffusion process* $(\mathsf{X}, \{\mathsf{P}_\mathsf{s}\}_{\mathsf{s}\in\mathsf{S}})$ *associated with the closure* $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$ *exists.*

A sufficient concrete condition for a point process μ to be a quasi-Gibbs measure was given by [42, 43]. We can prove all point processes but the Gaussian analytic functions (GAFs) in Section 9.4 in the present paper are quasi-Gibbs measures.

Assumption (**A1**) is used for the closability of the Dirichlet forms. (**A2**) is used for the proof of the quasi-regularity of the Dirichlet forms (see [28] for the definition of quasi-regularity). We thus obtain the existence of a L^2-Markovian semi-group from (**A1**), and the associated Markov processes from (**A2**). These two assumptions are obvious for Gibbs measures and sufficiently feasible for point processes arising from random matrices.

7.2. **Labeled dynamics.** In this section, we construct the labeled dynamics. Let

$$\mathsf{S}_{\mathrm{s,i}} = \{\mathsf{s}; \mathsf{s}(\{x\}) = 0 \text{ or } 1 \ \text{ for } \forall x \in \mathbb{R}^d,\ \mathsf{s}(\mathbb{R}^d) = \infty\}$$

and assume the following.

(**A3**) Particles $\{X^i\}$ do not collide with each other and the cardinality of particles are infinite:

$$\mathsf{P}_\mu(\mathsf{X}_t \in \mathsf{S}_{\mathrm{s,i}} \text{ for } \forall t \in [0,\infty)) = 1.$$

(**A4**) Each tagged particle does not explode:

$$\mathsf{P}_\mu(\bigcap_{i=1}^{\infty}\{\sup_{0\le u\le t} |X_u^i| < \infty \text{ for } \forall t \in [0,\infty)\}) = 1.$$

Although these are conditions for each tagged particle of the labeled dynamics $\mathbf{X}$, we can verify them by investigating the unlabeled dynamics X. Indeed, denoting by Cap the capacity of $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}, L^2(\mu))$, then (**A3**) follows from

$$\mathrm{Cap}(\mathsf{S}_{\mathrm{s,i}}^c) = 0. \tag{7.3}$$

A (Φ,Ψ)-quasi-Gibbs with locally bounded potentials in $\mathbb{R}^d$ with $d \ge 2$ always satisfies (7.3). In one dimension, we have a simple sufficient condition on Ψ [14]. If μ is a determinantal point process, then (**A3**) is deduced from the Hölder continuity of the diagonal part of the determinantal kernel [37].

As for the non-explosion condition, we obtain (**A4**) if the growth order of the one-point function ρ^1 of μ at infinity is such that for some constant $\alpha < 2$

$$\rho^1(x) = O(e^{|x|^\alpha}). \tag{7.4}$$

Theorem 7.2. *Assume* (**A1**)–(**A4**). *Then we can construct the labeled dynamics* $\mathbf{X} = \mathfrak{l}_{\mathrm{path}}(\mathsf{X})$ *from the unlabeled dynamics* X *in Theorem 7.1. The labeled dynamics* $\mathbf{X}$ *is an* $(\mathbb{R}^d)^{\mathbb{N}}$*-valued diffusion process.*

The m-Campbell measure $\mu^{[m]}$ of μ is by definition

$$\mu^{[m]}(d\mathbf{x}d\mathsf{s}) = \rho^m(\mathbf{x})d\mathbf{x}\mu_{\mathbf{x}}(d\mathsf{s}),$$

where $\mathbf{x} = (x_1,\dots,x_n)$, ρ^m is the m-point correlation function, and moreover $\mu_{\mathbf{x}}$ is the reduced Palm measure of μ conditioned at $\mathbf{x}$. $\mu_{\mathbf{x}}$ is informally given by

$$\mu_{\mathbf{x}}(d\mathsf{s}) = \mu(d\mathsf{s} - \mathsf{x}|\, \mathsf{s}(x_i) \ge 1 \text{ for all } i).$$

Here we write $\mathsf{x} = \sum_{i=1}^n \delta_{x_i}$. An analogy of Theorem 7.1 on the m-labeled stochastic dynamics holds for each m-Campbell measure $\mu^{[m]}$. We then denote by $\Xi^{[m]}(\mu)$ the Dirichlet space on $(\mathbb{R}^d)^m\times\mathsf{S}$ corresponding to $\mu^{[m]}$. $\Xi^{[m]}(\mu)$ is defined by (6.5) with the replacement of $\Lambda^{[m]}$ by $\mu^{[m]}$. Consistency investigated in Section 6, which is an existence of coupling in the present situation, holds for the unlabeled dynamics obtained in Theorem 7.1 through Theorem 7.2.

Theorem 7.3 ([40])**.** *Assume* (**A1**)–(**A4**). *Then, for* $\Xi^{[m]}(\mu)$*, we have the same coupling as in the case* Λ *in Section 6. That is, denoting by* $\mathbf{X}^{[m]}$ *the stochastic process associated with* $\Xi^{[m]}(\mu)$*, then we have*

$$\mathbf{X}^{[m]} = (X^1,\dots,X^m, \sum_{i=m+1}^{\infty} \delta_{X^i}) \quad \textit{in distribution,}$$

where each X^i *in the right hand side is a component of the labeled dynamics* $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ *given by Theorem 7.2.*

7.3. **Infinite-dimensional stochastic differential equations.** We next solve the ISDE (7.1). For this purpose we introduce the notion of the logarithmic derivative d^{μ} of μ.

