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ORIGINAL PAPER

Dynamical universality for random matrices

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Abstract
We establish an invariance principle corresponding to the universality of random matrices. More precisely, we prove the dynamical universality of random matrices in the sense that, if the random point fields μ^N of N-particle systems describing the eigenvalues of random matrices or log-gases with general self-interaction potentials V converge to some random point field μ, then the associated natural μ^N-reversible diffusions represented by solutions of stochastic differential equations (SDEs) converge to some μ-reversible diffusion given by the solution of an infinite-dimensional SDE (ISDE). Our results are general theorems that can be applied to various random point fields related to random matrices such as sine, Airy, Bessel, and Ginibre random point fields. In general, the representations of finite-dimensional SDEs describing N-particle systems are very complicated. Nevertheless, the limit ISDE has a simple and universal representation that depends on a class of random matrices appearing in the bulk, and at the soft- and at hard-edge positions. Thus, we prove that ISDEs such as the infinite-dimensional Dyson model and the Airy, Bessel, and Ginibre interacting Brownian motions are universal dynamical objects.

Keywords Random matrices · Dynamical universality · Dirichlet forms · Infinite-dimensional stochastic differential equations

Mathematics Subject Classification 60B20 · 60J40 · 60H10

Contents

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1 Introduction

The concept of universality in strongly correlated systems was envisioned by Wigner, who conjectured that the eigenvalue distribution of a large random matrix is universal, that is, the eigenvalue distribution depends only on the symmetry classes of matrices, rather than on the distributions of matrix components. The universality of random matrices is a central concept in random matrix theory, and has been studied intensively over the past two decades (see, e.g. [3,5–8,10,20–22,36]). However, its dynamical counterpart has been much less studied. In the present paper, we establish the concept of dynamical universality for random matrices in a general framework.

Let us recall some universality results for the sine$_\beta$ random point field derived by Deift et al. [5,6] following Deift and Gioev [8]. We consider the ensembles $\mathscr{M}^N$ with the distribution

$$\mathscr{P}^N_\beta(d\mathscr{M}^N) = \frac{1}{\mathscr{Z}^N_{V,\beta}} e^{-N\mathrm{tr}V(\mathscr{M}^N)} d\mathscr{M}^N$$

for $\beta = 1, 2$, and 4. Here, the ensembles $\mathscr{M}^N$ consist of $N \times N$ real symmetric matrices, $N \times N$ Hermitian matrices, and $2N \times 2N$ Hermitian self-dual matrices for $\beta = 1, 2$, and 4, respectively. For $\beta = 2$, the potential V is a real analytic function satisfying

$$\lim_{|x|\to\infty} \frac{V(x)}{\log|x|} = \infty. \tag{1.1}$$

For $\beta = 1, 4$, the potential V is a real polynomial such that

$$V(t) = \sum_{n=0}^{2m} v_n t^n, \quad v_{2m} > 0. \tag{1.2}$$

The density of the distribution of eigenvalues $\mathbf{x}_N$ of $\mathscr{M}^N$ is given by

$$P_{V,\beta}^N(\mathbf{x}_N) = \frac{1}{\mathscr{Z}_{V,\beta}^N} \prod_{i<j}^{N} |x_i - x_j|^\beta \prod_{k=1}^{N} e^{-NV_\beta(x_k)}, \tag{1.3}$$

where $\mathbf{x}_N = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $\mathscr{Z}_{V,\beta}^N$ is the normalizing constant, and we set

$$V_\beta(x) = \begin{cases} V(x), & \beta = 1, 2, \\ 2V(x) & \beta = 4. \end{cases} \tag{1.4}$$

Note that, according to the logarithmic interaction potential, particles repel each other. The logarithmic interaction potential has a strong long-range effect that causes special phenomena to occur in particle systems. One example is the convergence of the empirical distribution to a deterministic distribution that has non-degenerate density, typically known as Wigner's semi-circle law in the limit.

Let $\nu_{V,\beta}^N$ be the random point field such that its labeled density is given by $P_{V,\beta}^N$. The behavior of the system as the number of particles N tends to infinity has been extensively studied. We set $\mathfrak{x} = \sum_i \delta_{x_i}$, where δ_a denotes the delta measure at a. Then, there exists a probability density function $\boldsymbol{\rho}_V$ on $\mathbb{R}$ such that

$$\lim_{N\to\infty} \int \frac{1}{N}\mathfrak{x}((-\infty, s])\nu_{V,\beta}^N(d\mathfrak{x}) = \int_{-\infty}^{s} \boldsymbol{\rho}_V(x)\, dx. \tag{1.5}$$

If $V(x) = x^2$, then $\nu_{V,2}^N$ gives the eigenvalue distribution of the Gaussian unitary ensemble (GUE), which is the Hermitian random matrix whose entries follow an independent and identically distributed Gaussian distribution. The probability measure $\boldsymbol{\rho}_V dx$ is simply the Wigner semicircle law, which is given by $\boldsymbol{\rho}_V(x) = \frac{1}{\pi}\sqrt{2 - x^2}\mathbf{1}_{\{|x|<\sqrt{2}\}}$ (see, e.g., [2,23]).

The convergence in (1.5) is in the macroscopic regime. Next, we consider the microscopic scaling limit. More precisely, we consider a local fluctuation of (1.3) and obtain a random point field with infinitely many particles as the limit. Here, we take the bulk scaling limit. For a constant $\theta \in \mathbb{R}$ satisfying

$$\boldsymbol{\rho}_V(\theta) > 0, \tag{1.6}$$

we set the bulk scaling at θ as

$$x \mapsto \frac{s}{N\boldsymbol{\rho}_V(\theta)} + \theta. \tag{1.7}$$

Let $\mathbf{m}_{V,\beta,\theta}^N$ be the rescaled density function of $P_{V,\beta}^N$ in (1.3) under the scaling defined by (1.7). Then, $\mathbf{m}_{V,\beta,\theta}^N$ is given by

$$\mathbf{m}_{V,\beta,\theta}^N(\mathbf{s}_N) = \frac{1}{\mathscr{Z}_{V,\beta,\theta}^N} \prod_{i<j}^{N} |s_i - s_j|^\beta \prod_{k=1}^{N} \exp\Big(- NV_\beta\Big(\frac{s_k}{N\boldsymbol{\rho}_V(\theta)} + \theta\Big)\Big). \tag{1.8}$$

We define $\mu_{V,\beta,\theta}^N$ as the random point field whose labeled density is given by $\mathbf{m}_{V,\beta,\theta}^N$.

Let $\mu_{\sin,\beta}$ be the sine$_\beta$ random point field. Let $\rho^m_{\sin,\beta}$ be the m-point correlation function of $\mu_{\sin,\beta}$. If $\beta = 2$, then $\mu_{\sin,\beta}$ is the determinantal random point field whose kernel is given by

$$K_{\sin}(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)}.$$

Then, by definition,

$$\rho^m_{\sin,2}(x_1, \ldots, x_m) = \det[K_{\sin}(x_i, x_j)]_{1\le i,j\le m}.$$

Similar formulae are also known for $\beta = 1, 4$ (see [23]).

Bulk universality for log-gases asserts that, for a suitable and wide class of V and for any θ satisfying (1.6),

$$\lim_{N\to\infty} \mu^N_{V,\beta,\theta} = \mu_{\sin,\beta} \quad \text{weakly},$$

or, more strongly, for each $m \in \mathbb{N}$ the m-point correlation function $\rho^{N,m}_{V,\beta,\theta}$ of $\mu^N_{V,\beta,\theta}$ satisfies

$$\lim_{N\to\infty} \rho^{N,m}_{V,\beta,\theta} = \rho^m_{\sin,\beta} \tag{1.9}$$

uniformly on each compact set. Note that the limit $\mu_{\sin,\beta}$ is independent of V and θ, and, in this sense, the sine$_\beta$ random point field can be thought of as a universal object. We call such universality *static* or *geometric* because it involves no time evolution.

Consideration of the classical invariance principle yields a natural question: what is the dynamical counterpart of geometric universality?

We consider an N-dimensional stochastic differential equation (SDE) of $\mathbf{X}^N = (X^{N,i})_{i=1}^N$ corresponding to $\mu^N_{V,\beta,\theta}$ such that, for $1 \le i \le N$,

$$\begin{aligned} X^{N,i}(t) - X^{N,i}(0) = B^i(t) + \frac{\beta}{2}\int_0^t \sum_{j\neq i}^N \frac{1}{X^{N,i}(u) - X^{N,j}(u)} du \\ - \frac{1}{2}\int_0^t \frac{1}{\boldsymbol{\rho}_V(\theta)} V'_\beta\Big(\frac{X^{N,i}(u)}{N\boldsymbol{\rho}_V(\theta)} + \theta\Big) du. \end{aligned} \tag{1.10}$$

Here, B^i, $i = 1, \ldots, N$, are independent standard Brownian motions with $B^i(0) = 0$. We derive (1.10) from $\mu^N_{V,\beta,\theta}$ as follows. Let $\check{\mu}^N_{V,\beta,\theta}$ be the distribution of the labeled particle system of $\mu^N_{V,\beta,\theta}$. Then,

$$\check{\mu}^N_{V,\beta,\theta}(d\mathbf{x}_N) = \mathbf{m}^N_{V,\beta,\theta}(\mathbf{x}_N) d\mathbf{x}_N, \tag{1.11}$$

and consider the Dirichlet form on $L^2(\mathbb{R}^N, \check{\mu}^N_{V,\beta,\theta})$ such that

$$\mathscr{E}^{\mu^N_{V,\beta,\theta}}(f, g) = \int_{\mathbb{R}^N} \frac{1}{2}\sum_{i=1}^N \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} \mathbf{m}^N_{V,\beta,\theta}(\mathbf{x}_N) d\mathbf{x}_N.$$

Integrating by parts and using (1.11), we have that

$$\begin{aligned} \mathscr{E}^{\mu^N_{V,\beta,\theta}}(f, g) &= -\int_{\mathbb{R}^N} \frac{1}{2}\sum_{i=1}^N \Big\{ \frac{\partial^2 f}{(\partial x_i)^2} + \frac{\partial \log \mathbf{m}^N_{V,\beta,\theta}}{\partial x_i}\frac{\partial f}{\partial x_i}\Big\} g\, \mathbf{m}^N_{V,\beta,\theta}(\mathbf{x}_N) d\mathbf{x}_N \\ &= -\int_{\mathbb{R}^N} (A^N f) g\, \mathbf{m}^N_{V,\beta,\theta}(\mathbf{x}_N) d\mathbf{x}_N. \end{aligned} \tag{1.12}$$

Here, from (1.8) and (1.12), we see that A^N is given by

$$A^N = \frac{1}{2}\Delta + \frac{\beta}{2}\sum_{i=1}^N \sum_{j\neq i}^N \frac{1}{x_i - x_j}\frac{\partial}{\partial x_i} - \frac{1}{2}\sum_{k=1}^N \frac{1}{\boldsymbol{\rho}_V(\theta)} V_\beta'\Big(\frac{x_k}{N\boldsymbol{\rho}_V(\theta)} + \theta\Big)\frac{\partial}{\partial x_k}.$$

Because $\mathbf{m}^N_{V,\beta,\theta}(\mathbf{x}_N)$ is bounded and continuous, we easily find that the bilinear form $(\mathscr{E}^{\mu^N_{V,\beta,\theta}}, C_0^\infty(\mathbb{R}^N))$ is closable on $L^2(\mathbb{R}^N, \breve{\mu}^N_{V,\beta,\theta})$. Then, there exists a unique L^2-Markovian semi-group $\{T_t^N\}$ on $L^2(\mathbb{R}^N, \breve{\mu}^N_{V,\beta,\theta})$ associated with the closure of $(\mathscr{E}^{\mu^N_{V,\beta,\theta}}, C_0^\infty(\mathbb{R}^N))$. The semi-group $\{T_t^N\}$ is given by the solution $\mathbf{X}^N$ of (1.10):

$$T_t^N f(\mathbf{x}_N) = E_{\mathbf{x}_N}[f(\mathbf{X}^N(t))].$$

Here, $E_{\mathbf{x}_N}$ is the expectation with respect to the solution $\mathbf{X}^N$ starting at $\mathbf{x}_N$. By construction, $\mathbf{X}^N$ is reversible with respect to $\breve{\mu}^N_{V,\beta,\theta}$. We denote the unlabeled dynamics of $\mathbf{X}^N$ as $\mathfrak{X}^N$. Then, $\mathfrak{X}^N(t) = \sum_{i=1}^N \delta_{X^{N,i}(t)}$ by definition. It is clear that $\mathfrak{X}^N$ is reversible with respect to $\mu^N_{V,\beta,\theta}$.

If the initial conditions converge, it is interesting to determine whether the stochastic process $\mathbf{X}^N$ converges or not, and to exploit the infinite-dimensional SDE (ISDE) satisfied by the limit stochastic process $\mathbf{X}$.

We set $\mathbf{X}^{N,m} = (X^{N,i})_{i=1}^m$ and $\mathbf{X}^m = (X^i)_{i=1}^m$ for each $m \in \mathbb{N}$. We shall prove that, loosely speaking, if the initial distributions of $\mathbf{X}^{N,m}$ converge to that of $\mathbf{X}^m$ in distribution, then

$$\lim_{N\to\infty} \mathbf{X}^{N,m} = \mathbf{X}^m \quad \text{weakly in } C([0,\infty);\mathbb{R}^m).$$

Here, $\mathbf{X}^N$ is a solution to (1.26) for $\mu^N = \mu^N_{V,\beta,\theta}$. The limit ISDE is Dyson's model in infinite dimensions [35], which is an ISDE of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ such that, for $i \in \mathbb{N}$,

$$X^i(t) - X^i(0) = B^i(t) + \frac{\beta}{2}\int_0^t \lim_{R\to\infty} \sum_{\substack{j\neq i \\ |X^i(u)-X^j(u)|<R}} \frac{1}{X^i(u) - X^j(u)} du. \tag{1.13}$$

The associated unlabeled dynamics $\mathfrak{X}(t) = \sum_{i=1}^\infty \delta_{X^i(t)}$ are reversible with respect to the sine$_\beta$ random point field [27]. From the static universality defined in (1.9), we expect the limit of (1.10) as $N \to \infty$ to be given by (1.13). In particular, the limit does not depend on V and θ. In other words, ISDE (1.13) is expected to be a dynamical universal object, which is a consequence of the present paper.

The simplest case is $V(t) = t^2$, $\beta = 2$, and $|\theta| < \sqrt{2}$. Because $\boldsymbol{\rho}_V(\theta) = \frac{1}{\pi}\sqrt{2-\theta^2}$, (1.10) becomes

$$\begin{aligned} X^{N,i}(t) - X^{N,i}(0) &= B^i(t) + \int_0^t \sum_{j\neq i}^N \frac{1}{X^{N,i}(u) - X^{N,j}(u)} du \\ &\quad - \frac{\pi^2}{N(2-\theta^2)}\int_0^t X^{N,i}(u) du - \frac{\pi\theta}{\sqrt{2-\theta^2}} t. \end{aligned} \tag{1.14}$$

In [14], the authors proved that the solution of (1.14) converges to that of (1.13) in distribution in $C([0,\infty);\mathbb{R}^m)$ for each m if the dynamics start from reversible measures. Note that the term $-\frac{\pi\theta}{\sqrt{2-\theta^2}}t$ in (1.14) disappears in (1.13).

Suppose that V is a real analytic function satisfying (1.1) and $\beta = 2$. Set

$$c_1 := \frac{1}{2}\frac{1}{\boldsymbol{\rho}_V(\theta)}V'(\theta). \tag{1.15}$$

Then, as $N \to \infty$, the remaining part in the last term of (1.10) is $-c_1 t$, which vanishes in (1.13). Thus, we see that the convergence has an SDE gap. This phenomenon is a result of the long-range interaction of the logarithmic potential, and never happens for translation-invariant random point fields with potentials of Ruelle's class.

Our result is an invariance principle in the sense that weak convergence occurs in the path space $C([0,\infty);\mathbb{R}^{\mathbb{N}})$. As an application of the invariance principle, we see from (1.10), (1.15), and (1.26) that, for $a \geq 0$ and $i \leq n_N$,

$$\begin{aligned}
&\lim_{N\to\infty} P\Big(\max_{0\leq t\leq T} \Big\{ X^{N,i}(t) - X^{N,i}(0) - \int_0^t \sum_{j\neq i}^{n_N} \frac{1}{X^{N,i}(u) - X^{N,j}(u)} du \\
&\quad + c_1 t - \int_{S\setminus S_{r_N}} \frac{1}{X^{N,i}(t) - y} \rho^{N,1}(y) dy \Big\} \geq a \Big) \\
&= 2\int_{x\geq a} \frac{1}{\sqrt{2\pi T}} e^{-|x|^2/2T} dx.
\end{aligned} \tag{1.16}$$

Here, $\rho^{N,1}$ is the one-point correlation function of $\mu^N := \mu^N_{V,2,\theta}$ with respect to the Lebesgue measure. The process $(X^{N,i})_{i=1}^{n_N}$ is a solution of SDE (1.26) on $\{|x| < r_N\}^{n_N}$ for μ^N with V as above. The number of particles in S_{r_N} is denoted by n_N, and so $X^{N,i}(0) \in S_{r_N}$ for $i = 1,\dots,n_N$. The radius of the domain S_{r_N} satisfies $\lim_{N\to\infty} r_N = \infty$. Furthermore, (1.26) is given by (1.10) with the reflecting boundary condition and the free potential in (1.27) for $\Psi^N(x) = -2\log|x|$. We prove that (1.16) holds in Sect. 10.1.

The next example is the Ginibre random point field. The Ginibre random point field μ_{gin} is a determinantal random point field on $\mathbb{R}^2$ that has the kernel K_{gin} with respect to the Lebesgue measure such that

$$K_{\mathrm{gin}}(x,y) = \frac{1}{\pi}\exp\Big(-\frac{|x|^2+|y|^2}{2} + x\bar{y}\Big). \tag{1.17}$$

Here, we naturally regard $\mathbb{R}^2$ as $\mathbb{C}$ by $(x,y) \mapsto x + \sqrt{-1}y$. Then, by definition, the m-point correlation function ρ^m_{gin} with respect to the Lebesgue measure is given by

$$\rho^m_{\mathrm{gin}}(x_1,\dots,x_m) = \det[K_{\mathrm{gin}}(x_i,x_j)]^m_{i.j=1}.$$

The Ginibre random point field μ_{gin} is a limit of the random point fields μ^N_{gin} arising from the ensemble of the complex non-Hermitian Gaussian random matrices $M^N_{\mathrm{gin}} = (M^N_{\mathrm{gin}}(i,j))^N_{i.j=1}$ in which the $2N^2$ parameters are independent Gaussian random variables with mean zero and variance $1/2$. The labeled density $\mathbf{m}^N_{\mathrm{gin}}$ of the distribution of the eigenvalues of M^N_{gin} is given by

$$\mathbf{m}^N_{\mathrm{gin}}(x_1,\dots,x_N) = \frac{1}{\mathscr{Z}}\left\{\prod_{i<j}^N |x_i - x_j|^2\right\} e^{-\sum_{k=1}^N |x_k|^2}. \tag{1.18}$$

From (1.18), similar to the case for (1.10)–(1.12), we derive the SDE describing the labeled N-particle dynamics $\mathbf{X}^N = (X^{N,i})_{i=1,\dots,N}$ as follows. For $i = 1, \dots, N$,

$$X^{N,i}(t) - X^{N,i}(0) = B^i(t) - \int_0^t X^{N,i}(u)du + \int_0^t \sum_{j\neq i}^{N} \frac{X^{N,i}(u) - X^{N,j}(u)}{|X^{N,i}(u) - X^{N,j}(u)|^2} du. \tag{1.19}$$

It has been shown [13] that, under suitable assumptions regarding the initial distributions, the dynamics of the first m particles $\mathbf{X}^{N,m} = (X^{N,1}, \dots, X^{N,m})$ converge weakly in $C([0,\infty); (\mathbb{R}^2)^m)$ to those of the unique strong solutions $\mathbf{X} = (X^i)_{i=1}^{\infty}$ of the ISDE such that

$$X^i(t) - X^i(0) = B^i(t) - \int_0^t X^i(u)du + \int_0^t \lim_{R\to\infty} \sum_{\substack{j\neq i \\ |X^j(u)|<R}} \frac{X^i(u) - X^j(u)}{|X^i(u) - X^j(u)|^2} du. \tag{1.20}$$

It has also been shown [27,31] that the solution $\mathbf{X}$ of (1.20) satisfies the second ISDE

$$X^i(t) - X^i(0) = B^i(t) + \int_0^t \lim_{R\to\infty} \sum_{\substack{j\neq i \\ |X^i(u)-X^j(u)|<R}} \frac{X^i(u) - X^j(u)}{|X^i(u) - X^j(u)|^2} du. \tag{1.21}$$

Thus, both ISDEs have the same unique strong solution $\mathbf{X}$, and the solution $\mathbf{X}$ satisfies two different ISDEs [27,31]. Such multiple representations of ISDEs are the result of the long-range nature of the logarithmic interaction potential in the drift terms of (1.20) and (1.21).

The unlabeled dynamics $\mathfrak{X}(t) = \sum_{i=1}^{\infty} \delta_{X^i(t)}$ describe μ_{gin}-reversible diffusion [28]. The ISDEs are obtained from the general theory for random point fields and the associated ISDEs in [27,31]. Note that, even in such a typical case, the convergence of finite particle systems is a sensitive problem, as we see from (1.19)–(1.21).

Akemann–Cikovic–Venker [1] proved the following universality of the Ginibre random point field. Let $\mathscr{J}(N)$ be the space of the normal matrices of order N. For constants $\gamma \geq 0$, $K_p \in \mathbb{R}$, and $\omega \in [0, 1)$, consider the probability measure on $\mathscr{J}(N)$ whose density is given by

$$\sigma(J) = \frac{1}{\mathscr{Z}} \exp\Big\{ \frac{N}{1-\omega^2} \mathrm{Tr}(JJ^* - \frac{\omega}{2}(J^2 + J^{*2})) - \gamma(\mathrm{Tr} JJ^* - NK_p)^2 \Big\}.$$

Then, the joint density of the eigenvalues is proportional to

$$\Big\{ \prod_{i<j}^{N} |z_i - z_j|^2 \Big\} \times \exp\Big\{ -\frac{N}{1-\omega^2} \Big(\sum_{i=1}^{N} |z_i|^2 - \frac{\omega}{2} \sum_{i=1}^{N} (z_i^2 + \bar{z}_i^2) \Big) - \gamma \Big(\sum_{i=1}^{N} |z_i|^2 - NK_p \Big)^2 \Big\} \tag{1.22}$$

Let $\rho_{\mathrm{gin}}^{N,m}$ be the m-point correlation function of the eigenvalue density corresponding to (1.22). For positive constants c_2 and c_3, we set

$$E = \{ z \in \mathbb{C};\, c_2(\Re z)^2 + c_3(\Im z)^2 < 1 \}.$$

We quote a universality result for the Ginibre random point field from [1].

Proposition 1.1 ([1, Theorem 1]) *There exist positive constants c_2, c_3, c_4 depending on K_p, γ, ω such that, for any $\zeta \in \mathbb{C} \setminus \partial E$,*

$$\lim_{N\to\infty} \frac{1}{N} \rho_{\mathrm{gin}}^{N,1}(\zeta) = \frac{c_4}{\pi} 1_E(\zeta).$$

Furthermore, for each $m \in N$ and $\zeta \in E$,

$$\frac{1}{(c_4 N)^m} \rho_{\mathrm{gin}}^{N,m}\Big(\zeta + \frac{z_1}{\sqrt{c_4 N}}, \dots, \zeta + \frac{z_m}{\sqrt{c_4 N}}\Big) = \rho_{\mathrm{gin}}^{m}(z_1, \dots, z_m) + \mathscr{O}\Big(\frac{1}{\sqrt{N}}\Big).$$

Here, the error term $\mathscr{O}\Big(\frac{1}{\sqrt{N}}\Big)$ can be taken uniformly on each compact set in $\mathbb{C}^m$.

We shall investigate a dynamical counterpart of this result.

From the same calculation as for (1.11)–(1.12), we obtain from (1.22) the SDE of the N particles $\mathbf{X}^N = (X^{N,i})_{i=1}^N$ such that, for $i = 1, \dots, N$,

$$\begin{aligned} X^{N,i}(t) - X^{N,i}(0) &= B^i(t) + \int_0^t \sum_{j\neq i}^N \frac{X^{N,i}(u) - X^{N,j}(u)}{|X^{N,i}(u) - X^{N,j}(u)|^2} du \\ &- \int_0^t \frac{N}{1-\omega^2} \frac{1}{\sqrt{c_4 N}} \Big(\zeta + \frac{X^{N,i}(u)}{\sqrt{c_4 N}}\Big) du \\ &+ \int_0^t \frac{N}{1-\omega^2} \frac{\omega}{2} \frac{1}{\sqrt{c_4 N}} \Big\{ \Big(\zeta + \frac{X^{N,i}(u)}{\sqrt{c_4 N}}\Big) + \Big(\zeta + \frac{X^{N,i}(u)}{\sqrt{c_4 N}}\Big)^\dagger \Big\} du \\ &- \int_0^t \frac{2\gamma}{\sqrt{c_4 N}} \Big(\zeta + \frac{X^{N,i}(u)}{\sqrt{c_4 N}}\Big) \Big\{ \sum_{j=1}^N \Big|\zeta + \frac{X^{N,j}(u)}{\sqrt{c_4 N}}\Big|^2 - N K_p \Big\} du. \end{aligned} \tag{1.23}$$

Here, we set $(x, y)^\dagger = (x, -y) \in \mathbb{R}^2$. The limit ISDE corresponding to μ_{Gin} is

$$X^i(t) - X^i(0) = B^i(t) + \int_0^t \lim_{R\to\infty} \sum_{\substack{j\neq i \\ |X^i(u) - X^j(u)| < R}} \frac{X^i(u) - X^j(u)}{|X^i(u) - X^j(u)|^2} du \quad (i \in \mathbb{N}).$$

Thus, although the representation of the N-particle SDE is quite complicated, the limit ISDE is very simple and universal.

To prove such dynamical finite particle approximations, the authors have previously established a general theory [13]. The framework in [13] does not depend on the dimension of the underlying space, the inverse temperature, or the integrable structures; thus, the theory can be applied to many examples. A key point in our previous paper is the control of drift terms in finite-dimensional SDEs, which provide a sensitive estimate for the long-range interaction potential. Actually, we have proved the dynamical bulk scaling limit by completing such an estimate [14]. However, when the potentials become some general $V(x)$, the calculations are more difficult. In particular, when we consider an ISDE related to the Airy random point field, which arises from the soft-edge scaling limit of eigenvalue distributions of random matrices, this presents a more complicated problem. Indeed, the drift term in the corresponding finite-dimensional SDE includes a divergent term [32].

To overcome this difficulty, we construct a new method in the present paper. This approach uses the convergence concept of Dirichlet forms associated with finite or infinite particle systems. Let $\rho^{N,m}$ and ρ^m be the m-point correlation functions of μ^N and μ, respectively. Let σ_R^m be the m-point density function of μ on $S_R = \{|s| < R\}$. As well as the existence of infinite particle dynamics, we assume two main conditions for convergence:

(I) We assume that $\lim_{N\to\infty} \rho^{N,m} = \rho^m$ uniformly on each compact set, and the capacity of the zero points of σ_R^m vanishes.
(II) We assume the uniqueness in law of weak solutions to the limit ISDE.

Condition (I) is related to the static property of random point fields, while Condition (II) concerns the dynamical property of infinite-particle systems in the limit. These two ingredients are sufficient for dynamical universality.

Only condition (I) is related to finite-particle systems. This is purely because of the static property of such systems—we do not require any assumption regarding estimates related to the dynamics of finite-particle systems, such as the estimates of drift terms in [13]. Additionally, the method derived in the present paper is independent of the dimension of an underlying space, the inverse temperature, or the integrable structures, as in [13].

Consequently, if the limit ISDE has a unique weak solution, the strong convergence of random point fields automatically implies dynamical convergence. It has been proved that several ISDEs with coefficients given by the logarithmic interaction potential have unique solutions [31,37]. Therefore, the static strong universality of random matrices can be strengthened to dynamical universality, not only for Dyson's Brownian motion and the Ginibre random point field, but also for ISDEs related to the Airy random point field, the Bessel random point field, and so on.

We now explain some of our main results. There exists a natural correspondence among a random point field μ on S, an unlabeled diffusion $\mathfrak{X}$ in the configuration space $\mathfrak{S}$ over S, and an ISDE $\mathbf{X}$ on $S^{\mathbb{N}}$ (see [27,28]). For a given random point field μ, the associated unlabeled diffusion $\mathfrak{X}$ is given by distorted Brownian motion, which is a Dirichlet form whose energy and time change measures are common (see Sect. 2). The correspondence between $\mathfrak{X}$ and $\mathbf{X}$ is given by $\mathfrak{X}(t) = \sum_{i=1}^{\infty} \delta_{X^i(t)}$ and $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$. Furthermore, the labeled dynamics $\mathbf{X}$ have a representation as a solution of an ISDE. The ISDE is described in terms of the logarithmic derivative d^μ of μ as follows:

$$X^i(t) - X^i(0) = B^i(t) + \frac{1}{2}\int_0^t \mathrm{d}^\mu(X^i(u), \sum_{j\neq i}^{\infty} \delta_{X^j(u)})du. \tag{1.24}$$

See Definition 2.1 for the definition of the logarithmic derivative d^μ.

For a random point field μ^N, we consider a window $\overline{S}_{r_N} = \{|x| \le r_N\}$, $0 < r_N < \infty$, such that

$$\lim_{N\to\infty} r_N = \infty \tag{1.25}$$

and introduce the diffusion $(X_{r_N}^{N,i})_{i=1}^{n_N}$ with the state space $(\overline{S}_{r_N})^{n_N}$ such that, for $1 \le i \le n_N$, where n_N is the number of particles in $\overline{S}_{r_N}$,

$$\begin{aligned} X_{r_N}^{N,i}(t) - X_{r_N}^{N,i}(0) = B^i(t) + \frac{1}{2}\int_0^t \mathrm{d}^N(X_{r_N}^{N,i}(u), \sum_{j\neq i}^{n_N} \delta_{X_{r_N}^{N,j}(u)})du \\ + \frac{1}{2}\int_0^t U^N(X_{r_N}^{N,i}(u))du + \frac{1}{2}\mathrm{BC}^N(X_{r_N}^{N,i}(t)). \end{aligned} \tag{1.26}$$

Here, $\mathrm{BC}^N(X_{r_N}^{N,i}(t))$ comes from a boundary condition of ∂S_{r_N} and U^N is a free potential caused by the average of the outside particles. Furthermore, d^N is the logarithmic derivative of μ^N. If d^N is given by an interaction potential Ψ^N with inverse temperature β, then U^N becomes

$$U^N(x) = \frac{\beta}{2} \int_{S_{r_N}^c} \nabla \Psi^N(x, y) \rho^{N,1}(y) dy, \tag{1.27}$$

where $\rho^{N,1}$ is the one-point correlation function of μ^N (see (3.24)–(3.25)). By construction, the associated unlabeled diffusion $\mathfrak{X}^N(t) = \sum_{i=1}^{n_N} \delta_{X_{r_N}^{N,i}(t)}$ is $\mu^N \circ \pi_{r_N}^{-1}$-reversible. Here, $\pi_{r_N}(\mathfrak{s}) = \mathfrak{s}(\cdot \cap S_{r_N})$. We label the particles here in such a way that $|X_{r_N}^{N,i}(0)| \leq |X_{r_N}^{N,i+1}(0)|$ for all i. Note that the number n_N is, therefore, random.

The last term in (1.26) will vanish as $N \to \infty$ because of (1.25). The relation between d^N and d^μ is not transparent, as we saw in (1.10), (1.14), and (1.23). Nevertheless, in Corollary 3.3, we shall prove that there exists a sequence $\{r_N\}$ satisfying (1.25) such that $(X_{r_N}^{N,i})_{i=1}^m$ converge weakly in $C([0, \infty); S^m)$ to the first m components of $\mathbf{X}$. In this sense, the geometric universality in condition (I) to a random point field μ yields the dynamical universality to the solution $\mathbf{X}$ of ISDE (1.24).

The idea behind the proof of the dynamical universality is as follows. One of the main tools for the proof is the generalized Mosco convergence, in the sense of Kuwae–Shioya [18], of Dirichlet forms (cf. [17]). This convergence is equivalent to the strong convergence of semi-groups corresponding to Dirichlet forms. The Mosco convergence concept consists of two convergence relations related to Dirichlet forms (see Definition 4.4).

For a random point field μ, there are two canonical Dirichlet forms, called the upper and lower Dirichlet forms (see Sect. 2). Accordingly, two natural schemes of finite-volume Dirichlet forms exist, and each scheme converges to the limit Dirichlet form. We shall prove that these two schemes of Dirichlet forms realize the two convergence relations in the Mosco convergence definition, respectively. We use condition (I) at this stage.

In addition, the two canonical Dirichlet forms in the limit are the same under the uniqueness in law of weak solutions to the ISDE. Hence, we conclude the Mosco convergence from conditions (I) and (II).

In [15], we proved the uniqueness of Dirichlet forms applicable to the current situation. This uniqueness theorem is robust and can be applied to random point fields from random matrix theory despite the long-range interaction of these random point fields.

Note that, in general, dynamical convergence fails under only the weak convergence of measures, even in one-dimensional diffusion. A typical example is a homogenization problem. Hence, we must assume a stronger convergence of the random point fields μ^N, such as that specified by condition (I). Thus, the strong convergence in condition (I) is a valid assumption. We note that examples of the universality of random matrices satisfying the uniform convergence of correlation functions on each compact set can be found in [5–8,21,22,34].

In our argument, it is critical to take solutions of ISDEs as the limiting point of the stochastic dynamics. There are different constructions of infinite-dimensional stochastic dynamics arising from random matrices for $d = 1$ and $\beta = 2$. In [12], the dynamics were constructed by using spatial–temporal correlation functions. Using [29–31], we find that this construction defines the same stochastic dynamics as given by solutions of ISDEs. In [4], the dynamics were constructed using the Brownian Gibbs property. It is plausible, but has not yet been proved, that this construction also defines the same dynamics given by the solutions of ISDEs.

The remainder of this paper is organized as follows. In Sect. 2, we set up Dirichlet forms and recall the relations among random point fields, unlabeled diffusions, and an SDE representation. In particular, two types of unlabeled diffusions are presented. In Sect. 3, we state the main theorems (Theorems 3.1–3.5). In Sect. 4, we recall the concept of Mosco convergence in the sense of Kuwae-Shioya, before proving Theorem 3.1 in Sect. 5. In Sect. 6, we

prepare some results related to cut-off Dirichlet forms for the proof of Theorems 3.2–3.5. In Sect. 7, we prove Theorems 3.2–3.3, and then in Sect. 8, we prove Theorems 3.4–3.5. In Sect. 9, we present a sufficient condition for **(C3)**. Finally, Sect. 10 presents some examples of dynamical universality arising from random matrix theory.

2 Preliminaries

2.1 Two spatial approximations of Dirichlet forms

In this section, we prepare Dirichlet forms and the associated dynamics, following [15,24,27]. Let S be a connected open set in $\mathbb{R}^d$ such that $\overline{S}^{\text{int}} = S$. We take S as the underlying space, and denote the configuration space over S as $\mathfrak{S}$. By definition, $\mathfrak{S}$ is the set of Radon measures consisting of sums of point measures:

$$\mathfrak{S} = \left\{ \mathfrak{s} = \sum_i \delta_{s_i} \,;\, s_i \in S, \mathfrak{s}(K) < \infty \text{ for any compact set } K \text{ in } S \right\}.$$

Here, we regard the zero measure as an element of $\mathfrak{S}$. The set $\mathfrak{S}$ is equipped with the vague topology, under which $\mathfrak{S}$ is a Polish space. We set $S_R = \{|s| < R\}$ and

$$\mathfrak{S}_R^m = \{\mathfrak{s} \in \mathfrak{S} \,;\, \mathfrak{s}(S_R) = m\}.$$

For a set $A \subset S$, let $\pi_A : \mathfrak{S} \to \mathfrak{S}$ be the projection map given by $\pi_A(\mathfrak{s}) = \mathfrak{s}(\cdot \cap A)$. We often write $\pi_R = \pi_{S_R}$. A function f on $\mathfrak{S}$ is said to be local if f is $\sigma[\pi_K]$-measurable for some compact set K in S. For such a local function f on $\mathfrak{S}$, f is said to be smooth if $\check{f} = \check{f}_O$ is smooth for a relatively compact open set $O \subset S$ such that $K \subset O$. Here, $\check{f}_O$ is a function defined on $\cup_{k=0}^{\infty} O^k$ such that, for each k, $\check{f}_O(x_1, \dots x_k)$ restricted on O^k is symmetric in x_i $(i = 1, \dots, k)$ and $\check{f}_O(x_1, \dots, x_k) = f(\mathfrak{x})$, where $\sum_{i=1}^k \delta_{x_i} = \pi_O(\mathfrak{x})$. The case $k = 0$ corresponds to a constant function. Because $\mathfrak{x}$ is a configuration and O is relatively compact, the cardinality of the particles of $\mathfrak{x}$ is finite in O. Note that $\check{f}_O$ has the consistency property such that

$$\check{f}_O(x_1, \dots, x_k) = \check{f}_{O'}(x_1, \dots, x_k) \quad \text{for all } (x_1, \dots, x_k) \in O^k \cap O'^k.$$

Thus, we see that $f(\mathfrak{x}) = \check{f}(x_1, \dots x_k)$ is well-defined.