Definition 7.2. We say d^{μ} is the logarithmic derivative of μ if for each $f \in C_0(\mathbb{R}^d) \otimes \mathcal{D}_{\circ}$ we have

$$\int_{\mathbb{R}^d \times \mathsf{S}} \mathsf{d}^{\mu} f d\mu^{[1]} = -\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^{[1]}$$

We represent the logarithmic derivative by $\mathsf{d}^{\mu}(x,\mathsf{s}) = \nabla_x \log \mu^{[1]}(x,\mathsf{s})$.

In [41], we prepare a general theory to calculate the logarithmic derivative of μ. Using this, we can calculate the logarithmic derivative of all the examples in this article except for GAFs (see Section 9.4). As for the point processes with interaction potentials of Ruelle's class, the existence of logarithmic derivative is obvious. We assume:

(**A5**) μ has the logarithmic derivative d^{μ}.

(**A6**) μ satisfies the following **differential** equation.

$$2b(x,\mathsf{s}) = \nabla_x a(x,\mathsf{s}) + a(x,\mathsf{s}) \nabla_x \log \mu^{[1]}(x,\mathsf{s}). \tag{7.5}$$

Theorem 7.4 ([41])**.** *Assume* (**A1**)–(**A6**)*. Then, for a given label* $\mathfrak{l}$*, ISDE* (7.1) *has a solution* $(\mathbf{X},\mathbf{B})$ *for* $\mu^{\mathfrak{l}}$*-a.s. starting point* $\mathbf{s}$*. The labeled dynamics* $\mathbf{X}$ *is a* $(\mathbb{R}^d)^{\mathbb{N}}$*-valued diffusion, and the associated unlabeled dynamics is a* μ*-reversible diffusion.*

We can apply Theorem 7.4 to all the examples in this article.

The key point to solving the ISDE (7.1) is Theorem 7.3. Indeed, letting $\mathbf{x} = (x_i)_{i=1}^{\infty} \in (\mathbb{R}^d)^{\mathbb{N}}$, we see that each coordinate function x_i $(i \le m)$ is in the domain of the Dirichlet form $\Xi^{[m]}$ locally. Then, applying the Itô formula (the Fukushima decomposition and the Revue correspondence) in this instance, the resulting process satisfies SDE (7.1). Using the coupling obtained in Theorem 7.3, we can solve (7.1) not only for $i \le m$ but also for all $i \in \mathbb{N}$.

8. Analysis of tail σ-fields (the second theory): Existence of strong solutions and pathwise uniqueness

The results up to this point have found that ISDE (7.1) can be solved by the analysis of a geometric differential equation (7.5) for the point process μ. This solution is however a weak solution in the sense that it is a pair $(\mathbf{X},\mathbf{B})$ comprising a $S^{\mathbb{N}}$-valued process $\mathbf{X}$ and Brownian motion $\mathbf{B}$. Roughly speaking, if $\mathbf{X}$ is a functional of the Brownian motion $\mathbf{B}$, then $\mathbf{X}$ is called a strong solution. The construction in the previous section then does not provide the strong solution as it is. Furthermore, although the solution associated with the given Dirichlet form is certainly unique, we have not yet proved the uniqueness of Dirichlet forms associated with ISDE (7.1). We have thus seen that the uniqueness of the solution of ISDE (7.1) has not yet been proved in the first theory.

The difficulty of the problem is that to prove the existence of strong solutions and pathwise uniqueness, we need more or less a classical approach as for the Itô scheme. Hence, we have to find *the Lipschitz continuity of the coefficients of the ISDE* to use the Picard approximation. Although we may localize the ISDE in such a way that the coefficients are locally Lipschitz continuous outside a rough subset, localization should become very complicated. There is no way to carry out this directly in the current situation. Also, it seems difficult to arrive at the existence of strong solutions and pathwise uniqueness just by controlling the rough subset of the domain of the coefficients from the theory of Dirichlet forms.

Here we use again the sequence $\{(\mathbb{R}^d)^m \times \mathsf{S}\}_{m\in\{0\}\cup\mathbb{N}}$ from the tiny infinite-dimensional space S to the huge infinite-dimensional space $(\mathbb{R}^d)^{\mathbb{N}}$. We regard $(\mathbb{R}^d)^{\mathbb{N}}$ as the sequence of finite-dimensional spaces $(\mathbb{R}^d)^m$, where the existence of the strong solution and the pathwise uniqueness are established. Here, we consider a sequence of time-inhomogeneous, FSDEs on $(\mathbb{R}^d)^m$ for each m. Then we introduce the coupling among these FSDEs (8.1) on $(\mathbb{R}^d)^m$.

The point is the interpretation such that a single ISDE with symmetry is **an infinite system of FSDEs with consistency (IFC)**. With this interpretation, we then reduce the problem to the analysis of various tail σ-fields related to the scheme. To do this, we use the solution $(\mathbf{X}, \mathbf{B})$ obtained by the first theory. We think that the essential structure of this method is very general, and we expect that this method can be applied to many kinds of ISDE of infinite-particle type.