Next, we introduce carré du champ operators on $\mathfrak{S}$. Let $a = (a_{pq})_{p,q=1}^d$ be an $\mathbb{R}^{d^2}$-valued function defined on $S \times \mathfrak{S}$ such that $a_{pq} = a_{qp}$ and a is elliptic and bounded: there exists a constant c_5 such that, for all $(x, \mathfrak{s}) \in S \times \mathfrak{S}$,

$$c_5^{-1}|\xi|^2 \le (a\xi, \xi)_{\mathbb{R}^d} \le c_5|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d. \tag{2.1}$$

Because $a(x, \mathfrak{s}) = a(x, \sum_i \delta_{s_i})$ is symmetric in $(s_i)_i$, we can construct a function, denoted by the same symbol $a(x, s_1, s_2, \dots)$ such that a is symmetric in $(s_1, s_2, \dots)$ for each $x \in S$ and

$$a(x, s_1, s_2, \dots) = a(x, \sum_i \delta_{s_i}). \tag{2.2}$$

Let $\mathscr{D}_\circ$ be the set of all local, smooth functions on $\mathfrak{S}$. For $f, g \in \mathscr{D}_\circ$, we set

$$\mathbb{D}^a[f,g](\mathfrak{s}) = \frac{1}{2}\sum_{s_i\in S}(a(s_i,\mathfrak{s}_{i\diamond})\nabla_{s_i}\check{f}(\mathbf{s}),\nabla_{s_i}\check{g}(\mathbf{s}))_{\mathbb{R}^d}, \tag{2.3}$$

$$\mathbb{D}^a_R[f,g](\mathfrak{s}) = \frac{1}{2}\sum_{s_i\in S_R}(a(s_i,\mathfrak{s}_{i\diamond})\nabla_{s_i}\check{f}(\mathbf{s}),\nabla_{s_i}\check{g}(\mathbf{s}))_{\mathbb{R}^d}. \tag{2.4}$$

Here, for $\mathbf{s}=(s_i)_i$, we set $\mathfrak{s}=\sum_i\delta_{s_i}$ and $\mathfrak{s}_{i\diamond}=\sum_{j\neq i}\delta_{s_j}$. The right-hand sides of (2.3) and (2.4) are symmetric functions in $\mathbf{s}=(s_i)_i$. Hence, we regard them as functions in $\mathfrak{s}$. Let

$$\mathbb{D}^{a,m}_R[f,g](\mathfrak{s}) = 1_{\mathfrak{S}^m_R}(\mathfrak{s})\,\mathbb{D}^a_R[f,g](\mathfrak{s}). \tag{2.5}$$

Then, by construction,

$$\mathbb{D}^a_R=\sum_{m=1}^{\infty}\mathbb{D}^{a,m}_R,\quad \lim_{R\to\infty}\mathbb{D}^a_R[f,g]=\mathbb{D}^a[f,g]\quad \text{for } f,g\in\mathscr{D}_\circ.$$

A probability measure μ on $\mathfrak{S}$ is called a random point field. For a random point field μ, we set $L^2(\mu):=L^2(\mathfrak{S},\mu)$ and $\mathscr{D}^\mu_\circ=\{f\in\mathscr{D}_\circ\cap L^2(\mu)\,;\,\mathscr{E}(f,f)<\infty\}$. For each $R,m\in\mathbb{N}$, we define the bilinear forms on $L^2(\mu)$ such that

$$\mathscr{E}(f,g)=\int_{\mathfrak{S}}\mathbb{D}^a[f,g]d\mu,\quad \mathscr{E}_R(f,g)=\int_{\mathfrak{S}}\mathbb{D}^a_R[f,g]d\mu$$
$$\mathscr{E}^m_R(f,g)=\int_{\mathfrak{S}}\mathbb{D}^{a,m}_R[f,g]d\mu.$$

We assume the following.

(A1) $(\mathscr{E}^m_R,\mathscr{D}^\mu_\circ)$ is closable on $L^2(\mu)$ for each $R,m\in\mathbb{N}$.

Set $\mathscr{B}_R=\{f\,;\,f \text{ is } \sigma[\pi_R]\text{-measurable}\}$. If $f\in\mathscr{B}_R$, then $f(\mathfrak{s})$ is independent of $\pi^c_R(\mathfrak{s})$. Hence, for $f\in\mathscr{D}_\circ\cap\mathscr{B}_R$, we have that

$$\mathbb{D}^a[f,f](\mathfrak{s})=\mathbb{D}^a_R[f,f](\mathfrak{s})=\mathbb{D}^{a,m}_R[f,f](\mathfrak{s})\quad \text{for all } \mathfrak{s}\in\mathfrak{S}^m_R. \tag{2.6}$$

From (2.6), we obtain that, for $f\in\mathscr{D}_\circ\cap\mathscr{B}_R$,

$$\mathscr{E}(f,f)=\mathscr{E}_R(f,f)=\sum_{m=1}^{\infty}\mathscr{E}^m_R(f,f).$$

This obvious identity is one of the key points of the argument in [24]. In the following, we quote a sequence of results from [24].

Lemma 2.1 ([24, Lemma 2.2]) *Assume that **(A1)** is satisfied. Then, the following hold:*

(1) $(\mathscr{E}_R,\mathscr{D}^\mu_\circ)$ *is closable on* $L^2(\mu)$.
(2) $(\mathscr{E},\mathscr{D}^\mu_\circ\cap\mathscr{B}_R)$ *is closable on* $L^2(\mu)$.

Proof Claim (1) follows from Lemma 2.2 (1) in [24]. Claim (2) follows from Claim (1) and $\mathscr{D}^\mu_\circ\cap\mathscr{B}_R\subset\mathscr{D}^\mu_\circ$. □

We write $(\mathscr{E}^1,\mathscr{D}^1)\leq(\mathscr{E}^2,\mathscr{D}^2)$ if $\mathscr{D}^1\supset\mathscr{D}^2$ and $\mathscr{E}^1(f,f)\leq\mathscr{E}^2(f,f)$ for any $f\in\mathscr{D}^2$. For a sequence $\{(\mathscr{E}^n,\mathscr{D}^n)\}_{n\in\mathbb{N}}$ of positive definite, symmetric bilinear forms on $L^2(\mu)$, we say that $\{(\mathscr{E}^n,\mathscr{D}^n)\}$ is increasing if $(\mathscr{E}^n,\mathscr{D}^n)\leq(\mathscr{E}^{n+1},\mathscr{D}^{n+1})$ for any $n\in\mathbb{N}$, and decreasing if $(\mathscr{E}^n,\mathscr{D}^n)\geq(\mathscr{E}^{n+1},\mathscr{D}^{n+1})$ for any $n\in\mathbb{N}$.

Taking Lemma 2.1 into account, we denote the closures of $(\mathscr{E}_R,\mathscr{D}^\mu_\circ)$ and $(\mathscr{E},\mathscr{D}^\mu_\circ\cap\mathscr{B}_R)$ on $L^2(\mu)$ as $(\mathscr{E}_R,\underline{\mathscr{D}}_R)$ and $(\mathscr{E}_R,\mathscr{D}_R)$, respectively. Note that $(\mathscr{E}_R,\underline{\mathscr{D}}_R)$ is an extension of $(\mathscr{E}_R,\mathscr{D}_R)$ in the sense that $\mathscr{D}_R\subset\underline{\mathscr{D}}_R$ and the value $\mathscr{E}_R(f,f)$ for $f\in\mathscr{D}_R$ of $(\mathscr{E}_R,\mathscr{D}_R)$ coincides with that of $(\mathscr{E}_R,\underline{\mathscr{D}}_R)$.

Lemma 2.2 ([24, Lemma 2.2]) *Assume that **(A1)** is satisfied. Then, the following hold:*

(1) $\{(\mathscr{E}_R, \underline{\mathscr{D}}_R)\}_{R\in\mathbb{N}}$ *is increasing.*
(2) $\{(\mathscr{E}_R, \mathscr{D}_R)\}_{R\in\mathbb{N}}$ *is decreasing.*

By definition, the largest closable part $((\tilde{\mathscr{E}})_{\mathrm{reg}}, (\tilde{\mathscr{D}})_{\mathrm{reg}})$ of a given positive symmetric form $(\tilde{\mathscr{E}}, \tilde{\mathscr{D}})$ with a dense domain is a closable form such that $((\tilde{\mathscr{E}})_{\mathrm{reg}}, (\tilde{\mathscr{D}})_{\mathrm{reg}})$ is the largest element of closable forms dominated by $(\tilde{\mathscr{E}}, \tilde{\mathscr{D}})$. Such a form exists and is unique [33, Theorem S.15].

Let $\mathscr{D}_\infty = \bigcup_{R\in\mathbb{N}} \mathscr{D}_R$. Let $(\mathscr{E}_\infty, \mathscr{D}_\infty)$ be the symmetric form such that

$$\mathscr{E}_\infty(f, f) = \lim_{R\to\infty} \mathscr{E}_R(f, f).$$

From Lemma 2.1 and Lemma 2.2, we obtain the following.

Lemma 2.3 ([24]) *Assume **(A1)**. Then, the following hold:*

(1) $(\mathscr{E}, \mathscr{D}_\circ^\mu)$ *is closable on* $L^2(\mu)$.
(2) *The closure of* $((\mathscr{E}_\infty)_{\mathrm{reg}}, (\mathscr{D}_\infty)_{\mathrm{reg}})$ *on* $L^2(\mu)$ *coincides with that of* $(\mathscr{E}, \mathscr{D}_\circ^\mu)$ *on* $L^2(\mu)$.

From Lemma 2.3 (1), we denote the closure of $(\mathscr{E}, \mathscr{D}_\circ^\mu)$ on $L^2(\mu)$ as $(\mathscr{E}, \mathscr{D})$.

By Lemma 2.2 (2), $\{(\mathscr{E}_R, \underline{\mathscr{D}}_R)\}_{R\in\mathbb{N}}$ is increasing. Let $(\mathscr{E}, \underline{\mathscr{D}})$ be the closed symmetric form given by the increasing limit of $\{(\mathscr{E}_R, \underline{\mathscr{D}}_R)\}_{R\in\mathbb{N}}$ as $R \to \infty$. Then, by definition,

$$\mathscr{E}(f, f) = \lim_{R\to\infty} \mathscr{E}_R(f, f), \quad \underline{\mathscr{D}} = \left\{ f \in \bigcap_{R=1}^{\infty} \underline{\mathscr{D}}_R \,;\, \lim_{R\to\infty} \mathscr{E}_R(f, f) < \infty \right\}. \tag{2.7}$$

We say a sequence of positive closed bilinear forms $\{(\mathscr{E}^N, \mathscr{D}^N)\}$ on $L^2(\mu)$ converges to a positive closed bilinear form $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$ in the strong resolvent sense if the sequence of their resolvents $\{R_\alpha^N\}$ converges to the resolvent R_α of $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$ strongly in $L^2(\mu)$ for each $\alpha > 0$.

Summarizing the above, we obtain the following lemma.

Lemma 2.4 *Assume that **(A1)** is satisfied. Then, the following hold:*

(1) $(\mathscr{E}, \mathscr{D})$ *is the strong resolvent limit of* $\{(\mathscr{E}_R, \mathscr{D}_R)\}_{R\in\mathbb{N}}$ *as* $R \to \infty$.
(2) $(\mathscr{E}, \underline{\mathscr{D}})$ *is the strong resolvent limit of* $\{(\mathscr{E}_R, \underline{\mathscr{D}}_R)\}_{R\in\mathbb{N}}$ *as* $R \to \infty$.
(3) $(\mathscr{E}, \underline{\mathscr{D}}) \le (\mathscr{E}, \mathscr{D})$.

Proof The first two statements follow from Theorem 3 in [24]. The third follows from Remark (3) in [24, 120 p]. □

Taking Lemma 2.4 (3) into account, we call $(\mathscr{E}, \underline{\mathscr{D}})$ and $(\mathscr{E}, \mathscr{D})$ the lower and upper Dirichlet forms, respectively. We make an assumption.

(A2) $(\mathscr{E}, \underline{\mathscr{D}}) = (\mathscr{E}, \mathscr{D})$.

We shall present a sufficient condition for **(A2)** in Lemma 2.9.

2.2 A diffusion associated with the upper Dirichlet form

In Sect. 2.2, we introduce the $\mathfrak{S}$-valued diffusion $\mathfrak{X}$ given by the Dirichlet form $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$, and the associated $S^{\mathbb{N}}$-valued labeled process $\mathbf{X}$.

We denote the density function of μ on $\mathfrak{S}_R^m$ with respect to the Lebesgue measure on S_R^m as σ_R^m, that is, σ_R^m is the symmetric function such that

$$\frac{1}{m!}\int_{S_R^m} f_R^m(\mathbf{x}_m)\sigma_R^m(\mathbf{x}_m)d\mathbf{x}_m = \int_{\mathfrak{S}_R^m} f(\mathfrak{x})d\mu$$

for any bounded $\sigma[\pi_R]$-measurable functions f, where f_R^m is a symmetric function on S_R^m such that $f_R^m(\mathbf{x}_m) = f(\mathfrak{x})$ for $\mathbf{x}_m = (x_1, \dots, x_m)$ and $\mathfrak{x} = \sum_{i=1}^m \delta_{x_i} \in \mathfrak{S}_R^m$. We set the following condition:

(A3) μ has a density function σ_R^m for each $R, m \in \mathbb{N}$, and μ satisfies

$$\sum_{m=1}^{\infty} m\mu(\mathfrak{S}_R^m) < \infty \text{ for each } R \in \mathbb{N}.$$

We recall the concepts of quasi-regularity and locality of Dirichlet forms [9,19]. Quasi-regularity and locality guarantee the existence of the associated diffusion. Here, a diffusion process is a strong Markov process with continuous sample paths starting at each point in the state space.

We say that a diffusion $(\mathfrak{X}, \mathfrak{P}) = (\{\mathfrak{X}(t)\}, \{\mathfrak{P}_\mathfrak{s}\})$ is associated with the Dirichlet form $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$ if $T_t f(\mathfrak{s}) = \mathfrak{E}_\mathfrak{s}[f(\mathfrak{X}(t))]$ for any $f \in L^2(\mu)$, where $\{T_t\}$ is the $L^2(\mu)$-semi-group given by the Dirichlet form $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$ and $\mathfrak{E}_\mathfrak{s}$ is the expectation with respect to $\mathfrak{P}_\mathfrak{s}$. By definition, $\mathfrak{P}_\mathfrak{s}(\mathfrak{X}(0) = \mathfrak{s}) = 1$. We say that $\mathfrak{X}(t)$ is μ-reversible if T_t is μ-symmetric and μ is an invariant probability measure of T_t. The unlabeled diffusion $(\mathfrak{X}, \mathfrak{P})$ associated with $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$ is constructed in [15,24].

Lemma 2.5 ([24, Theorem 1, Corollary 1], [15, Lemma 2.5]) *Assume that **(A1)** and **(A3)** hold. Then, $(\mathscr{E}, \mathscr{D})$ is a local, quasi-regular Dirichlet form on $L^2(\mu)$. In particular, there exists an $\mathfrak{S}$-valued, μ-reversible diffusion $(\mathfrak{X}, \mathfrak{P})$ associated with $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$.*

Let $\mathfrak{u}$ be the map defined on $\{\cup_{m=0}^\infty S^m\} \cup S^\mathbb{N}$ such that $\mathfrak{u}(\mathbf{s}) = \sum_i \delta_{s_i}$ for $\mathbf{s} = (s_i)_i$. Here, $S^0 = \{\emptyset\}$ and $\mathfrak{u}(\emptyset) = \mathfrak{o}$, where $\mathfrak{o}$ is the zero measure. We call $\mathfrak{u}$ the unlabeling map.

Let $\mathfrak{S}_\mathrm{si}$ be the subset of $\mathfrak{S}$ consisting of the single and infinite configurations. By definition, $\mathfrak{S}_\mathrm{si} = \mathfrak{S}_\mathrm{s} \cap \mathfrak{S}_\mathrm{i}$, where $\mathfrak{S}_\mathrm{s}$ and $\mathfrak{S}_\mathrm{i}$ are given by

$$\mathfrak{S}_\mathrm{s} = \{\mathfrak{s} \in \mathfrak{S}\,;\, \mathfrak{s}(\{x\}) \le 1 \text{ for all } x \in S\}, \quad \mathfrak{S}_\mathrm{i} = \{\mathfrak{s} \in \mathfrak{S}\,;\, \mathfrak{s}(S) = \infty\}. \tag{2.8}$$

We denote the set consisting of $\mathfrak{A}$-valued continuous paths on $[0, \infty)$ as $W(\mathfrak{A})$. Each $\mathfrak{w} \in W(\mathfrak{S}_\mathrm{s})$ can be written as $\mathfrak{w}(t) = \sum_i \delta_{w^i(t)}$, where w^i is an S-valued continuous path defined on an interval I_i of the form $[0, b_i)$ or (a_i, b_i), where $0 \le a_i < b_i \le \infty$. Note that intervals of this form are unique up to labeling. Additionally, note that if $\partial S = \emptyset$, then $\lim_{t \downarrow a_i} |w^i(t)| = \infty$ and $\lim_{t \uparrow b_i} |w^i(t)| = \infty$ for $b_i < \infty$ for all i. For $\mathfrak{A} \subset \mathfrak{S}_\mathrm{s}$, we set

$$W_\mathrm{NE}(\mathfrak{A}) = \{\mathfrak{w} \in W(\mathfrak{A})\,;\, I_i = [0, \infty) \text{ for all } i\}. \tag{2.9}$$

We say that the tagged path w^i of $\mathfrak{w}$ *does not explode* if $b_i = \infty$, and *does not enter* if $I_i = [0, b_i)$. By definition, $W_\mathrm{NE}(\mathfrak{S}_\mathrm{si})$ is the set consisting of non-exploding and non-entering paths of infinitely many non-colliding particles.