With these ideas, Tanemura and the author have developed the study of an ISDE with symmetry and have proved the existence of strong solutions and pathwise uniqueness of solutions of the ISDE in a sequence of papers [49, 50, 52, 53, 51], which we now explain.

8.1. Existence of strong solutions and pathwise uniqueness of solutions of ISDE.

In [49], Tanemura and the author proved the existence of strong solutions and the pathwise uniqueness of solutions of ISDE (7.1) under almost the same assumption as Theorem 7.4 with an additional assumption. This result can be applied to all examples except GAFs to be described later.

The idea is introducing the concept called an IFC solution for the ISDE. Here, as described in the previous section, an IFC is an infinite system of FSDEs with consistency. This concept is specific to the infinite particle system. It can be proved that it is equivalent to the conventional solution, and it is more appropriate for analyzing ISDE in infinite-dimensional spaces. We emphasize that the essential observation is the equivalence between ISDE and IFC.

The scheme solving IFC is as follows: At each step of finite-dimension, we use the conventional method. Then we lift the sequence of solutions to infinite dimensions using the consistency of solutions of FSDEs. We have already obtained the weak solution of ISDE by the first theory. From this, we prove the consistency of FSDEs. The point of the second theory is the usage of weak solutions and analysis of tail σ-field to complete the scheme.

Suppose that we have a weak solution $(\mathbf{X}, \mathbf{B})$ for ISDE (7.1). Then we introduce the sequence consisting of the $(\mathbb{R}^d)^m$-valued SDE. For each $m \in \mathbb{N}$ we consider the SDE for $\mathbf{Y}^m = (Y^{m,i})_{i=1}^m$ such that

$$
\begin{aligned}
dY_t^{m,i} &= \sigma(Y_t^{m,i}, \mathsf{Y}_t^{m,i\diamondsuit} + \mathsf{X}_t^{m*})dB_t^i + b(Y_t^{m,i}, \mathsf{Y}_t^{m,i\diamondsuit} + \mathsf{X}_t^{m*})dt \\
\mathbf{Y}_0 &= \mathbf{s}^m.
\end{aligned}
\tag{8.1}
$$

Here we set $\mathbf{s}^m = (s_1, \dots, s_m)$ for $\mathbf{s} = (s_i)_{i\in\mathbb{N}}$, and we put

$$\mathsf{Y}^{m,i\diamondsuit} = \sum_{j\neq i}^{m} \delta_{Y^{m,j}}, \quad \mathsf{X}^{m*} = \sum_{k=m+1}^{\infty} \delta_{X^k}, \quad \mathbf{X}^{m*} = (X^k)_{k=m+1}^{\infty}.$$

For each $\mathbf{X}$ the m-dimensional SDE (8.1) is time-inhomogeneous. Because (8.1) is finite-dimensional, if $\mathbf{X}$ is well behaved, then (8.1) has a pathwise-unique, strong solution for each $m \in \mathbb{N}$. This solution is a functional of $(\mathbf{B}^m, \mathbf{X}^{m*})$ and initial staring point $\mathbf{s}^m$. We then denote it by

$$\mathbf{Y}^m = \mathbf{Y}^m(\mathbf{s}^m, \mathbf{B}^m, \mathbf{X}^{m*}) = \mathbf{Y}^m(\mathbf{s}, \mathbf{B}, \mathbf{X}^{m*}).$$

$\mathbf{Y}^m$ is $\sigma[\mathbf{s}, \mathbf{B}, \mathbf{X}^{m*}]$-measurable by construction. From the fact that $(\mathbf{X}, \mathbf{B})$ is a solution of FSDE (8.1) and solution of (8.1) is pathwise-unique (recalling the existence of strong solution and the pathwise uniqueness together imply the uniqueness in the sense of distribution and the coincidence of weak solution and strong solution), we see that for each $m \in \mathbb{N}$

$$\mathbf{X}^m = \mathbf{Y}^m. \tag{8.2}$$

We thus see that the limit $\lim_{m\to\infty} \mathbf{Y}^m$ obviously exists. That is, the following relation holds.

$$\mathbf{X} = \lim_{m\to\infty} \mathbf{Y}^m(\mathbf{s}, \mathbf{B}, \mathbf{X}^{m*}). \tag{8.3}$$

From (8.2) we see that the solution $\mathbf{X}$ is a "fixed point". We consider the more general case in [49] not necessarily satisfying (8.2).

We define the tail σ-field $\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ of the labeled path space with respect to the label by

$$\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}}) = \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}].$$

Furthermore, we set

$$\widetilde{\mathcal{T}\mathcal{T}}_{\mathrm{path}} = \mathcal{B}((\mathbb{R}^d)^{\mathbb{N}}) \times \mathcal{B}(W((\mathbb{R}^d)^{\mathbb{N}})) \times \mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}}).$$

Then from (8.3) we see that $\mathbf{X}$ is $\widetilde{\mathcal{T}\mathcal{T}}_{\mathrm{path}}$-measurable.

We denote by P the distribution of the solution $(\mathbf{X}, \mathbf{B})$, and define the regular conditional probability conditioned at the initial starting point $\mathbf{s}$ and Brownian motion $\mathbf{B}$ by

$$P_{\mathbf{s},\mathbf{B}} = P(\,\cdot\,|(\mathbf{s}, \mathbf{B})).$$

Let P_{Br}^{∞} be the distribution of the Brownian motion $\mathbf{B}$ and put

$$\Upsilon = (\mu \circ \mathfrak{l}^{-1}) \times P_{\mathrm{Br}}^{\infty}.$$

We assume
(**P1**) ISDE (7.1) has a solution $(\mathbf{X}, \mathbf{B})$ for $\mu \circ \mathfrak{l}^{-1}$-a.s. $\mathbf{s}$.
(**P2**) (8.1) has a pathwise-unique, strong solution for each $m \in \mathbb{N}$.
(**P3**) $P_{\mathbf{s},\mathbf{B}}|_{\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})}$ is trivial and unique for Υ-a.s. $(\mathbf{s}, \mathbf{B})$.

The uniqueness in (**P3**) means that the distribution $P_{\mathbf{s},\mathbf{B}}|_{\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})}$ is independent of the particular choice of solutions $(\mathbf{X}, \mathbf{B})$. Because $P_{\mathbf{s},\mathbf{B}}|_{\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})}$ is trivial, this is equivalent to the independence of the set of $P_{\mathbf{s},\mathbf{B}}|_{\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})}$-measure 1.

Theorem 8.1 ([49])**.** *Assume* (**P1**)–(**P3**)*. Then ISDE* (7.1) *has a pathwise-unique, strong solution for $\mu \circ \mathfrak{l}^{-1}$-a.s.* $\mathbf{s}$*. In particular, an arbitrary solution coincides with the strong solution.*

Condition (**P1**) follows from the first theory. For other conditions, we explain in the following subsections.

8.2. **Sufficient conditions for** (**P2**)**.**

(**P2**) also has a sufficient condition to be established in a wide range [49]. We can apply this to all examples in Section 2.

8.3. **Sufficient conditions for** (**P3**)**.**

Generally, it is not easy to check (**P3**), but it can be derived again from the geometric information of the configuration space. Indeed, the results described below were obtained in this manner.

We denote by $\mathcal{T}(\mathsf{S})$ the tail σ-field of the configuration space S of $\mathbb{R}^d$.

$$\mathcal{T}(\mathsf{S}) = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}],$$

where $S_r^c = \{s \in \mathbb{R}^d;\, |s| \geq r\}$ and $\pi_A : \mathsf{S} \to \mathsf{S}$ is the projection $\pi_A(\mathsf{s}) = \mathsf{s}(\cdot \cap A)$ for a set $A \subset \mathbb{R}^d$ as before. We make assumptions:
(**Q1**) The tail σ-field $\mathcal{T}(\mathsf{S})$ is μ-trivial, that is, $\mu(A) \in \{0, 1\}$ for all $A \in \mathcal{T}(\mathsf{S})$.
(**Q2**) $P_\mu \circ \mathsf{X}_t^{-1} \prec \mu$ for all t, (absolute continuity condition).
(**Q3**) $P_\mu(\cap_{r=1}^{\infty} \{\mathsf{m}_r(\mathsf{X}) < \infty\}) = 1$, (no big jump condition),
where $\mathsf{m}_r = \inf\{m \in \mathbb{N}; X^i \in C([0,T]; S_r^c) \text{ for } m < \forall i \in \mathbb{N}\}$ for $\mathsf{X} = \sum_{i \in \mathbb{N}} \delta_{X^i}$.

Theorem 8.2 ([49])**.** *Assume* (**Q1**)–(**Q3**)*. Then* (**P3**) *holds.*

Theorem 8.2 asserts that tail triviality of the labeled path space with respect to the distribution of the solution of ISDE follows from triviality of the tail σ-field of the configuration space with respect to μ.

Remark 8.1. (i) Determinantal point processes satisfy (**Q1**) [45]. In particular, Sine_2, Airy_2, Bessel_2, and Ginibre point processes are all tail trivial, and thus satisfy (**Q1**).
(ii) Because the unlabeled dynamics X is μ-reversible, (**Q2**) is obvious.
(iii) (**Q3**) follows from (7.4).
(iv) Let $\mu^{\mathsf{a}}(\cdot) = \mu(\cdot|\mathcal{T}(\mathsf{S}))(\mathsf{a})$ be the regular conditional probability of μ. Suppose that μ is a quasi-Gibbs measure. Then we can take a version of μ^{a} in such a way that each μ^{a} satisfies (**Q1**). Moreover, μ^{a} fulfills the assumption of Theorem 8.1 [49]. We thus see that we can apply Theorem 8.1 to (7.1) even if μ is not tail trivial. We remark that we say nothing about the moving tail events of the stochastic dynamics X. Recently, Kawamoto [21] proved that the Dyson Brownian motion with $\beta = 2$ does not change the tail events in a time evolutionary manner.