The measurable map $\mathfrak{l} : \mathfrak{S}_\mathrm{s} \backslash \{\mathfrak{o}\} \to \{\cup_{m=1}^\infty S^m\} \cup S^\mathbb{N}$ is called a label if $\mathfrak{u} \circ \mathfrak{l}(\mathfrak{s}) = \mathfrak{s}$ for all $\mathfrak{s} \in \mathfrak{S}_\mathrm{s} \backslash \{\mathfrak{o}\}$. For a label $\mathfrak{l}$, we set

$$\mathfrak{l}_\mathrm{path} : W_\mathrm{NE}(\mathfrak{S}_\mathrm{s} \backslash \{\mathfrak{o}\}) \to C([0, \infty); \{\cup_{m=1}^\infty S^m\} \cup S^\mathbb{N}) \tag{2.10}$$

by

$$\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(0) = \mathfrak{l}(\mathfrak{w}(0)), \quad \mathfrak{u}(\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(t)) = \mathfrak{w}(t). \tag{2.11}$$

The map $\mathfrak{l}_{\mathrm{path}}$ is uniquely determined by $\mathfrak{l}$.

Let $\mathfrak{P}_\mu$ be the distribution of the unlabeled diffusion with $\mathfrak{X}(0) \overset{\mathrm{law}}{=} \mu$, where $X \overset{\mathrm{law}}{=} \mu$ denotes that the distribution of a random variable X equals μ. We assume that the stochastic process $\mathfrak{X}$ is defined on $W(\mathfrak{S})$ such that $\mathfrak{X}(\mathfrak{w})(t) = \mathfrak{w}(t)$ for $\mathfrak{w} = \{\mathfrak{w}(t)\} \in W(\mathfrak{S})$. We make the following assumption.

(A4) $\mathfrak{P}_\mu(W_{\mathrm{NE}}(\mathfrak{S}_{\mathrm{si}})) = 1$.

Lemma 2.6 *Assume that* ***(A1)****,* ***(A3)****, and* ***(A4)*** *hold. Let* $(\mathfrak{X}, \mathfrak{P})$ *be the diffusion in Lemma* 2.5*. Then, there exists a unique labeled process* $\mathbf{X} \in C([0,\infty); S^{\mathbb{N}})$ *such that* $\mathbf{X}(0) = \mathfrak{l}(\mathfrak{s})$ *for* μ*-a.s.* $\mathfrak{s}$ *and* $\mathfrak{u}(\mathbf{X}(t)) = \mathfrak{X}(t)$.

Proof Because of **(A4)**, $\mathbf{X} := \mathfrak{l}_{\mathrm{path}}(\mathfrak{X})$ is well-defined for $\mathfrak{P}_\mu$-a.s. Let $\mathfrak{P}_\mu(\cdot|\mathfrak{X}(0) = \mathfrak{s})$ be the regular conditional probability. Then, $\mathbf{X} = \mathfrak{l}_{\mathrm{path}}(\mathfrak{X})$ under $\mathfrak{P}_\mu(\cdot|\mathfrak{X}(0) = \mathfrak{s})$ satisfies the claim. □

Remark 2.1 Recall that S is an open set. If $\partial S \neq \emptyset$, then assumption **(A4)** implies that none of the tagged particles X^i of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ hits the boundary ∂S.

2.3 ISDE describing two limit stochastic dynamics, and identity of the upper and lower Dirichlet forms

Once we have established the labeled dynamics, the next task is to describe the dynamics more explicitly. We shall present an ISDE representation of the limit labeled dynamics $\mathbf{X}$ in Lemma 2.7. For this, we recall the concept of the logarithmic derivative of μ in Definition 2.1.

A symmetric and locally integrable function $\rho^n : S^n \to [0,\infty)$ is called the n-point correlation function of μ with respect to the Lebesgue measure if ρ^n satisfies

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(\mathbf{x}_n)d\mathbf{x}_n = \int_{\mathfrak{S}} \prod_{i=1}^m \frac{\mathfrak{s}(A_i)!}{(\mathfrak{s}(A_i)-k_i)!}d\mu$$

for any sequence of disjoint bounded measurable sets $A_1, \ldots, A_m \in \mathscr{B}(S)$ and a sequence of natural numbers $k_1, \ldots, k_m$ satisfying $k_1 + \cdots + k_m = n$.

Let $\tilde{\mu}^{[1]}$ be the measure on $(S \times \mathfrak{S}, \mathscr{B}(S \times \mathfrak{S}))$ determined by

$$\tilde{\mu}^{[1]}(A \times B) = \int_B \mathfrak{s}(A)\mu(d\mathfrak{s}), \quad A \in \mathscr{B}(S),\ B \in \mathscr{B}(\mathfrak{S}).$$

If μ has a one-point correlation function ρ^1, then there exists a regular conditional probability $\tilde{\mu}_x$ of μ satisfying

$$\int_A \tilde{\mu}_x(B)\rho^1(x)dx = \tilde{\mu}^{[1]}(A \times B), \quad A \in \mathscr{B}(S),\ B \in \mathscr{B}(\mathfrak{S}).$$

The measure $\tilde{\mu}_x$ is called the Palm measure of μ [11]. In this paper, we use the probability measure $\mu_x(\cdot) := \tilde{\mu}_x(\cdot - \delta_x)$ instead of $\tilde{\mu}_x$. We call μ_x the reduced Palm measure of μ conditioned at x. Informally, μ_x is given by

$$\mu_x = \mu(\cdot - \delta_x | \mathfrak{s}(\{x\}) \geq 1).$$

We consider the Radon measure $\mu^{[1]}$ on $S \times \mathfrak{S}$ such that

$$\mu^{[1]}(dxd\mathfrak{s}) = \rho^1(x)\mu_x(d\mathfrak{s})dx.$$

We take $\mu^{[1]}$ instead of $\tilde{\mu}^{[1]}$, and call this the reduced one-Campbell measure of μ.

We write $f \in L^p_{\mathrm{loc}}(\mu^{[1]})$ if $f \in L^p(S_R \times \mathfrak{S}, \mu^{[1]})$ for all $R \in \mathbb{N}$. We set

$$C_0^\infty(S) \otimes \mathscr{D}_\circ = \left\{ \sum_{i=1}^N f_i(x)g_i(\mathfrak{y}) \,;\; f_i \in C_0^\infty(S),\ g_i \in \mathscr{D}_\circ,\ N \in \mathbb{N} \right\}.$$

Let $\mathscr{B}^b_R$, $R \in \mathbb{N}$, be a set consisting of $\sigma[\pi_R]$-measurable, bounded functions. Let $\mathscr{B}^b_\infty$ be the set of bounded $\mathscr{B}(\mathfrak{S})$-measurable functions. We can naturally regard $\mathscr{B}^b_R$ and $\mathscr{D}_\circ \cap \mathscr{B}^b_R$ as function spaces on $\mathfrak{S}_R$. For $0 < R \le \infty$, we set

$$\mu_R = \mu \circ \pi_R^{-1}. \tag{2.12}$$

We denote the reduced one-Campbell measure of μ_R as $\mu^{[1]}_R$. If $R = \infty$, then $\mu_R = \mu$ and $\mu^{[1]}_R = \mu^{[1]}$. We recall the concept of the logarithmic derivative d^μ of μ in [27].

Definition 2.1 Let $0 < R \le \infty$. An $\mathbb{R}^d$-valued function $\mathrm{d}^\mu_R \in L^1_{\mathrm{loc}}(\mu^{[1]}_R)^d$ is called a *logarithmic derivative* d^μ_R of μ on S_R if, for all $h \in C_0^\infty(S_R) \otimes \{\mathscr{D}_\circ \cap \mathscr{B}^b_R\}$,

$$\int_{S_R \times \mathfrak{S}} \mathrm{d}^\mu_R(x,\mathfrak{y})h(x,\mathfrak{y})\mu^{[1]}_R(dxd\mathfrak{y}) = -\int_{S_R \times \mathfrak{S}} \nabla_x h(x,\mathfrak{y})\mu^{[1]}_R(dxd\mathfrak{y}).$$

We write $\mathrm{d}^\mu = \mathrm{d}^\mu_R$ for $R = \infty$.

Let a be as in (2.2). We set $\nabla a(x,\mathfrak{s}) = (\sum_{j=1}^d \frac{\partial a_{ij}}{\partial x_j}(x,\mathfrak{s}))_{i=1}^d$, where $x = (x_1, \dots, x_d)$. We make the following assumption.

(A5) The logarithmic derivative d^μ_R of μ on S_R exists for each $R \in \mathbb{N} \cup \{\infty\}$ and $\nabla a \in L^1_{\mathrm{loc}}(\mu^{[1]}_R)^d$.

Let $\sigma(x,\mathfrak{s})$ be a matrix-valued function such that $\sigma^t\sigma = a$. We consider the ISDE of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ on $S^{\mathbb{N}}$ defined by

$$dX^i(t) = \sigma(X^i(t), \mathfrak{X}^{i\diamond}(t))dB^i(t) + \frac{1}{2}\{\nabla a + a\mathrm{d}^\mu\}(X^i(t), \mathfrak{X}^{i\diamond}(t))dt. \tag{2.13}$$

Here, $i \in \mathbb{N}$ and $\mathfrak{X}^{i\diamond}(t)$ denotes $\sum_{j\neq i}^\infty \delta_{X^j(t)}$. We say that a continuous process $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ defined on a filtered space $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\})$ is a weak solution of (2.13) if there exists a Brownian motion $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ on the same space such that $(\mathbf{X}, \mathbf{B})$ satisfies the following for all t:

$$X^i(t) - X^i(0) = \int_0^t \sigma(X^i(u), \mathfrak{X}^{i\diamond}(u))dB^i(u) + \int_0^t \frac{1}{2}\{\nabla a + a\mathrm{d}^\mu\}(X^i(u), \mathfrak{X}^{i\diamond}(u))du.$$

Here, $i \in \mathbb{N}$ and we implicitly assume that $\partial S = \emptyset$, or no tagged particles hit the boundary. Otherwise, a boundary term generally appears in (2.13). As we saw in Remark 2.1, the assumption stated above follows from **(A4)**.

Lemma 2.7 ([27]) *Assume that **(A1)** and **(A3)–(A5)** hold. Then, the labeled process* $\mathbf{X} = \mathfrak{l}_{\mathrm{path}}(\mathfrak{X})$ *under* $\mathfrak{P}_\mathfrak{s}$ *given by Lemma 2.6 is a weak solution of (2.13) such that* $\mathbf{X}(0) = \mathfrak{l}(\mathfrak{s})$ *for* μ*-a.s.* $\mathfrak{s}$.

Proof If $S = \mathbb{R}^d$, then the claim follows from Theorem 26 in [27]. The proof of the case $S \neq \mathbb{R}^d$ is the same as for the case $\mathbb{R}^d$ because of **(A4)**, and is therefore omitted. □

Note that the solution $\mathbf{X}$ in Lemma 2.7 is associated with the upper Dirichlet form $(\mathscr{E}, \mathscr{D})$ in the sense that the $L^2(\mu)$-semi-group T_t given by $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$ satisfies $T_t f(\mathfrak{s}) = \mathfrak{E}_{\mathfrak{s}}[f(\mathfrak{u}(\mathbf{X}_t))]$ for any $f \in L^2(\mu)$, where $\mathfrak{E}_{\mathfrak{s}}$ is the expectation with respect to $\mathfrak{P}_{\mathfrak{s}}$.

In [15, Theorem 3.1], it was proved that the lower Dirichlet form $(\mathscr{E}, \underline{\mathscr{D}})$ is also associated with a weak solution $\mathbf{X}$ of (2.13) under mild additional constraints. Specifically, in [15], σ was assumed to be the identity matrix. The generalization of the result in [15, Theorem 3.1] to the present case is easy.

We make the following assumption.

(A6) For each $r, s, p \in \mathbb{N}$ with $r < s$, there exists $\mathrm{d}^{\mu}_{r,s,p} \in C_b(S \times \mathfrak{S})$ such that

$$\lim_{r\to\infty} \lim_{s\to\infty} \lim_{p\to\infty} \sup_{R\geq r+s+1} \| \mathrm{d}^{\mu}_{r,s,p} - \mathrm{d}^{\mu} \|_{L^1_{\mathrm{loc}}(S\times\mathfrak{S},\, \mu^{[1]}_{R,\mathfrak{s}})} = 0 \quad \text{for } \mu\text{-a.s}\mathfrak{s}. \tag{2.14}$$

Here, $\mu^{[1]}_{R,\mathfrak{s}}$ is the reduced one-Campbell measure of $\mu_{R,\mathfrak{s}} = \mu(\cdot|\pi^c_R(\mathfrak{s}))$.

Condition **(A6)** is not difficult to check in practice; see [15, Lemma 6.1] for a sufficient condition. The roles of the parameters (r, s, p) in **(A6)** are discussed in Section 6 of [15]. Recall that $(\mathscr{E}, \underline{\mathscr{D}})$ on $L^2(\mu)$ is a Dirichlet form. Hence, we have the associated Markovian $L^2(\mu)$-semi-group $\underline{T}_t$. We quote a result from [15].

Lemma 2.8 ([15, Theorem 3.1]) *Assume that **(A1)** and **(A3)–(A6)** hold. Then, there exist a continuous $\mathfrak{S}$-valued process $\underline{\mathfrak{X}}$ and a family of probability measures $\{\underline{\mathfrak{P}}_{\mathfrak{s}}\}$ associated with $(\mathscr{E}, \underline{\mathscr{D}})$ on $L^2(\mu)$. Furthermore, there exists an $S^{\mathbb{N}}$-valued continuous process $\underline{\mathbf{X}}$ such that $\underline{\mathbf{X}}$ under $\underline{\mathfrak{P}}_{\mathfrak{s}}$ is a solution of (2.13) with $\underline{\mathbf{X}}(0) = \mathfrak{l}(\mathfrak{s})$ for μ-a.s. $\mathfrak{s}$.*

A key point of Lemma 2.8 is that $(\mathscr{E}, \underline{\mathscr{D}})$ on $L^2(\mu)$ is not necessarily a quasi-regular Dirichlet form. If $(\mathscr{E}, \underline{\mathscr{D}})$ on $L^2(\mu)$ is quasi-regular, then the conclusion in Lemma 2.8 follows immediately from the result in [27]. We do not know how to prove the quasi-regularity directly.

The quasi-regularity can be proved from the identity $(\mathscr{E}, \underline{\mathscr{D}}) = (\mathscr{E}, \mathscr{D})$ in Lemma 2.9. The proof of Lemma 2.9 follows from Lemma 2.8 and the uniqueness in law of solutions of (2.13) (see [15]). To state the uniqueness result, we introduce two conditions for solutions $\mathbf{X}$ of ISDE (2.13) defined on $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\})$.

μ-**(AC)** $P \circ \mathfrak{u}(\mathbf{X}_t)^{-1}$ is absolutely continuous with respect to μ for each $t > 0$.

(NBJ) $P(\mathfrak{m}_{R,T}(\mathbf{X}) < \infty) = 1$ for each $R, T \in \mathbb{N}$.

Here, $\mathfrak{m}_{R,T}$ is such that, for $\mathbf{w} = (w^n)$,

$$\mathfrak{m}_{R,T}(\mathbf{w}) = \inf\{m \in \mathbb{N};\ \min_{t\in[0,T]} |w^n(t)| > R \text{ for all } n \text{ such that } n > m\}. \tag{2.15}$$

Condition **(NBJ)** plays an important role in controlling the labeled dynamics $\mathbf{X} = \mathfrak{l}_{\mathrm{path}}(\mathfrak{X})$ using the unlabeled dynamics $\mathfrak{X}$ (see [31, Section 5]). Note that **(NBJ)** obviously holds if the cardinality of the particles is finite. Even if the number of particle is infinite, **(NBJ)** is easy to check. Lemma 10.3. in [31] states a sufficient condition for **(NBJ)**. We make the following assumption for ISDE (2.13).

(A7) The uniqueness in law of weak solutions of ISDE (2.13) with the initial distribution $\mu \circ \mathfrak{l}^{-1}$ holds under constraints μ-**(AC)** and **(NBJ)**.

References [16,31] provide sufficient conditions for **(A7)**.

We quote a result from [15].

Lemma 2.9 ([15, Theorem 3.2]) *Assume that **(A1)** and **(A3)–(A7)** hold. Then, **(A2)** holds.*

2.4 A sufficient condition for (A1): quasi-Gibbs measures

We introduce a Hamiltonian on a bounded Borel set S_R. For Borel-measurable functions $\Phi : S \to \mathbb{R} \cup \{\infty\}$ and $\Psi : S \times S \to \mathbb{R} \cup \{\infty\}$ with $\Psi(x, y) = \Psi(y, x)$, let

$$\mathscr{H}_R^{\Phi,\Psi}(\mathfrak{x}) = \sum_{x_i \in S_R} \Phi(x_i) + \sum_{x_i, x_j \in S_R, i<j} \Psi(x_i, x_j), \quad \text{where } \mathfrak{x} = \sum_i \delta_{x_i}. \tag{2.16}$$

The functions Φ and Ψ are called free and interaction potentials, respectively.

For a symmetric open set $O_R^m \subset S_R^m$, we set

$$\mathfrak{O}_R^m = \pi_R^{-1}(\mathfrak{u}(O_R^m)), \quad \mathfrak{O}_R = \bigcup_{m=1}^{\infty} \mathfrak{O}_R^m, \quad \mathfrak{O} = \bigcap_{R=1}^{\infty} \mathfrak{O}_R.$$

Here, $\mathfrak{u}$ is the unlabeling map defined after Lemma 2.5. We regard $\mathfrak{O}_R^m$, $\mathfrak{O}_R$, and $\mathfrak{O}$ as subsets of $\mathfrak{S}$. We note that $\mathfrak{O}_R^m$ and $\mathfrak{O}_R$ are also subsets of $\mathfrak{S}_R := \cup_{m=0}^{\infty} \mathfrak{S}_R^m$. We write $\mu_1 \le \mu_2$ if two measures μ_1 and μ_2 on a measurable space $(\Omega, \mathscr{B})$ satisfy $\mu_1(A) \le \mu_2(A)$ for all $A \in \mathscr{B}$. We introduce the concept of a quasi-Gibbs measure.