We next expound on the idea behind the proof of Theorem 8.2. We begin with some preparatory notation. Let

$$\mathbf{T} = \{\mathbf{t} = (t_1, \ldots, t_m)\,;\, t_i \in [0,T], m \in \mathbb{N}\}, \quad \mathbf{X}_{\mathbf{t}}^{n*} = (\mathbf{X}_{t_1}^{n*}, \ldots, \mathbf{X}_{t_m}^{n*})$$

$$\tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S}) = \bigvee_{\mathbf{t} \in \mathbf{T}} \bigcap_{r=1}^{\infty} \sigma[\pi_r^c(\mathsf{X}_{\mathbf{t}})], \quad \tilde{\mathcal{T}}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}}) = \bigvee_{\mathbf{t} \in \mathbf{T}} \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_{\mathbf{t}}^{n*}].$$

By definition, $\tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S})$ is the cylindrical tail σ-field of the unlabeled path space, and $\tilde{\mathcal{T}}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ is the cylindrical tail σ-field of the labeled path space $W((\mathbb{R}^d)^{\mathbb{N}})$. The meaning of tails is different in these tail σ-fields. We shall deduce tail triviality of $\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ from that of $\tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S})$. Here is a scheme of it.

$$\begin{array}{ccccccc}
\mathcal{T}(\mathsf{S}) & \xrightarrow[(\mathbf{Q1}),(\mathbf{Q2})]{\text{(Step I)}} & \tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S}) & \xrightarrow[(\mathbf{Q3})]{\text{(Step II)}} & \tilde{\mathcal{T}}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}}) & \xrightarrow[\text{(IFC)}]{\text{(Step III)}} & \mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}}) \\
\mu & & \mathsf{P}_{\mu} & & \mathbf{P}_{\mu^{\mathfrak{l}}} = \int \bar{P}_{\mathbf{s}}(\mathbf{X}\in\cdot)d\mu^{\mathfrak{l}} & & \bar{P}_{\mathbf{s},\mathbf{B}} \text{ a.s.} (\mathbf{s},\mathbf{B}).
\end{array}$$

This diagram shows the conditions and concepts used at each stage. The upper row consists of the tail σ-fields. The lower row consists of the probability measures for which we prove tail triviality. We shall therefore move from the smallest infinite dimensions to the biggest infinite dimensions one by one.

The two theorems above can be applied to examples in the present article and, as a result, the pathwise-unique, strong solution can be obtained. In these theorems, the existence of potentials in the sense of the DLR equation is not presumed. The quasi-Gibbs property of the point process μ and the existence of a logarithmic derivative are sufficient for these theorems.

We have many applications of the pathwise uniqueness of solutions of ISDE. Specifically, the uniqueness of Dirichlet forms [52, 51], SDE gaps [22], finite-particle approximation of ISDE [24, 53, 51], coincidence of algebraic and analytic constructions of stochastic dynamics [49, 50, 52, 53, 51], and the uniqueness of the martingale problem.

8.4. **IFC solution.** The point of the previous discussion is consistency in (8.2). Generally, we do not need exact consistency but only asymptotic consistency suffices. We have hence arrived at the notion of an IFC solution. This notion has various levels and corresponds to the classical “weak solution”, “strong solution”, and “pathwise uniqueness”. Below we shall explain the IFC solutions and the correspondences among them. We assume (**P2**) in this subsection.

Let $\mathbf{W_0} = \{\mathbf{X} \in W((\mathbb{R}^d)^{\mathbb{N}})\,;\, \mathbf{X}_0 = \mathbf{0}\}$, and $\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}} = \{\mathbf{X} \in \mathbf{W}^{\mathrm{sol}}; \mathbf{X}_0 = \mathbf{s}\}$. We define the map $F_{\mathbf{s}}^m : \mathbf{W}_{\mathbf{s}}^{\mathrm{sol}} \times \mathbf{W_0} \to \mathbf{W}_{\mathbf{s}}^{\mathrm{sol}}$ by

$$F_{\mathbf{s}}^m(\mathbf{X},\mathbf{B}) = \{(Y_t^{m,1},\dots,Y_t^{m,m},X_t^{m+1},X_t^{m+2},\dots)\}_{0\le t\le T}.$$

Here $\mathbf{Y}^m = (Y^{m,i})_{i=1}^m$ is the unique solution (8.1) given by (**P2**).

Fix $(\mathbf{s},\mathbf{B})$. Then $F_{\mathbf{s}}^m(\,\cdot\,,\mathbf{B})$ defines the map from $\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}}$ to $\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}}$. Suppose that a probability measure $\bar{P}_{\mathbf{s}}$ on $\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}} \times \mathbf{W_0}$ is given. We say

$$F_{\mathbf{s}}^\infty(\mathbf{X},\mathbf{B}) = \lim_{m\to\infty} F_{\mathbf{s}}^m(\mathbf{X},\mathbf{B}) \text{ in } \mathbf{W}^{\mathrm{sol}} \text{ under } \bar{P}_{\mathbf{s}} \tag{8.4}$$

holds if the following are satisfied. $F_{\mathbf{s}}^\infty(\mathbf{X},\mathbf{B}) \in \mathbf{W}^{\mathrm{sol}}$ and for each $i \in \mathbb{N}$ there exist limits (8.5)–(8.7) in $W(\mathbb{R}^d) := C([0,T];\mathbb{R}^d)$ for $\bar{P}_{\mathbf{s}}$-a.s. $(\mathbf{X},\mathbf{B})$