Definition 2.2 A random point field μ is said to be a (Φ, Ψ)-quasi-Gibbs measure with $\{O_R^m\}_{R,m\in\mathbb{N}}$ if (1), (2), and (3) hold.
(1) $\{O_R^m\}_{R,m\in\mathbb{N}}$ is a sequence of symmetric open sets such that $O_R^m \subset S_R^m$ for each $R, m \in \mathbb{N}$.
(2) There exists a sequence of measures $\{\mu_{(k)}\}$ on $\mathfrak{S}$ such that

$$\mu_{(k)} \le \mu_{(k+1)} \le \mu \quad \text{for each } k \in \mathbb{N},$$
$$\lim_{k\to\infty} \mu_{(k),R} = \mu_R \quad \text{weakly for each } R \in \mathbb{N}.$$

Here, we set measures $\mu_{(k),R} = \mu_{(k)} \circ \pi_R^{-1}$ and $\mu_R = \mu \circ \pi_R^{-1}$ on $\mathfrak{S}$ as in (2.12).
(3) The regular conditional probability measures

$$\mu_{(k),R,\mathfrak{s}}^m = \mu_{(k)}(\pi_R(\mathfrak{x}) \in \cdot \mid \mathfrak{x}(S_R) = m,\ \pi_R^c(\mathfrak{x}) = \pi_R^c(\mathfrak{s}))$$

satisfy, for all $k, R, m \in \mathbb{N}$ and $\mu_{(k)}$-a.e. $\mathfrak{s} \in \mathfrak{S}$,

$$c_6^{-1} 1_{\mathfrak{O}_R^m} e^{-\mathscr{H}_R^{\Phi,\Psi}} \Lambda_R^m(d\mathfrak{x}) \le 1_{\mathfrak{O}_R^m} \mu_{(k),R,\mathfrak{s}}^m(d\mathfrak{x}) \le c_6 1_{\mathfrak{O}_R^m} e^{-\mathscr{H}_R^{\Phi,\Psi}} \Lambda_R^m(d\mathfrak{x}). \tag{2.17}$$

Here, $\pi_R^c(\mathfrak{s}) = \mathfrak{s}(\cdot \cap S_R^c)$. We set $c_6 = c_6(k, R, m, \pi_R^c(\mathfrak{s}))$ to be a positive constant and $\Lambda_R^m = \Lambda_R(\cdot \cap \mathfrak{S}_R^m)$, where Λ_R is the Poisson random point field with intensity $1_{S_R} dx$.

Definition 2.3 ([28]) We call μ a (Φ, Ψ)-quasi-Gibbs measure if we take $O_R^m = S_R^m$ in Definition 2.2. In this case, $\mathfrak{O}_R^m = \mathfrak{S}_R^m$ and $\mathfrak{O} = \mathfrak{S}$.

We say that a random point field μ satisfies **(QG)** with $\{O_R^m\}_{R,m\in\mathbb{N}}$ if the following hold:
(QG1) μ is a (Φ, Ψ)-quasi-Gibbs measure with $\{O_R^m\}_{R,m\in\mathbb{N}}$.
(QG2) There exists a potential (Φ_0, Ψ_0) such that (Φ_0, Ψ_0) is locally bounded from below, upper semi-continuous, and satisfies

$$c_7^{-1}\Phi_0(x) \le \Phi(x) \le c_7\Phi_0(x),\ x \in S_R,$$
$$c_7^{-1}\Psi_0(x-y) \le \Psi(x,y) \le c_7\Psi_0(x-y),\ x, y \in S_R, \tag{2.18}$$

for each R with a constant $c_7 := c_7(R) > 0$ depending on R.

Note that (Φ, Ψ) is unbounded in general, and that $e^{-\mathscr{H}_R^{\Phi,\Psi}(\mathfrak{u}(\mathbf{x}))}$ is bounded and lower semi-continuous on S_R^m for each $R, m \in \mathbb{N}$. Replacing μ by $\mu(\cdot \cap \mathfrak{O})$ in the definition of $(\mathscr{E}_R, \mathscr{D}_\circ^\mu)$, we set $(\mathscr{E}_R^{\mathfrak{O}}, \mathscr{D}_\circ^{\mathfrak{O}})$. The following result was essentially obtained in [28].

Lemma 2.10 *Assume that μ satisfies **(QG)** with $\{O_R^m\}_{R,m\in\mathbb{N}}$. Then, $(\mathscr{E}_R^{\mathfrak{O}}, \mathscr{D}_\circ^{\mathfrak{O}})$ is closable on $L^2(\mu(\cdot\cap\mathfrak{O}))$ and $L^2(\mu)$ for each $R\in\mathbb{N}$.*

Proof In the same manner as for Lemmas 3.4 and 3.5 in [28], we can prove that $(\mathscr{E}_R^{\mathfrak{O}}, \mathscr{D}_\circ^{\mathfrak{O}})$ is closable on $L^2(\mu(\cdot\cap\mathfrak{O}))$, which implies the first claim. Note that, if a sequence $\{f_n\}$ strongly converges to zero in $L^2(\mu)$, then $\{f_n\}$ strongly converges to zero in $L^2(\mu(\cdot\cap\mathfrak{O}))$. Using this, we deduce the second claim from the first. □

3 Main results

In this section, we set up the problem and state the main results (Theorems 3.1–3.5).

3.1 Universality of unlabeled dynamics

Let S, $\mathfrak{S}$, and $\mathscr{D}_\circ$ be as in Sect. 2.1. Let $\{\mu^N\}_{N\in\mathbb{N}}$ be a sequence of random point fields on S. Let $\mathbb{D}_R^{a,m}$ be the carré du champ operator in (2.5). We set

$$\mathscr{E}_R^{N,m}(f,g)=\int_{\mathfrak{S}}\mathbb{D}_R^{a,m}[f,g]d\mu^N,$$
$$\mathscr{D}_\circ^{N,m}=\{f\in\mathscr{D}_\circ\cap L^2(\mu^N)\,;\,\mathscr{E}_R^{N,m}(f,f)<\infty\}.$$

The bilinear forms $(\mathscr{E}^N,\mathscr{D}_\circ^N)$ and $(\mathscr{E}_R^N,\mathscr{D}_\circ^N)$ are given by

$$\mathscr{E}^N(f,g)=\int_{\mathfrak{S}}\mathbb{D}^a[f,g]d\mu^N,\quad \mathscr{E}_R^N(f,g)=\int_{\mathfrak{S}}\mathbb{D}_R^a[f,g]d\mu^N,$$
$$\mathscr{D}_\circ^N=\{f\in\mathscr{D}_\circ\cap L^2(\mu^N)\,;\,\mathscr{E}^N(f,f)<\infty\}. \tag{3.1}$$

For the existence of μ^N-reversible diffusion, we assume the following.
(B1) (1) $(\mathscr{E}_R^{N,m},\mathscr{D}_\circ^{N,m})$ is closable on $L^2(\mu^N)$ for each $N,R,m\in\mathbb{N}$.
(2) μ^N has an m-density function on S_R for each $N,R,m\in\mathbb{N}$, and μ^N satisfies

$$\sum_{m=1}^{\infty}m\mu^N(\mathfrak{S}_R^m)<\infty\quad\text{for each } N,R\in\mathbb{N}.$$

To take $\mathscr{D}_\circ^\mu$ as $\mathscr{C}$ in Definition 4.1, we assume the following:
(B2) $\mathscr{D}_\circ^\mu\subset\bigcap_{N\in\mathbb{N}}\mathscr{D}_\circ^N$.

Note that **(B2)** is a mild assumption. Indeed, if $\mu^N(\mathfrak{s}(\mathfrak{S})\le C_N)=1$ for some $C_N\in\mathbb{N}$ for each $N\in\mathbb{N}$, then $\mathscr{D}_\circ^N=\mathscr{D}_\circ^\mu$. Hence, **(B2)** holds.

To state the main theorems, we introduce cut-off Dirichlet forms and the associated unlabeled diffusions. From **(B1)**, we see that $(\mathscr{E}_R^N,\mathscr{D}_\circ^N)$ is closable on $L^2(\mu^N)$. Then, we denote the closure by $(\mathscr{E}_R^N,\underline{\mathscr{D}}_R^N)$. Let $\overline{S}_R=\{s\in S;\,|s|\le R\}$. We set

$$\widetilde{\underline{\mathscr{D}}}_R^N=\{f\in\underline{\mathscr{D}}_R^N\,;\,f\text{ is }\sigma[\pi_{\overline{S}_R}]\text{-measurable}\}. \tag{3.2}$$

Then, $(\mathscr{E}_R^N,\widetilde{\underline{\mathscr{D}}}_R^N)$ is a Dirichlet form on $L^2(\mu^N)$. Clearly, $\underline{\mathscr{D}}_R^N\supset\widetilde{\underline{\mathscr{D}}}_R^N$. Hence, we have

$$(\mathscr{E}_R^N,\underline{\mathscr{D}}_R^N)\le(\mathscr{E}_R^N,\widetilde{\underline{\mathscr{D}}}_R^N). \tag{3.3}$$

Let $\mu_R^N = \mu^N \circ \pi_R^{-1}$. Note that $\mu^N(\{\mathfrak{s} \in \mathfrak{S};\ \mathfrak{s}(\partial S_R) > 0\}) = 0$. Then, we can regard μ_R^N as a probability measure on $\mathrm{Cf}(\overline{S}_R)$, where $\mathrm{Cf}(A)$ denotes the configuration space over A for a topological space A. We regard $(\mathscr{E}_R^N, \widetilde{\mathscr{D}}_R^N)$ as a closed form on

$$L^2(\mu_R^N) := L^2(\mathrm{Cf}(\overline{S}_R), \mu_R^N).$$

It is easy to show that $(\mathscr{E}_R^N, \widetilde{\mathscr{D}}_R^N)$ is a quasi-regular Dirichlet form on $L^2(\mu_R^N)$. Hence, we have a diffusion $\mathfrak{X}_R^N$ with the state space $\mathrm{Cf}(\overline{S}_R)$ associated with the Dirichlet form $(\mathscr{E}_R^N, \widetilde{\mathscr{D}}_R^N)$ on $L^2(\mu_R^N)$. The associated labeled process $\mathfrak{l}_{\mathrm{path}}(\mathfrak{X}_R^N)$ is described by SDE (1.26). Note that the concept of quasi-regularity depends on the topology equipped on the measurable space. Hence, we take $\mathrm{Cf}(\overline{S}_R)$ as the state space of the diffusion.

Let $\{r_N\}_{N\in\mathbb{N}}$ be a non-decreasing sequence in $\mathbb{N} \cup \{\infty\}$. Let $(\mathfrak{X}, \mathfrak{P})$ be the diffusion in Lemma 2.5. We make the following assumptions.

(B3) The distributions of $\mathfrak{X}_{r_N}^N(0)$ and $\mathfrak{X}(0)$ have densities $\xi_{r_N}^N \in L^2(\mu_{r_N}^N)$ and $\xi \in L^2(\mu)$, respectively. The functions $\xi_{r_N}^N$ and ξ satisfy $\lim_{N\to\infty} \xi_{r_N}^N = \xi$ strongly in the sense of Definition 4.2.

(B4) For each $R, m \in \mathbb{N}$,

$$\lim_{N\to\infty} \left\| \frac{\sigma_R^{N,m}}{\sigma_R^m} - 1 \right\|_{S_R^m} = 0. \tag{3.4}$$

Here, σ_R^m and $\sigma_R^{N,m}$ are the m-density functions of μ and μ^N on S_R, respectively. Additionally, $\|\cdot\|_{S_R^m}$ is the $L^\infty(S_R^m, d\mathbf{x})$-norm.

Theorem 3.1 *Assume that **(A1)–(A3)**, and **(B1)–(B4)** hold. Then, there exists a non-decreasing sequence $\{r_N\}_{N\in\mathbb{N}}$ in $\mathbb{N} \cup \{\infty\}$ satisfying the following:*

$$\lim_{N\to\infty} r_N = \infty, \tag{3.5}$$

$$\lim_{N\to\infty} \mathfrak{X}_{r_N}^N = \mathfrak{X} \text{ in finite-dimensional distributions.} \tag{3.6}$$

Here, using the natural inclusion $\mathrm{Cf}(\overline{S}_{r_N}) \subset \mathfrak{S}$, we regard $\mathfrak{X}_{r_N}^N$ as a $\mathfrak{S}$-valued process.

Remark 3.1 We shall give a concrete value of r_N in (5.7) and the subsequent sentence. With this choice of r_N, we have $r_N = \infty$ if and only if $N \geq \sup\{N_n; n \in \mathbb{N}\}$. Furthermore, $r_N = \infty$ implies $\mu^N = \mu$. Thus, if $r_N = \infty$, then (3.6) yields a trivial result.

We present a sufficient condition such that both $r_N = \infty$ for all $N \in \mathbb{N}$ and (3.6) hold for μ^N such that $\mu^N \neq \mu$. Let $c_8(N)$ be such that

$$c_8(N) = \sup\left\{ \left\| \frac{\sigma_R^{N,m}}{\sigma_R^m} - 1 \right\|_{S_R^m};\ 1 \leq m < \infty,\ 1 \leq R < \infty \right\}. \tag{3.7}$$

We introduce the condition stronger than **(B4)** as follows.

$$\lim_{N\to\infty} c_8(N) = 0. \tag{3.8}$$

Let c_{11} be as in (5.4). By definition, we deduce for any $N, \kappa, R, R' \in \mathbb{N}$

$$c_{11}(N, \kappa, R, R') \leq c_8(N). \tag{3.9}$$

Let c_{12} be as in (5.8). Then $c_{12}(N) \leq c_8(N)$ from (3.9). Replacing $c_{12}(N)$ by $c_8(N)$, we obtain (3.6) with $r_N = \infty$ for all $N \in \mathbb{N}$ with a slight modification of the proof in Sect. 5 using $c_{12}(N) \leq c_8(N)$.

Combining Theorem 3.1 and Lemma 2.9, we obtain the following.

Corollary 3.1 *Assume that **(A1)**, **(A3)–(A7)**, and **(B1)–(B4)** hold. Then, we have the same conclusion as in Theorem 3.1.*

Next, we present variants of assumption **(B4)**.

In general, the density σ_R^m in (3.4) may vanish. To control the set $\{\sigma_R^m = 0\}$, we use the concept of capacity. See [9, 66p] for the definition of capacity.

Note that $S_R = \{s \in S;\ |s| < R\}$ is an open set in $\mathbb{R}^d$. Let $\mathrm{Cf}(S_R)$ be the configuration space over S_R. We equip $\mathrm{Cf}(S_R)$ with the vague topology. Note that the topology of $\mathrm{Cf}(S_R)$ is different from the relative topology as a topological subspace of $\mathfrak{S}$.

Let $(\mathscr{E}_R, \mathscr{D}_R)$ be the Dirichlet form on $L^2(\mu)$ defined before Lemma 2.2. Recall that $(\mathscr{E}_R, \mathscr{D}_R)$ is the closure of $(\mathscr{E}, \mathscr{D}_\circ^\mu \cap \mathscr{B}_R)$ on $L^2(\mu)$. Then, each $f \in \mathscr{D}_R$ is $\sigma[\pi_R]$-measurable. Hence, we regard f as a function on $\mathrm{Cf}(S_R)$.

We regard μ_R as a probability measure on $\mathrm{Cf}(S_R)$ instead of $\mathfrak{S}$, and $(\mathscr{E}_R, \mathscr{D}_R)$ as a Dirichlet form on $L^2(\mathrm{Cf}(S_R), \mu_R)$. Then, we can prove that $(\mathscr{E}_R, \mathscr{D}_R)$ is a quasi-regular Dirichlet form on $L^2(\mathrm{Cf}(S_R), \mu_R)$. The proof is similar to that given in [24] for $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$, and is therefore omitted.

Let Cap_R be the capacity given by the Dirichlet form $(\mathscr{E}_R, \mathscr{D}_R)$ on $L^2(\mathrm{Cf}(S_R), \mu_R)$. Let $\mathrm{Cf}^m(S_R) = \{\mathfrak{s} \in \mathrm{Cf}(S_R)\,;\ \mathfrak{s}(S_R) = m\}$. We make the following assumption.

(ZC) For each $R, m \in \mathbb{N}$, the capacity Cap_R satisfies

$$\mathrm{Cap}_R\big(\{\mathfrak{s} \in \mathrm{Cf}^m(S_R)\,;\ \sigma_R^m(\mathfrak{s}) = 0\}\big) = 0. \tag{3.10}$$

Here, σ_R^m is regarded as a function on $\mathrm{Cf}^m(S_R)$ in an obvious manner, that is, we set $\sigma_R^m(\mathfrak{s}) = \sigma_R^m(s_1, \dots, s_m)$ for $\mathfrak{s} = \sum_{i=1}^m \delta_{s_i}$.

For $R, \nu, m \in \mathbb{N}$, let $O_{R,\nu}^m$ be a symmetric open set in S_R^m. We set

$$\mathfrak{O}_{R,\nu}^m = \pi_R^{-1}(\mathfrak{u}(O_{R,\nu}^m)), \quad \mathfrak{O}_{R,\nu} = \bigcup_{m=1}^\infty \mathfrak{O}_{R,\nu}^m, \quad \mathfrak{O}_\nu = \bigcap_{R=1}^\infty \mathfrak{O}_{R,\nu}. \tag{3.11}$$

We introduce the following conditions for $\{O_{R,\nu}^m\}_{R,\nu,m\in\mathbb{N}}$. For each $R, \nu, m \in \mathbb{N}$,

$$\mathfrak{O}_{R,\nu} = \pi_R^{-1}(\pi_R(\mathfrak{O}_{R+1,\nu})), \tag{3.12}$$

$$\mathfrak{O}_{R,\nu} \subset \mathfrak{O}_{R,\nu+1}, \tag{3.13}$$

$$\mu\Big(\bigcup_{\nu=1}^\infty \mathfrak{O}_{R,\nu}\Big) = 1, \tag{3.14}$$

$$\pi_R\Big(\mathfrak{S} \setminus \bigcup_{\nu=1}^\infty \mathfrak{O}_{R,\nu}\Big) = \bigcup_{m=1}^\infty \big\{\mathfrak{s} \in \mathfrak{S}_R^m\,;\ \sigma_R^m(\mathfrak{s}) = 0\big\},\ \mu \circ \pi_R^{-1}\text{-a.s.}, \tag{3.15}$$

$$0 < \inf\{\sigma_R^m(\mathfrak{s})\,;\ \mathfrak{s} \in \mathfrak{O}_{R,\nu}^m\}, \quad \sup\{\sigma_R^m(\mathfrak{s})\,;\ \mathfrak{s} \in \mathfrak{O}_{R,\nu}^m\} < \infty. \tag{3.16}$$

We make the following assumptions regarding the density and correlation functions.

(B4)′ The density function σ_R^m is continuous for each $R, m \in \mathbb{N}$ and satisfies

$$\lim_{N\to\infty} \big\|\sigma_R^{N,m} - \sigma_R^m\big\|_{S_R^m} = 0 \quad \text{for each } R, m \in \mathbb{N}. \tag{3.17}$$

(B4)″ The correlation functions $\rho^{N,m}$ and ρ^m are continuous and satisfy

$$\lim_{N\to\infty} \left\| \rho^{N,m} - \rho^m \right\|_{S_R^m} = 0 \quad \text{for each } R, m \in \mathbb{N}, \tag{3.18}$$

$$\sup_{N\in\mathbb{N},\, \mathbf{x}_m \in S_R^m} \rho^{N,m}(\mathbf{x}_m) \le c_9^m m^{c_{10} m} \quad \text{for all } m \in \mathbb{N}. \tag{3.19}$$

Here, $c_9 = c_9(R)$ and $c_{10} = c_{10}(R)$ are constants satisfying $0 < c_9 < \infty$ and $0 < c_{10} < 1$, and $\rho^{N,m}$ and ρ^m are the m-point correlation functions of μ^N and μ, respectively.

Theorem 3.2 *Assume that **(A1)–(A3)** and **(B1)–(B3)** hold, and that **(ZC)** is satisfied. Assume that μ and μ^N satisfy **(QG)** with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$ and that $\{O^m_{R,\nu}\}_{R,\nu,m\in\mathbb{N}}$ satisfies* (3.12)–(3.16)*. Finally, assume that either **(B4)′** or **(B4)″** holds. Then, the same conclusion as for Theorem* 3.1 *holds.*

Remark 3.2 (1) Let $(\overline{\mathfrak{P}}_R, \overline{\mathfrak{X}}_R)$ be the diffusion associated with the Dirichlet form $(\mathscr{E}_R, \mathscr{D}_R)$ on $L^2(\mathrm{Cf}(S_R), \mu_R)$. From the general theory of Dirichlet forms in [9], (3.10) implies that

$$\overline{\mathfrak{P}}_R(\overline{\mathfrak{X}}_R(t) \in \{\mathfrak{s} \in \mathrm{Cf}^m(S_R)\,;\, \sigma_R^m(\mathfrak{s}) = 0\} \text{ for some } t) = 0. \tag{3.20}$$

(2) Let Cap be the capacity associated with $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$. Note that (3.10) implies $\mathrm{Cap}\big(\{\mathfrak{s} \in \mathfrak{S}_R^m\,;\, \sigma_R^m(\mathfrak{s}) = 0\}\big) = 0$ for each $R, m \in \mathbb{N}$. This follows from the relation $(\mathscr{E}, \mathscr{D}) \le (\mathscr{E}_R, \mathscr{D}_R)$ by Lemma 2.2, the variational formula of capacity, and the natural identification between $\mathrm{Cf}^m(S_R)$ and $\mathfrak{S}_R^m$.

We set $(\mathscr{E}^N_{r_N}, \mathscr{D}^N_{r_N})$ similarly to $(\mathscr{E}_R, \mathscr{D}_R)$ in Lemma 2.2, replacing μ and R by $\mu^N_{r_N}$ and r_N, respectively. Note that

$$(\mathscr{E}^N_{r_N}, \widetilde{\mathscr{D}}^N_{r_N}) \le (\mathscr{E}^N_{r_N}, \mathscr{D}^N_{r_N}).$$

Let T_t be the Markovian semi-group associated with $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$. Let $(\mathfrak{X}, \mathfrak{P})$ be the diffusion associated with $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$. The following demonstrates the universality of boundary conditions on $\partial S_{r_N} = \{|x| = r_N\}$.

Theorem 3.3 *Under the same assumptions as for either Theorem* 3.1 *or Theorem* 3.2*, let $(\mathscr{E}^N_{r_N}, \widehat{\mathscr{D}}^N_{r_N})$ be a (not necessarily quasi-regular) Dirichlet form on $L^2(\mu^N_{r_N})$ such that*

$$(\mathscr{E}^N_{r_N}, \widetilde{\mathscr{D}}^N_{r_N}) \le (\mathscr{E}^N_{r_N}, \widehat{\mathscr{D}}^N_{r_N}) \le (\mathscr{E}^N_{r_N}, \mathscr{D}^N_{r_N}). \tag{3.21}$$

Let $\widehat{T}^N_{r_N,t}$ be the Markovian semi-group associated with $(\mathscr{E}^N_{r_N}, \widehat{\mathscr{D}}^N_{r_N})$ on $L^2(\mu^N_{r_N})$. Then, $\widehat{T}^N_{r_N,t}$ converges to T_t for each t strongly in the sense of Definition 4.2*. In particular, if there exists a Markov process $\widehat{\mathfrak{X}}^N_{r_N}$ satisfying **(B3)** associated with $\widehat{T}^N_{r_N,t}$, then $\widehat{\mathfrak{X}}^N_{r_N}$ converges to $\mathfrak{X}$ in finite-dimensional distributions.*

3.2 Universality of labeled dynamics and SDEs of finite-particle systems

In this section, we state several results on the convergence of labeled dynamics. We describe the limit dynamics in terms of solutions to an ISDE and strengthen the meaning of convergence at the path-space level.

Let $\mathfrak{l}^N$ be a label. Let $\mathfrak{l}^N_{\mathrm{path}}$ be the label path map given by $\mathfrak{l}^N$ as in (2.10) and (2.11). For the sequence $\{\mathfrak{l}^N\}_{N\in\mathbb{N}}$ and $\mathfrak{l}$, we define the discontinuity set by

$$\mathrm{Disc}[\mathfrak{l}] = \{\mathfrak{s} \in \mathfrak{S}\,;\ \mathfrak{s} \notin \mathfrak{S}_{\mathrm{s}} \text{ or there exists } \{\mathfrak{s}_N\}_{N\in\mathbb{N}} \text{ in } \mathfrak{S}_{\mathrm{s}}$$
$$\text{such that } \lim_{N\to\infty} \mathfrak{s}_N = \mathfrak{s} \text{ and } \lim_{N\to\infty} \mathfrak{l}^N(\mathfrak{s}_N) \neq \mathfrak{l}(\mathfrak{s})\}.$$

We make the following assumptions.
(C1) $\mu(\mathrm{Disc}[\mathfrak{l}]) = 0$.
From **(B3)** and **(C1)**, we have that, for each $m \in \mathbb{N}$,

$$\lim_{N\to\infty} \mu^N_{r_N} \circ (\mathfrak{l}^{N,1}, \dots, \mathfrak{l}^{N,m})^{-1} = \mu \circ (\mathfrak{l}^1, \dots, \mathfrak{l}^m)^{-1} \quad \text{weakly.} \tag{3.22}$$

Without loss of generality, we can take a non-decreasing label as follows:

$$|\mathfrak{l}^{N,i}| \le |\mathfrak{l}^{N,i+1}| \text{ for all } i \in \mathbb{N}.$$

Let $(\mathfrak{X}^N_{r_N}, \underline{\widetilde{\mathfrak{P}}}^N_{r_N})$ be the diffusion given by the Dirichlet form $(\mathscr{E}^N_{r_N}, \underline{\widetilde{\mathscr{D}}}^N_{r_N})$ with the initial distribution $\mu^N_{r_N}$. Let $W_{\mathrm{NE}}(\mathfrak{S}_{\mathrm{s}})$ be as in (2.9).
(C2) $\underline{\widetilde{\mathfrak{P}}}^N_{r_N}(\mathfrak{X}^N_{r_N} \in W_{\mathrm{NE}}(\mathfrak{S}_{\mathrm{s}})) = 1$ for each $N \in \mathbb{N}$.
We set the labeled process $\mathbf{X}^N_{r_N} = (X^{N,i}_{r_N})_{i=1}^{\infty}$ to

$$\mathbf{X}^N_{r_N} = (\mathfrak{l}^N_{\mathrm{path}}(\mathfrak{X}^N_{r_N}), o, o, , \dots) \in C([0,\infty); S^{\mathbb{N}}),$$

where o denotes a constant path whose value is denoted by the same symbol o. Here, $o \in S$ is a dummy point that has no importance. From **(C2)**, we see that $\mathbf{X}^N_{r_N}$ under $\underline{\widetilde{\mathfrak{P}}}^N_{r_N}$ is well-defined.
Let $\mathfrak{m}_{R,T}$ be as in (2.15).
(C3) For each $R, T \in \mathbb{N}$ and $\epsilon > 0$, there exists $l \in \mathbb{N}$ such that

$$\sup_{N\in\mathbb{N}} \underline{\widetilde{\mathfrak{P}}}^N_{r_N}(\{\mathfrak{m}_{R,T}(\mathfrak{l}^N_{\mathrm{path}}(\mathfrak{X}^N_{r_N})) > l\}) < 1 - \epsilon.$$

To prove Theorem 3.4, we need the tightness of $\{\mathfrak{X}^N_{r_N}\}_{N\in\mathbb{N}}$ in $W(\mathfrak{S})$. We deduce this from the tightness of $\{\mathbf{X}^N_{r_N}\}_{N\in\mathbb{N}}$ in $C([0,\infty); S^{\mathbb{N}})$. We use **(C3)** for this.

Theorem 3.4 *Assume that **(A2)–(A4)** and **(B1)–(B3)** hold. Assume that either **(B4)**, **(B4)′**, or **(B4)″** is satisfied. Assume that μ and μ^N satisfy **(QG)** with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$ and that $\{O^m_{R,\nu}\}_{R,\nu,m\in\mathbb{N}}$ satisfies (3.12)–(3.16). Finally, assume that **(ZC)** and **(C1)–(C3)** hold. Then, $(\mathfrak{X}^N_{r_N}, \underline{\widetilde{\mathfrak{P}}}^N_{r_N})$ satisfy*

$$\lim_{N\to\infty} \mathbf{X}^N_{r_N} = \mathbf{X} \quad \textit{in law in } C([0,\infty); S^{\mathbb{N}})$$

and, in particular, for each $m \in \mathbb{N}$,

$$\lim_{N\to\infty} (X^{N,1}_{r_N}, \dots, X^{N,m}_{r_N}) = (X^1, \dots, X^m) \quad \textit{in law in } C([0,\infty); S^m). \tag{3.23}$$

Here, $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ is a solution of (2.13) with the initial distribution $(\xi\, d\mu) \circ \mathfrak{l}^{-1}$.

The most critical assumption in Theorems 3.1–3.4 is **(A2)**, which asserts the uniqueness of Dirichlet forms. Combining Theorem 3.4 and Lemma 2.9, we immediately obtain the following.

Corollary 3.2 *Under the same assumptions as for Theorem 3.4 and replacing **(A2)** by **(A5)–(A7)**, the same conclusion as in Theorem 3.4 holds.*

We present an SDE describing finite particle systems in Theorem 3.4.

Assume that μ^N has the logarithmic derivative d_R^N on S_R in the sense of Definition 2.1. Let n_N be the number of particles in $\overline{S}_R$ at time zero. Note that n_N is unchanged as t increases. Hence, we have that $n_N = \mathfrak{X}_{r_N}^N(t)(\overline{S}_{r_N})$. Let

$$\mathfrak{X}_{r_N}^{N,i\diamond}(t) = \sum_{j\neq i}^{n_N} \delta_{X_{r_N}^{N,j}(t)}.$$

Then, we introduce the SDE of $(X_{r_N}^{N,i})_{i=1}^{n_N}$ as follows. For each $1 \le i \le n_N$,

$$\begin{aligned} X_{r_N}^{N,i}(t) - X_{r_N}^{N,i}(0) &= \int_0^t \sigma(X_{r_N}^{N,i}(u), \mathfrak{X}_{r_N}^{N,i\diamond}(u))dB^{N,i}(u) \\ &+ \frac{1}{2}\int_0^t \{\nabla a + a\mathrm{d}_{r_N}^N\}(X_{r_N}^{N,i}(u), \mathfrak{X}_{r_N}^{N,i\diamond}(u))du \\ &+ \frac{1}{2}\int_0^t a(X_{r_N}^{N,i}(u), \mathfrak{X}_{r_N}^{N,i\diamond}(u))\mathbf{n}^{r_N}(X_{r_N}^{N,i}(u))L_{r_N}^{N,i}(du). \end{aligned} \tag{3.24}$$

Here, $\mathbf{n}^{r_N}(x)$ is the inward normal unit vector at $x \in \partial S_{r_N}$ and $\mathrm{d}_{r_N}^N$ is the logarithmic derivative of μ^N on S_{r_N}. Furthermore, $L_{r_N}^{N,i}$ is the local time of $X_{r_N}^{N,i}$ on the boundary ∂S_{r_N}. That is, $L_{r_N}^{N,i}$ is a continuous non-decreasing process such that

$$L_{r_N}^{N,i}(t) = \int_0^t \mathbf{1}_{\partial S_{r_N}}(X_{r_N}^{N,i}(u))L_{r_N}^{N,i}(du).$$

If $r_N = \infty$, then we delete the term $L_{r_N}^{N,i}$ from (3.24). It is easy to show that if d^N is given by a pair of potentials (Φ^N, Ψ^N) such that

$$\mathrm{d}^N(x,\mathfrak{s}) = -\nabla\Phi^N(x) - \sum_i \nabla\Psi^N(x - s_i),$$

then the logarithmic derivative $\mathrm{d}_{r_N}^N(x,\mathfrak{s})$ of μ^N on S_{r_N} is given by

$$\begin{aligned} \mathrm{d}_{r_N}^N(x,\mathfrak{s}) &= -\nabla\Phi^N(x) - \sum_{s_i\in S_{r_N}} \nabla\Psi^N(x - s_i) \\ &- \int_{S\setminus S_{r_N}} \nabla\Psi^N(x-y)\rho^{N,1}(y)dy, \end{aligned} \tag{3.25}$$

where $\rho^{N,1}$ is the one-point correlation function of μ^N, as before. Thus, under the reduced one-Campbell measure $\mu_{r_N}^{N,[1]}$ of $\mu_{r_N}^N = \mu^N \circ \pi_{r_N}^{-1}$, we have that

$$\mathrm{d}_{r_N}^N(x,\mathfrak{s}) = \mathrm{d}^N(x,\mathfrak{s}) - \int_{S\setminus S_{r_N}} \nabla\Psi^N(x-y)\rho^{N,1}(y)dy.$$

From (3.24) and (3.25), we obtain (1.26).

From **(A4)** and **(B1)**, we find that $\mathbf{X}_{r_N}^N$ is a solution of (3.24). Taking this into account, we make the following assumption.

(C4) The uniqueness in law of weak solutions of SDE (3.24) with the initial distribution $\mu_{r_N}^N \circ (\mathfrak{l}^N)^{-1}$ holds under constraints $\mu_{r_N}^N$-**(AC)** and **(NBJ)** for each $N \in \mathbb{N}$.

Here, conditions $\mu_{r_N}^N$-**(AC)** and **(NBJ)** were given in Sect. 2.3. Clearly, we do not need condition **(NBJ)** in **(C4)** if $n_N < \infty$.

From Theorem 3.4, we obtain the convergence of solutions of SDEs.

Corollary 3.3 *Consider the same assumptions as for Theorem 3.4. In addition, we assume that **(C4)** holds. Let $\mathbf{X}^N_{r_N}$ be the solution of (3.24) in **(C4)**. Then, the first m components of $\mathbf{X}^N_{r_N}$ satisfy (3.23) for each $m \in \mathbb{N}$.*

Proof Applying Lemma 2.7 to the Dirichlet form $(\mathscr{E}^N_{r_N}, \widetilde{\mathscr{D}}^N_{r_N})$ on $L^2(\mu^N_{r_N})$, we see that $\mathbf{X}^N_{r_N} = \Gamma^N_{\text{path}}(\mathfrak{X}^N_{r_N})$ is a solution of (3.24). By **(C4)**, all weak solutions are equivalent in law to $\mathbf{X}^N_{r_N}$ if they have common initial distributions. Hence, Corollary 3.3 follows from Theorem 3.4. □

Finally, we strengthen the statement of Theorem 3.1.

Theorem 3.5 *Under the same assumptions as for Theorem 3.4, there exists a non-decreasing sequence $\{r_N\}_{N\in\mathbb{N}}$ in $\mathbb{N} \cup \{\infty\}$ satisfying (3.5) and*

$$\lim_{N\to\infty} \mathfrak{X}^N_{r_N} = \mathfrak{X} \text{ weakly in } W(\mathfrak{S}).$$

4 Generalized Mosco convergence

We now introduce the concept of generalized Mosco convergence in the sense of Kuwae-Shioya [18]. Using this concept, we shall prove the main results in Sect. 3.

Definition 4.1 Let H_N ($N \in \mathbb{N}$) and H be real Hilbert spaces. We say that $\{H_N\}_{N\in\mathbb{N}}$ converges to H if there exists a dense subspace $\mathscr{C} \subset H$ and a sequence of operators

$$\Phi_N : \mathscr{C} \to H_N$$

such that, for any $u \in \mathscr{C}$,

$$\lim_{N\to\infty} ||\Phi_N u||_{H_N} = ||u||_H.$$

Definition 4.2 (1) We say that a sequence $\{u_N\}$ with $u_N \in H_N$ strongly converges to $u \in H$ if there exists $\{\tilde{u}_M\} \subset \mathscr{C}$ such that

$$\lim_{M\to\infty} ||\tilde{u}_M - u||_H = 0, \tag{4.1}$$

$$\lim_{M\to\infty} \limsup_{N\to\infty} ||\Phi_N \tilde{u}_M - u_N||_{H_N} = 0. \tag{4.2}$$

(2) We say that $\{u_N\}$ with $u_N \in H_N$ weakly converges to $u \in H$ if

$$\lim_{N\to\infty} (u_N, v_N)_{H_N} = (u, v)_H \tag{4.3}$$

for any sequence $\{v_N\}$ with $v_N \in H_N$ that strongly converges to $v \in H$.