$$\lim_{m\to\infty} F_{\mathbf{s}}^{m,i}(\mathbf{X},\mathbf{B}) = F_{\mathbf{s}}^{\infty,i}(\mathbf{X},\mathbf{B}), \tag{8.5}$$

$$\lim_{m\to\infty} \int_0^{\cdot} \sigma^i(F_{\mathbf{s}}^m(\mathbf{X},\mathbf{B})_u)dB_u^i = \int_0^{\cdot} \sigma^i(F_{\mathbf{s}}^\infty(\mathbf{X},\mathbf{B})_u)dB_u^i, \tag{8.6}$$

$$\lim_{m\to\infty} \int_0^{\cdot} b^i(F_{\mathbf{s}}^m(\mathbf{X},\mathbf{B})_u)du = \int_0^{\cdot} b^i(F_{\mathbf{s}}^\infty(\mathbf{X},\mathbf{B})_u)du, \tag{8.7}$$

where $\sigma^i(\mathbf{Z}_t) = \sigma(Z_t^i, \mathsf{Z}_t^{i,\diamondsuit})$, and b^i is defined in a similar fashion.

Definition 8.1 (IFC solution)**.** A probability measure $\bar{P}_{\mathbf{s}}$ on $W((\mathbb{R}^d)^{\mathbb{N}}) \times \mathbf{W_0}$ is called an IFC solution of (7.1) if $\bar{P}_{\mathbf{s}}$ satisfies (8.4) and (8.8):

$$\bar{P}_{\mathbf{s}}(\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}} \times \mathbf{W_0}) = 1, \quad \bar{P}_{\mathbf{s}}(\mathbf{B} \in \cdot) = P_{\mathrm{Br}}^{\infty}. \tag{8.8}$$

First, we construct a weak solution of (7.1) from the IFC solution $\bar{P}_{\mathbf{s}}$.

Lemma 8.3 ([49])**.** *Fix initial starting point* $\mathbf{s}$ *and assume* (**P2**)*. Suppose that* $\bar{P}_{\mathbf{s}}$ *is an IFC solution of* (7.1)*. Set* $\mathbf{Y} = F_{\mathbf{s}}^{\infty}(\mathbf{X}, \mathbf{B})$*, where* $F_{\mathbf{s}}^{\infty}$ *is given by* (8.4)*. Then* $(\mathbf{Y}, \mathbf{B})$ *under* $\bar{P}_{\mathbf{s}}$ *is a weak solution of* (7.1)*.*

The next theorem clarifies the relation between IFC solution and strong solution. It also explains relation among the pathwise uniqueness, the uniqueness of strong solutions, and tail triviality of the labeled path space with respect to the label.

Theorem 8.4 ([49])**.** *Fix initial starting point* $\mathbf{s}$*. Assume* (**P2**)*. Then we have*
(1) *Suppose that* $\bar{P}_{\mathbf{s}}$ *is an IFC solution of* (7.1)*, and set* $\mathbf{Y} = F_{\mathbf{s}}^{\infty}(\mathbf{X}, \mathbf{B})$*. Then* $(\mathbf{Y}, \mathbf{B})$ *under* $\bar{P}_{\mathbf{s}}$ *is a strong solution of* (7.1) *if and only if the tail* σ*-field* $\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ *is* $\bar{P}_{\mathbf{s},\mathbf{B}}$*-trivial for* P_{Br}^{∞}*-a.s.* $\mathbf{B}$*.*
(2) *Suppose that* $\mathbf{X}$ *and* $\mathbf{X}'$ *are strong solutions of* (7.1) *defined for the same Brownian motion* $\mathbf{B}$*. Denote by* $\bar{P}_{\mathbf{s}}$ *and* $\bar{P}'_{\mathbf{s}}$ *the distributions of* $(\mathbf{X}, \mathbf{B})$ *and* $(\mathbf{X}', \mathbf{B})$*, respectively. Then* (8.9) *and* (8.10) *are equivalent.*

$$P_{\mathrm{Br}}^{\infty}(\mathbf{X} = \mathbf{X}') = 1 \tag{8.9}$$

$$\mathcal{T}_{\mathrm{path}}^{[1]}((\mathbb{R}^d)^{\mathbb{N}}; \bar{P}_{\mathbf{s},\mathbf{B}}) = \mathcal{T}_{\mathrm{path}}^{[1]}((\mathbb{R}^d)^{\mathbb{N}}; \bar{P}'_{\mathbf{s},\mathbf{B}}) \quad \textit{for } P_{\mathrm{Br}}^{\infty}\textit{-a.s. } \mathbf{B}. \tag{8.10}$$

(3) (7.1) *has a unique strong solution* $\mathbf{X}$ *if and if, for* P_{Br}^{∞}*-a.s.* $\mathbf{B}$*, the tail* σ*-field* $\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ *is* $\bar{P}_{\mathbf{s},\mathbf{B}}$*-trivial, and* $\mathcal{T}_{\mathrm{path}}^{[1]}((\mathbb{R}^d)^{\mathbb{N}}; \bar{P}_{\mathbf{s},\mathbf{B}})$ *is independent of the distribution* $\bar{P}_{\mathbf{s}}$ *of the solution* $(\mathbf{X}, \mathbf{B})$ *of* (7.1)*.*

We refer to [13] for the general theory on the existence of strong solutions and pathwise uniqueness of solutions of SDEs. We remark that, in the present article, we change the state space of the solutions of SDEs from $\mathbb{R}^d$ to the symmetric subset of $(\mathbb{R}^d)^{\mathbb{N}}$, and we modify the details of the general theory such as measurability of the initial starting points and others were suitably appropriate.

We explain the concept of IFC. This idea is to use classical theory (finite-dimensional result) at each stage, using the consistency of the scheme resulting from the symmetry of the original ISDE. Conditions (**P1**)–(**P3**) may appear as difficult assumptions to accept at first glance, but we can successfully check these assumptions using the existence of the small infinite-dimensional space and the unlabeled dynamics/Dirichlet space on it. In addition, we can develop what we need based on the classical theory.

Finally, to validate such finite-dimensional schemes, we must demonstrate the triviality of the tail σ-field in the labeled path space and make this scheme self-contained. We treat the tail σ-field as the boundary condition of the strong solution. Then, if the tail σ-field is trivial, the boundary condition consisting of "a single point", and hence the uniqueness of solution is reduced to the triviality of the tail σ-field with respect to the restriction of the distribution of the solution to the tail σ-field. Therefore, triviality of the tail σ-field of the labeled path space determines the uniqueness of the solution of the original ISDE.

In addition to the existence and uniqueness of the strong solutions, we also expect that the idea of this scheme is to extend the results of various other classical theories to the infinite particle system, such as the construction of stochastic flow on $(\mathbb{R}^d)^{\mathbb{N}}$, and ergodic decomposition of $(\mathbb{R}^d)^{\mathbb{N}}$.

9. Further development

This ongoing research is currently taking various directions. Here, we explain some of them.

9.1. **Dynamical universality.** Various kinds of universality about the point processes arising from random matrix theory have been examined by Soshnikov [59], Tao [64], Yau [2], and many others. These are the counterparts of the classical universality of the limit of the sum of identically, independent random variables; that is, one can obtain the law of large numbers and the central limit theorem under those assumptions given only by the moments of the random variable. The universality of random matrices is analogous to the eigenvalues of random matrices and Coulomb gases. Once static universality is established, it is natural to pursue its dynamical counterpart, that is, the universality of the natural stochastic dynamics associated with the limit universal point processes. This is now being developed in [22] and [23].

9.2. **Dynamical rigidity of the Ginibre interacting Brownian motion and phase transition conjectures.** The Ginibre point process features various geometric rigidities, as we see in Section 9.3. Hence, it can be expected that the Ginibre interacting Brownian motion has dynamical rigidity reflected in it. In that regard, the author conjectures that that tagged particles are sub-diffusive, that is,

$$\lim_{\epsilon \to 0} \epsilon X^i_{t/\epsilon^2} = 0. \tag{9.1}$$

A phase transition is also conjectured for the self-diffusion matrix. A toy model of this conjecture asserts that the effective matrix describing the homogenization of the periodic Coulomb potentials missing a particle at the origin has a phase transition [31]. This strongly supports the conjecture regarding the self-diffusion matrix.

Suppose now that the solution of (2.6) exists for the general reverse temperature $\beta > 0$. Then the critical point β_c for the inverse temperature β with respect to the degeneracy/non-degeneracy of the limit coefficient of the diffusive scale limit (the self-diffusion matrix) seems to have a bound such that

$$1 \leq \beta_c \leq 2.$$

The upper bound follows from the conjecture (9.1). The lower bound follows from the observation for the lower bound of effective constants of homogenization problem in the periodic Coulomb environment.

Tagged particles of interacting Brownian motions with Ruelle's class potentials with convex hard core in $\mathbb{R}^d$ $(d \geq 2)$ are always non-degenerate under the diffusive scaling [35]. That is, each tagged particle X^i behaves like Brownian motion under the diffusive scaling [32, 34, 35, 46]. Here we consider the translation-invariant equilibrium states.

Therefore, the asymptotic behavior in the diffusive scaling of the Ginibre interacting Brownian motions would be quite different from the usual interacting

Brownian motions. This reveals the strength of the long-distance effect of logarithmic potentials. If the conjecture (9.1) is settled, then this would also be a dynamical rigidity in the sense that the movement of each particle slows down.

Translation-invariant point processes on $\mathbb{R}^d$ interacting through a d-dimensional Coulomb potential are called strict Coulomb point processes ([44, 48]). The Ginibre point process is the most representative and, at the present time, the only strict Coulomb point process. It would be interesting to consider the above-mentioned conjecture for the strict Coulomb point process for general $d \geq 2$.

9.3. **Rigidity of the Ginibre point process.** Rigidity of the Ginibre point process has been successively developed by Ghosh [8], Ghosh and Peres [9], O.–Shirai [47, 48], Shirai [57]. Simulations of the Poisson and Ginibre point processes are given below.