Definition 4.3 Let $L(H)$ denote the set consisting of linear operators on H. We say that a sequence of bounded operators $\{B_N\}$ with $B_N \in L(H_N)$ strongly converges to an operator $B \in L(H)$ if, for any sequence $\{u_N\}$ with $u_N \in H_N$ that strongly converges to $u \in H$, $\{B_N u_N\}$ strongly converges to Bu.

Let $(\mathscr{E}, \mathscr{D})$ be a non-negative, symmetric bilinear form $\mathscr{E} : \mathscr{D} \times \mathscr{D} \to \mathbb{R}$, where $\mathscr{D}$ is a subspace of the Hilbert space H. We identify the bilinear form $\mathscr{E}(\cdot, *)$ with the function $\mathscr{E}\langle\cdot\rangle$ on H such that

$$\mathscr{E}\langle u\rangle = \begin{cases} \mathscr{E}(u,u), & u \in \mathscr{D}, \\ \infty, & u \notin \mathscr{D}. \end{cases} \tag{4.4}$$

We say that $\mathscr{E}$ is a bilinear form on H if the domain of $\mathscr{E}$ is a subset of H.

Definition 4.4 We say that a sequence of bilinear forms $(\mathscr{E}^N, \mathscr{D}^N)$ on H_N, $N \in \mathbb{N}$, is Mosco convergent to a bilinear form $(\mathscr{E}, \mathscr{D})$ on H if the following conditions hold.
(1) If a sequence $\{u_N\}$ with $u_N \in H_N$ weakly converges to $u \in H$, then

$$\mathscr{E}\langle u\rangle \le \liminf_{N\to\infty} \mathscr{E}^N\langle u_N\rangle. \tag{4.5}$$

(2) For any $u \in H$, there exists a strongly convergent sequence $\lim_{N\to\infty} u_N = u$ with $u_N \in H_N$ such that

$$\mathscr{E}\langle u\rangle = \lim_{N\to\infty} \mathscr{E}^N\langle u_N\rangle. \tag{4.6}$$

Lemma 4.1 ([17]) *Let $(\mathscr{E}^N, \mathscr{D}^N)$ and $(\mathscr{E}, \mathscr{D})$ be Dirichlet forms on $L^2(\mu^N)$ and $L^2(\mu)$, respectively. Let T_t^N and T_t be the associated semi-groups on $L^2(\mu^N)$ and $L^2(\mu)$, respectively. Then, the following are equivalent.*

(1) $\lim_{N\to\infty} \mathscr{E}^N = \mathscr{E}$ *in the sense of Mosco convergence.*
(2) $\lim_{N\to\infty} T_t^N = T_t$ *strongly for all* $t > 0$.

Thus, we see that the Mosco convergence of Dirichlet forms is equivalent to the strong convergence of the associated semi-groups, which implies the convergence of finite-dimensional distributions.

5 Proof of Theorem 3.1

In this section, we prove Theorem 3.1. Throughout this section, we assume that **(A1)**, **(A3)**, and **(B1)**–**(B4)** hold. We utilize the concept of Mosco convergence in Definition 4.4. We take $H_N = L^2(\mu_{r_N}^N)$, $H = L^2(\mu)$, $\mathscr{C} = \mathscr{D}_\circ^\mu$, and $\Phi_N = \chi_N \cdot$ in Definition 4.1, where r_N is given by (5.7) and χ_N is given before (5.39). Then, we have that $\lim_{N\to\infty} H_N = H$ in the sense of Definition 4.1.

We take the Dirichlet forms $\mathscr{E}^N$ and $\mathscr{E}$ in Sect. 3 as the bilinear forms $\mathscr{E}^N$ and $\mathscr{E}$ in Sect. 4. Hereafter, $\mathscr{E}(f)$ and $\mathbb{D}^a[f]$ denote $\mathscr{E}(f, f)$ and $\mathbb{D}^a[f, f]$, respectively. We shall use the same convention for bilinear objects when they appear. The difference between $\mathscr{E}(f)$ and $\mathscr{E}\langle f\rangle$ should be clearly distinguished. Note that $\mathscr{E}(f)$ is only defined for $f \in \mathscr{D}$, while $\mathscr{E}\langle f\rangle$ is defined for all $f \in L^2(\mu)$ and

$$\mathscr{E}\langle f\rangle = \begin{cases} \infty & \text{for } f \notin \mathscr{D} \\ \mathscr{E}(f, f) & \text{for } f \in \mathscr{D}. \end{cases}$$

Let $\{a_{\kappa,R}\}_{\kappa,R\in\mathbb{N}}$ be a sequence of natural numbers satisfying

$$a_{\kappa,R} < a_{\kappa,R+1}, \quad a_{\kappa,R} < a_{\kappa+1,R} \quad \text{for all } \kappa, R \in \mathbb{N}. \tag{5.1}$$

For $\{a_{\kappa,R}\}_{\kappa,R\in\mathbb{N}}$, we set

$$\mathfrak{K}_{\kappa,R} = \{\mathfrak{s} \in \mathfrak{S};\ \mathfrak{s}(S_R) \le a_{\kappa,R}\}, \quad \mathfrak{K}_\kappa = \cap_{R=1}^\infty \mathfrak{K}_{\kappa,R}. \tag{5.2}$$

Lemma 5.1 *There exists a sequence $\{a_{\kappa,R}\}_{\kappa,R\in\mathbb{N}}$ satisfying (5.1) and*

$$\inf_{N\in\mathbb{N}} \mu^N(\mathfrak{K}_\kappa) \ge 1 - \frac{1}{\kappa}, \quad \mu(\mathfrak{K}_\kappa) \ge 1 - \frac{1}{\kappa} \quad \textit{for each } \kappa \in \mathbb{N}. \tag{5.3}$$

Proof Recall that a subset $\mathfrak{A}$ in $\mathfrak{S}$ is relatively compact if and only if there exists an increasing sequence of natural numbers M_R such that $\mathfrak{A} \subset \{\mathfrak{s}; \mathfrak{s}(S_R) \leq M_R \text{ for all } R\}$. Because μ^N converges weakly to μ, $\{\mu^N\}$ is tight. Hence, there exists a sequence $\{a_{\kappa,R}\}_{\kappa,R\in\mathbb{N}}$ satisfying (5.1) and (5.3). □

For $N, \kappa, R, R' \in \mathbb{N}$, we set $c_{11}(N, \kappa, R, R')$ to

$$c_{11}(N,\kappa,R,R') = \max\left\{\left\|\frac{\sigma_r^{N,m}}{\sigma_r^m} - 1\right\|_{S_r^m};\ 1 \leq m \leq a_{\kappa,R},\ 1 \leq r \leq R'\right\}. \tag{5.4}$$

Lemma 5.2 *For each $n \in \mathbb{N}$,*

$$\lim_{N\to\infty} c_{11}(N,n,n,n) = 0. \tag{5.5}$$

Proof From **(B4)**, we have that $\lim_{N\to\infty} c_{11}(N,\kappa,R,R') = 0$ for each $\kappa, R, R' \in \mathbb{N}$. Then, taking $\kappa = R = R' = n$, we have (5.5). □

For each $n \in \mathbb{N}$, let

$$N_n = \min\{N;\ c_{11}(N',n,n,n) \leq 2^{-n} \text{ for all } N' \geq N\}. \tag{5.6}$$

From (5.5) and (5.6), we find that $N_n < \infty$ for each $n \in \mathbb{N}$. Furthermore, it is easy to see that $\{N_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence. Hence, we denote the inverse function of N_n on $\mathbb{N}$ as r_N . Indeed, we take $r_1 = 1$ for $N < N_1$ and

$$r_N = 1 + \sup\{n \in \mathbb{N};\ N_n \leq N \leq N_{n+1}\}. \tag{5.7}$$

If $\{N_n\}_{n\in\mathbb{N}}$ is bounded, then we set $r_N = \infty$ for $N \geq \sup\{N_n; n \in \mathbb{N}\}$. We set

$$c_{12}(N) = \begin{cases} c_{11}(N, r_N, r_N, r_N) & \text{if } r_N < \infty, \\ 0 & \text{if } r_N = \infty. \end{cases} \tag{5.8}$$

Lemma 5.3 *The sequences r_N and $c_{12}(N)$, $N \in \mathbb{N}$, satisfy*

$$\lim_{N\to\infty} r_N = \infty, \tag{5.9}$$

$$\lim_{N\to\infty} c_{12}(N) = 0. \tag{5.10}$$

Proof It is sufficient to assume that $r_N < \infty$ for all $N \in \mathbb{N}$. From (5.5), we can deduce that $N_n < \infty$. From this and (5.7), we have (5.9). From (5.8), we deduce

$$c_{12}(N) = c_{11}(N, r_N, r_N, r_N) = c_{11}(N, n, n, n) \leq 2^{-n} = 2^{-r_N}. \tag{5.11}$$

From (5.9) and (5.11), we obtain (5.10). □

5.1 Lower schemes of Dirichlet forms

In Sect. 5.1, we check Definition 4.4 (1).

Let $\mathfrak{K}_{\kappa,R}$ and r_N be as in (5.2) and (5.7), respectively. We set

$$\widehat{f} = 1_{\mathfrak{K}_{r_N,r_N}} f. \tag{5.12}$$

Lemma 5.4 *Assume that* $c_{12}(N) \le 1/2$ *and* $f \in \widetilde{\mathscr{D}}^N_{r_N}$. *Then,* $\widehat{f} \in \widetilde{\mathscr{D}}^N_{r_N}$ *and*

$$|\mathscr{E}_{r_N}(\widehat{f}) - \mathscr{E}^N_{r_N}(\widehat{f})| \le c_{12}(N)\, \mathscr{E}_{r_N}(\widehat{f}). \tag{5.13}$$

Proof Because $f \in \widetilde{\mathscr{D}}^N_{r_N}$, we have that f is $\sigma[\pi_{\overline{S}_{r_N}}]$-measurable. Hence, $\widehat{f}$ is also $\sigma[\pi_{\overline{S}_{r_N}}]$-measurable. Using this and (5.12), we have

$$\mathscr{E}^N_{r_N}(\widehat{f}) \le \mathscr{E}^N_{r_N}(f) < \infty. \tag{5.14}$$

This implies that $\widehat{f} \in \widetilde{\mathscr{D}}^N_{r_N}$. Note that $\widetilde{\mathscr{D}}^N_{r_N} \cap \mathscr{D}_\circ$ is dense in $\widetilde{\mathscr{D}}^N_{r_N}$ with respect to

$$\mathscr{E}^N_{r_N,(1)} := \mathscr{E}^N_{r_N} + (\cdot,*)_{L^2(\mu^N_{r_N})}.$$

Then, for each $f \in \widetilde{\mathscr{D}}^N_{r_N}$, we have an $\mathscr{E}^N_{r_N,(1)}$-Cauchy sequence $\{f_p\}$ in $\widetilde{\mathscr{D}}^N_{r_N} \cap \mathscr{D}_\circ$ such that

$$\lim_{p\to\infty} \mathscr{E}^N_{r_N,(1)}(f_p - f) = 0. \tag{5.15}$$

Note that $\widehat{f_p} - \widehat{f} = \widehat{f_p - f}$. Then from (5.12), (5.14), and (5.15), we deduce that $\{\widehat{f_p}\}$ is an $\mathscr{E}^N_{r_N,(1)}$-Cauchy sequence satisfying

$$\lim_{p\to\infty} \mathscr{E}^N_{r_N,(1)}(\widehat{f_p} - \widehat{f}) = \lim_{p\to\infty} \mathscr{E}^N_{r_N,(1)}(\widehat{f_p - f}) \le \lim_{p\to\infty} \mathscr{E}^N_{r_N,(1)}(f_p - f) = 0. \tag{5.16}$$

Hence, we have, from (5.14) and (5.16),

$$\lim_{p\to\infty} \mathscr{E}^N_{r_N,(1)}(\widehat{f_p}) = \mathscr{E}^N_{r_N,(1)}(\widehat{f}) < \infty. \tag{5.17}$$

Next, we assume that $f \in \widetilde{\mathscr{D}}^N_{r_N} \cap \mathscr{D}_\circ$. Then, from (5.12), we have that

$$\begin{aligned}
\mathscr{E}^N_{r_N}(\widehat{f}) &= \sum_{m=1}^{\infty} \frac{1}{m!} \int_{S^m_{r_N}} \mathbb{D}^a[\widehat{f}]\sigma^{N,m}_{r_N} d\mathbf{x}_m \\
&= \sum_{m=1}^{a_{r_N,r_N}} \frac{1}{m!} \int_{S^m_{r_N}} \mathbb{D}^a[f]\sigma^{N,m}_{r_N} d\mathbf{x}_m \\
&= \sum_{m=1}^{a_{r_N,r_N}} \frac{1}{m!} \int_{S^m_{r_N}} \mathbb{D}^a[f]\sigma^m_{r_N} \left\{ \frac{\sigma^{N,m}_{r_N}}{\sigma^m_{r_N}} - 1 \right\} d\mathbf{x}_m + \mathscr{E}_{r_N}(\widehat{f}).
\end{aligned}$$

Together with (5.4) and (5.8), this implies that

$$\begin{aligned}
|\mathscr{E}_{r_N}(\widehat{f}) - \mathscr{E}^N_{r_N}(\widehat{f})| &= \left| \sum_{m=1}^{a_{r_N,r_N}} \frac{1}{m!} \int_{S^m_{r_N}} \mathbb{D}^a[f]\sigma^m_{r_N} \left\{ \frac{\sigma^{N,m}_{r_N}}{\sigma^m_{r_N}} - 1 \right\} d\mathbf{x}_m \right| \\
&\le c_{12}(N)\, \mathscr{E}_{r_N}(\widehat{f}).
\end{aligned} \tag{5.18}$$

Applying (5.18) to $f_p - f_q$ and noting that $\widehat{f_p - f_q} = \widehat{f_p} - \widehat{f_q}$, we obtain

$$|\mathscr{E}_{r_N}(\widehat{f_p} - \widehat{f_q}) - \mathscr{E}^N_{r_N}(\widehat{f_p} - \widehat{f_q})| \le c_{12}(N)\, \mathscr{E}_{r_N}(\widehat{f_p} - \widehat{f_q}).$$

Using this, $c_{12}(N) \le 1/2$, the fact that $\{\widehat{f_p}\}$ is an $\mathscr{E}^N_{r_N,(1)}$-Cauchy sequence satisfying (5.16), and **(B4)**, we deduce that $\{\widehat{f_p}\}$ is an $\mathscr{E}_{r_N,(1)}$-Cauchy sequence satisfying

$$\lim_{p\to\infty} \mathscr{E}_{r_N,(1)}(\widehat{f_p} - \widehat{f}) = 0.$$

Hence, we have

$$\lim_{p\to\infty} \mathscr{E}_{r_N,(1)}(\widehat{f}_p) = \mathscr{E}_{r_N,(1)}(\widehat{f}) < \infty. \tag{5.19}$$

Applying (5.18) to $\{f_p\}$, we have

$$|\mathscr{E}_{r_N}(\widehat{f}_p) - \mathscr{E}^N_{r_N}(\widehat{f}_p)| \le c_{12}(N)\,\mathscr{E}_{r_N}(\widehat{f}_p). \tag{5.20}$$

Using (5.17) and (5.19) in (5.20), we obtain (5.13) for $\widehat{f}$. □

Proposition 5.1 *Assume that $\{f_N\}_{N\in\mathbb{N}}$ with $f_N \in L^2(\mu^N_{r_N})$ weakly converges to $f \in L^2(\mu)$ in the sense of Definition 4.2. Then, $(\mathscr{E}, \underline{\mathscr{D}})$ and $\{(\mathscr{E}^N_{r_N}, \underline{\widetilde{\mathscr{D}}}^N_{r_N})\}_{N\in\mathbb{N}}$ satisfy*

$$\mathscr{E}\langle f\rangle \le \liminf_{N\to\infty} \mathscr{E}^N_{r_N}\langle f_N\rangle. \tag{5.21}$$

Proof If $\liminf_{N\to\infty} \mathscr{E}^N_{r_N}\langle f_N\rangle = \infty$, then (5.21) is obvious. Hence, we assume that

$$\liminf_{N\to\infty} \mathscr{E}^N_{r_N}\langle f_N\rangle < \infty.$$

From this and (4.4), we have

$$\liminf_{N\to\infty} \mathscr{E}^N_{r_N}(f_N) < \infty. \tag{5.22}$$

Assume that $c_{12}(N) \le 1/2$ and $f_N \in \underline{\widetilde{\mathscr{D}}}^N_{r_N}$. Using Lemma 5.4, we have from (5.13) that

$$(1 - c_{12}(N))\mathscr{E}_{r_N}(\widehat{f}_N) \le \mathscr{E}^N_{r_N}(\widehat{f}_N) \le (1 + c_{12}(N))\mathscr{E}_{r_N}(\widehat{f}_N). \tag{5.23}$$

Note that $\mathscr{E}^N_{r_N}(f_N) < \infty$ implies $f_N \in \underline{\widetilde{\mathscr{D}}}^N_{r_N}$. Hence, using (5.22), we deduce that $f_N \in \underline{\widetilde{\mathscr{D}}}^N_{r_N}$ for infinitely many N. From this, we see that (5.23) holds for infinitely many N. Combining this with (5.10) and (5.14), we deduce from (5.23) that

$$\liminf_{N\to\infty} \mathscr{E}_{r_N}(\widehat{f}_N) = \liminf_{N\to\infty} \mathscr{E}^N_{r_N}(\widehat{f}_N) \le \liminf_{N\to\infty} \mathscr{E}^N_{r_N}(f_N) < \infty. \tag{5.24}$$

Because $\widehat{f}_N$ is $\sigma[\pi_{\overline{S}_{r_N}}]$-measurable, we see that $\mathscr{E}(\widehat{f}_N) = \mathscr{E}_{r_N}(\widehat{f}_N)$. Hence, (5.24) yields

$$\liminf_{N\to\infty} \mathscr{E}(\widehat{f}_N) < \infty. \tag{5.25}$$

Furthermore, from (5.4) and (5.8), we obtain

$$(1 - c_{12}(N))\|\widehat{f}_N\|_{L^2(\mu)} \le \|\widehat{f}_N\|_{L^2(\mu^N_{r_N})} \le (1 + c_{12}(N))\|\widehat{f}_N\|_{L^2(\mu)}.$$

Combining this with (5.10), we deduce

$$\limsup_{N\to\infty} \|\widehat{f}_N\|_{L^2(\mu)} = \limsup_{N\to\infty} \|\widehat{f}_N\|_{L^2(\mu^N_{r_N})}. \tag{5.26}$$

Because $\{f_N\}$ with $f_N \in L^2(\mu^N)$ weakly converges to $f \in L^2(\mu)$, we have

$$\sup_{N\in\mathbb{N}} \|f_N\|_{L^2(\mu^N_{r_N})} < \infty. \tag{5.27}$$

Clearly, $\|\widehat{f}_N\|_{L^2(\mu^N_{r_N})} \le \|f_N\|_{L^2(\mu^N_{r_N})}$. Hence, we have from (5.26) and (5.27) that

$$\limsup_{N\to\infty} \|\widehat{f}_N\|_{L^2(\mu)} \le \sup_{N\in\mathbb{N}} \|f_N\|_{L^2(\mu^N_{r_N})} < \infty. \tag{5.28}$$

We set $\mathscr{E}_{(p)}(\cdot,\cdot)=\mathscr{E}(\cdot,\cdot)+p^{-1}(\cdot,\cdot)_{L^2(\mu)}$ for $p\in\mathbb{N}$. Then, $\underline{\mathscr{D}}$ is a Hilbert space with the inner product $\mathscr{E}_{(p)}$ for each $p\in\mathbb{N}$. From (5.25) and (5.28), we deduce

$$\liminf_{N\to\infty}\mathscr{E}_{(p)}(\widehat{f}_N)<\infty.$$

Hence, we can choose an $\mathscr{E}_{(p)}$-bounded subsequence from $\{\widehat{f}_N\}$.