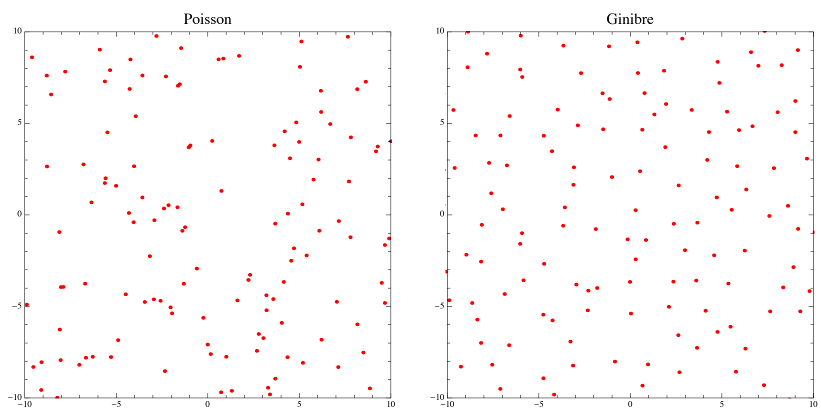

FIGURE 1. The Poisson and Ginibre point processes

As is evident, the Ginibre point process is certainly random, but at the same time, it appears understandable in that it has somewhat orderly random crystal structure.

Below we present examples of rigidity of the Ginibre point process.

• Shirai [57] proved that the variance in the number of particles in the disk of radius r is of order r as $r \to \infty$. We remark that the Poisson point process is of order r^2.

• Ghosh and Peres [9] proved that, if we fixed the bounded subset A and the configuration of the particles outside it, then the number of particles inside A is uniquely determined. We remark that, for a Poisson point process, the inside and outside distributions of the particles are independent.

• Considering the reduced Palm measure of the Ginibre point process, we see that the necessary and sufficient condition for the reduced Palm measures mutually becoming absolutely continuous arises as a coincidence in the number of conditioned particles [48]. [48] also found another dichotomy in that two reduced Palm measures are singular with each other if and only if the number of conditioned particles is different. These results show that we can determine the number of removed particles of reduced Palm measures with probability one.

The rigidity above is the same as for periodic point processes. In this regard, the Ginibre point process has a random periodic (crystal-like) property. Furthermore, the Ginibre point process is a quasi-Gibbs measure, and has a local density function conditioned outside the configuration. This property is similar to the Poisson point process. In this way, the Ginibre point process has the interesting property of enjoying both randomness and non-randomness.

9.4. **GAF: beyond the logarithmic interaction potential.** The above story starts from the point process μ, constructs the label probability dynamics of the space of the infinite particle system, expresses it as an ISDE, and examines its nature. Such a story is valid for point processes not only with interaction potentials but also without potentials. Indeed, recently, interesting point processes other than those with interacting potential have emerged.

A typical example is a point process consisting of the zeros of a Gaussian random analytic function (Gaussian Analytic functions (GAF)). Although there are various types, we introduce the planer GAF point process μ_{GAF}. This is a point process on $\mathbb{C}$ consisting of zero points of the entire function $F(z)$ with Gaussian random coefficients.

$$F(z) = \sum_{k=0}^{\infty} \frac{\xi_k}{\sqrt{k!}} z^k.$$

Here $\{\xi_k\}_{k=0}^{\infty}$ is iid, ξ_1 has a mean free Gaussian distribution with unit variance on $\mathbb{C}$. Let μ_{GAF} be the distribution of zero points of F. Then, μ_{GAF} is rotation and translation-invariant, and thus resembles the Ginibre point process. We refer to [12] for simulation and other information. Peres and Ghosh proved μ_{GAF} has a stricter rigidity than the Ginibre point process [9]. Indeed, if the configuration outside the disk S_r is conditioned, then the number of the particles in S_r is determined in the same fashion as the Ginibre point process. In addition, the mean of the particles inside S_r μ_{GAF} is also determined.

The unlabeled dynamics associated with μ_{GAF} has been constructed in a similar fashion as the first theory. In ongoing work, Ghosh, Shirai, and the author have considered a logarithmic derivative $\mathsf{d}_{\mu_{\mathrm{GAF}}}$ and therefore have a representation of the dynamics with ISDE

$$X_t^i = dB_t^i + \frac{1}{2}\mathsf{d}_{\mu_{\mathrm{GAF}}}(X_t^i, \mathsf{X}_t^{i\diamond})dt.$$

We have not yet obtained any explicit representation of $\mathsf{d}_{\mu_{\mathrm{GAF}}}$. The difficulty is that μ_{GAF} is determined by the relation between roots and coefficients of the algebraic equation, and is not simple as for the Ginibre point process with the structure given by a two-body potential. Of course, if cancellations were to be found, then there is a possibility that $\mathsf{d}_{\mu_{\mathrm{GAF}}}$ has a clear expression, but we do not know at this moment.

The world of infinite particle systems is vast and there are various point processes other than the foregoing, such as the determinantal point process, the α-determinant point process, and the GAF. We expect further developments in the research on various point processes. To do that, it is essential to study the rigidity of these point processes, as well as various dynamical and geometric properties. The first thing we need is the **quasi-Gibbs property** and **the logarithmic derivatives** of the point processes.

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