We can choose an $\mathscr{E}_{(p)}$-weak convergent subsequence with the limit f from an arbitrary, $\mathscr{E}_{(p)}$-bounded subsequence $\{\widehat{f}_{N'}\}$ of $\{\widehat{f}_N\}$. Furthermore, we obtain $\lim_{N\to\infty}1_{\mathfrak{K}_{r_N,r_N}}(\mathfrak{s})=1$ for μ-a.s.$\mathfrak{s}$ from (5.3). Hence, for μ-a.s.$\mathfrak{s}$, we have that

$$f(\mathfrak{s})=\lim_{N\to\infty}\widehat{f}_N(\mathfrak{s}),\quad \mathbb{D}^a[\lim_{N\to\infty}\widehat{f}_N](\mathfrak{s})=\lim_{N\to\infty}\mathbb{D}^a[\widehat{f}_N](\mathfrak{s}).$$

Collecting these results together, we can deduce for all $p\in\mathbb{N}$ that

$$\mathscr{E}_{(p)}(f)=\mathscr{E}_{(p)}(\lim_{N\to\infty}\widehat{f}_N)\le\liminf_{N\to\infty}\mathscr{E}_{(p)}(\widehat{f}_N). \tag{5.29}$$

Then, from (5.28) and (5.29), we obtain

$$\begin{aligned}\mathscr{E}(f)\le\mathscr{E}_{(p)}(f)&\le\liminf_{N\to\infty}\left\{\mathscr{E}(\widehat{f}_N)+\frac{1}{p}\|\widehat{f}_N\|^2_{L^2(\mu)}\right\}\\&\le\liminf_{N\to\infty}\mathscr{E}(\widehat{f}_N)+\frac{1}{p}\limsup_{N\to\infty}\|\widehat{f}_N\|^2_{L^2(\mu)}.\end{aligned} \tag{5.30}$$

Note that (5.30) holds for all $p\in\mathbb{N}$. From (5.28) and (5.30), we then obtain

$$\mathscr{E}(f)\le\liminf_{N\to\infty}\mathscr{E}(\widehat{f}_N). \tag{5.31}$$

Putting these together, we obtain

$$\begin{aligned}\mathscr{E}(f)&\le\liminf_{N\to\infty}\mathscr{E}(\widehat{f}_N)\quad\text{by (5.31)}\\&=\liminf_{N\to\infty}\mathscr{E}_{r_N}(\widehat{f}_N)\quad\text{by the }\sigma[\pi_{\overline{S}_{r_N}}]-\text{measurability of }\widehat{f}_N\\&\le\liminf_{N\to\infty}\mathscr{E}^N_{r_N}(f_N)\quad\text{by (5.24).}\end{aligned} \tag{5.32}$$

Finally, we have (5.21) from (5.32). □

5.2 Upper schemes of Dirichlet forms and proof of Theorem 3.1

In Sect. 5.2, we check Definition 4.4 (2). Let $a_{\kappa,R}$ be as in (5.1). For $a_{\kappa,R}$ and a label $\mathfrak{l}=(\mathfrak{l}_j)_j$ satisfying $|\mathfrak{l}_j(\mathfrak{s})|\le|\mathfrak{l}_{j+1}(\mathfrak{s})|$ for all j, we set

$$\begin{aligned}J_{\kappa,R}(\mathfrak{s})&=\{j\,;\,j>a_{\kappa,R},\mathfrak{l}_j(\mathfrak{s})\in S_R\},\\d_{\kappa,R}(\mathfrak{s})&=\Big\{\sum_{j\in J_{\kappa,R}(\mathfrak{s})}(R-|\mathfrak{l}_j(\mathfrak{s})|)^2\Big\}^{\frac{1}{2}}.\end{aligned}$$

Let $\rho\in C^\infty(\mathbb{R})$ be a function satisfying $\rho(t)\in[0,1]$ for any $t\in\mathbb{R}$, $\rho(t)=1$ for $t\le 0$, $\rho(t)=0$ for $t\ge 1$, and $\rho'(t)\le\sqrt{2}$ for any $t\in\mathbb{R}$. We set

$$\chi_{\kappa,R}(\mathfrak{s})=\rho\circ d_{\kappa,R}(\mathfrak{s}).$$

Let $\mathfrak{K}_{\kappa,R}=\{\mathfrak{s}\in\mathfrak{S}\,;\,\mathfrak{s}(S_R)\le a_{\kappa,R}\}$ be as in (5.2). We set

$$\mathfrak{K}^+_{\kappa,R}=\{\mathfrak{s}\in\mathfrak{S}\,;\,\mathfrak{s}(S_{R-1})\le a_{\kappa,R}\}.$$

Then, $\mathfrak{K}_{\kappa,R} \subset \mathfrak{K}^+_{\kappa,R}$ by construction. The next lemma shows that $\chi_{\kappa,R}$ is a cut-off function of $\mathfrak{K}_{\kappa,R}$. Let $\mathscr{E}_{(1)}(f,g) = \mathscr{E}(f,g) + (f,g)_{L^2(\mu)}$. We can prove Lemma 5.5 in a similar manner to [24, Lemma 2.5].

Lemma 5.5 *Let c_5 be as in (2.1). For each κ, $R \in \mathbb{N}$, the following hold:*

(1) $\chi_{\kappa,R} f \in \mathscr{D}^\mu_\circ$ *for each* $f \in \mathscr{D}^\mu_\circ$.
(2) $\chi_{\kappa,R} = 1$ *on* $\mathfrak{K}_{\kappa,R}$ *and* $\chi_{\kappa,R} = 0$ *on* $\mathfrak{S}\backslash\mathfrak{K}^+_{\kappa,R}$.
(3) $\mathbb{D}^a[\chi_{\kappa,R}] = 0$ *on* $(\mathfrak{K}^+_{\kappa,R}\backslash\mathfrak{K}_{\kappa,R})^c$ *and* $0 \le \mathbb{D}^a[\chi_{\kappa,R}] \le c_5$ *on* $\mathfrak{K}^+_{\kappa,R}\backslash\mathfrak{K}_{\kappa,R}$.
(4) $\mathscr{E}_{(1)}(\chi_{\kappa,R} f) \le 2\mathscr{E}(f) + 3c_5\|f\|^2_{L^2(\mu)}$ *for each* $f \in \mathscr{D}^\mu_\circ$.

Proof A straightforward calculation shows that

$$\begin{aligned}\mathbb{D}^a[\chi_{\kappa,R}] &\le c_5\mathbb{D}[\chi_{\kappa,R}]\\ &= \frac{c_5}{2}\left\{\frac{\rho'(d_{\kappa,R}(\mathfrak{s}))}{d_{\kappa,R}(\mathfrak{s})}\right\}^2 \sum_{j\in J_{\kappa,R}(\mathfrak{s})} (R - |\mathfrak{l}_j(\mathfrak{s})|)^2\\ &= \frac{c_5}{2}\rho'(d_{\kappa,R}(\mathfrak{s}))^2 \le c_5. \end{aligned} \tag{5.33}$$

From (5.33), we have

$$\begin{aligned}\mathbb{D}^a[\chi_{\kappa,R} f] &= \chi^2_{\kappa,R}\mathbb{D}^a[f] + f^2\mathbb{D}^a[\chi_{\kappa,R}] + 2\chi_{\kappa,R} f\, \mathbb{D}^a[\chi_{\kappa,R}, f]\\ &\le 2\{\chi^2_{\kappa,R}\mathbb{D}^a[f] + f^2\mathbb{D}^a[\chi_{\kappa,R}]\}\\ &\le 2\{\mathbb{D}^a[f] + c_5 f^2\}. \end{aligned} \tag{5.34}$$

From (5.34), we see that $\mathscr{E}(\chi_{\kappa,R} f) < \infty$, which implies (1). (2) is clear by construction. (3) follows from (2) and (5.33). (4) follows from (5.34). □

Let $(\mathscr{E}^N, \mathscr{D}^N_\circ)$ be as in (3.1). Then, using Lemma 2.3 (1), we see from **(B1)** (1) that $(\mathscr{E}^N, \mathscr{D}^N_\circ)$ is closable on $L^2(\mu^N)$. We denote the closure of $(\mathscr{E}^N, \mathscr{D}^N_\circ)$ on $L^2(\mu^N)$ as $(\mathscr{E}^N, \mathscr{D}^N)$.

Proposition 5.2 *Let r_N be as in (5.7). For each $f \in L^2(\mu)$, there exists a sequence $\{g_N\}_{N\in\mathbb{N}}$ satisfying the following:*

$$g_N \text{ is } \sigma[\pi_{r_N}] - \text{measurable}, \limsup_{N\to\infty} \|g_N\|_{L^2(\mu^N_{r_N})} < \infty, \tag{5.35}$$

$$\lim_{N\to\infty} g_N = f \text{ strongly in the sense of Definition 4.2}, \tag{5.36}$$

$$\lim_{N\to\infty} \mathscr{E}^N\langle g_N\rangle = \mathscr{E}\langle f\rangle. \tag{5.37}$$

Proof If $f \notin \mathscr{D}$, then $\mathscr{E}\langle f\rangle = \infty$ and we can easily take g_N so as to satisfy (5.35)–(5.37).

We next suppose that $f \in \mathscr{D}$. Then, $\mathscr{E}\langle f\rangle = \mathscr{E}(f) < \infty$. Furthermore, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathscr{D}^\mu_\circ$ such that

$$\lim_{n\to\infty} \mathscr{E}_{(1)}(f_n - f) = 0. \tag{5.38}$$

Because each $f_n \in \mathscr{D}^\mu_\circ$ is a local function, we can assume that f_n is $\sigma[\pi_n]$-measurable.

Let $\{q_N\}$ be a non-decreasing sequence of natural numbers such that $q_N + 1 \le r_N$ for $r_N \ge 2$ and $\lim_{N\to\infty} q_N = \infty$. We set $\chi_N = \chi_{q_N+1,q_N}$. Let

$$g_N = \chi_N f_{q_N}. \tag{5.39}$$

Because χ_N and f_{q_N} are $\sigma[\pi_{q_N}]$-measurable and $q_N + 1 \le r_N$, we find that g_N is $\sigma[\pi_{r_N}]$-measurable. From $f_{q_N} \in L^2(\mu)$, we have that $g_N \in L^2(\mu)$. Combining these results with **(B4)**, we find that $\limsup_{N\to\infty} \|g_N\|_{L^2(\mu^N_{r_N})} < \infty$. Thus, we obtain (5.35).

We next check (5.36). Recall that we take $\mathscr{C} = \mathscr{D}_\circ^\mu$. For $g_N = \chi_N f_{q_N}$, we take $\widetilde{g}_M = f_{q_M}$. Then, $\widetilde{g}_M \in \mathscr{C}$. Using (5.38) and $\lim_{M\to\infty} q_M = \infty$, we have

$$\lim_{M\to\infty} \|\widetilde{g}_M - f\|^2_{L^2(\mu)} = \lim_{M\to\infty} \|f_{q_M} - f\|^2_{L^2(\mu)} = 0. \tag{5.40}$$

Let $\Phi_N(\cdot) = \chi_N \cdot$. Then, $\Phi_N(\widetilde{g}_M) = \Phi_N(f_{q_M}) = \chi_N f_{q_M}$ by construction and (5.39). Hence, from this, (5.4), and Lemma 5.5, we have that

$$\begin{aligned}
&\lim_{M\to\infty} \limsup_{N\to\infty} \|\Phi_N(\widetilde{g}_M) - g_N\|_{L^2(\mu^N_{r_N})} \\
&\quad = \lim_{M\to\infty} \limsup_{N\to\infty} \|\chi_N f_{q_M} - \chi_N f_{q_N}\|_{L^2(\mu^N_{r_N})} \quad \text{by definition} \\
&\quad \le \lim_{M\to\infty} \limsup_{N\to\infty} c_{11}(N, q_N+1, q_N+1, q_N) \|\chi_N f_{q_M} - \chi_N f_{q_N}\|_{L^2(\mu)} \text{ by (5.4)} \\
&\quad \le \lim_{M\to\infty} \limsup_{N\to\infty} c_{12}(N) \|\chi_N f_{q_M} - \chi_N f_{q_N}\|_{L^2(\mu)} = 0.
\end{aligned} \tag{5.41}$$

Here, we have used $c_{11}(N, q_N+1, q_N+1, q_N) \le c_{12}(N)$, which follows from $q_N + 1 \le r_N$. The last equality follows from Lemma 5.5 and (5.38).

Thus, we see that $\{g_N\}$ and $\{\widetilde{g}_M\}$ satisfy (4.1) and (4.2) from (5.40) and (5.41), respectively. We have already checked $\widetilde{g}_M \in \mathscr{C}$. Hence, we obtain (5.36).

By Lemma 5.5, (5.38), and (5.39), we have that

$$\sup_{N\in\mathbb{N}} \mathscr{E}_{(1)}(g_N) = \sup_{N\in\mathbb{N}} \mathscr{E}_{(1)}(\chi_N f_{q_N}) < \infty. \tag{5.42}$$

Obviously, we have

$$|\mathscr{E}^N(g_N) - \mathscr{E}(f)| \le |\mathscr{E}^N(g_N) - \mathscr{E}(g_N)| + |\mathscr{E}(g_N) - \mathscr{E}(f)|. \tag{5.43}$$

From Lemma 5.3, (5.4) and (5.42), we see that

$$\begin{aligned}
|\mathscr{E}^N(g_N) - \mathscr{E}(g_N)| &\le c_{11}(N, q_N+1, q_N+1, q_N)\mathscr{E}(g_N) \\
&\le c_{12}(N)\mathscr{E}(g_N) \xrightarrow[N\to\infty]{} 0.
\end{aligned} \tag{5.44}$$

By a straightforward calculation, Lemma 5.5, (5.40), and (5.42) imply that

$$\begin{aligned}
|\mathscr{E}(g_N) - \mathscr{E}(f)| &= \left| \int_{\mathfrak{S}} \mathbb{D}^a[\chi_N f_{q_N}] - \mathbb{D}^a[f] d\mu \right| \\
&= \left| \int_{\mathfrak{S}} \mathbb{D}^a[\chi_N] f_{q_N}^2 + 2\mathbb{D}^a[\chi_N, f_{q_N}]\chi_N f_{q_N} + \chi_N^2 \mathbb{D}^a[f_{q_N}] - \mathbb{D}^a[f] d\mu \right| \\
&\le \left| \int_{\mathfrak{S}} \mathbb{D}^a[\chi_N] f_{q_N}^2 + 2\mathbb{D}^a[\chi_N, f_{q_N}] d\mu \right| + \left| \int_{\mathfrak{S}} \chi_N^2 \mathbb{D}^a[f_{q_N}] - \mathbb{D}^a[f] d\mu \right| \\
&\xrightarrow[N\to\infty]{} 0.
\end{aligned} \tag{5.45}$$

From (5.43), (5.44), and (5.45), we obtain (5.37). This completes the proof. □

Proof of Theorem 3.1 Assume that $\{f_N\}_{N\in\mathbb{N}}$ with $f_N \in L^2(\mu_{r_N}^N)$ weakly converges to $f \in L^2(\mu)$ in the sense of Definition 4.2. From Proposition 5.1, we see that $(\mathscr{E}, \underline{\mathscr{D}})$ and $\{(\mathscr{E}_{r_N}^N, \underline{\widetilde{\mathscr{D}}}_{r_N}^N)\}_{N\in\mathbb{N}}$ satisfy

$$\mathscr{E}\langle f\rangle \le \liminf_{N\to\infty} \mathscr{E}_{r_N}^N\langle f_N\rangle. \tag{5.46}$$

By **(A2)**, we have $(\mathscr{E}, \mathscr{D}) = (\mathscr{E}, \underline{\mathscr{D}})$. From this and (5.46), we see that $(\mathscr{E}, \mathscr{D})$ and $\{(\mathscr{E}_{r_N}^N, \underline{\widetilde{\mathscr{D}}}_{r_N}^N)\}_{N\in\mathbb{N}}$ satisfy (4.5). Thus, we obtain Definition 4.4 (1).

Let g_N be as in Proposition 5.2. From (5.37), we see that

$$\mathscr{E}\langle f\rangle = \lim_{N\to\infty} \mathscr{E}^N\langle g_N\rangle. \tag{5.47}$$

Because g_N is $\sigma[\pi_{r_N}]$-measurable, we find that $(\mathscr{E}_{r_N}^N, \underline{\widetilde{\mathscr{D}}}_{r_N}^N)$ satisfies

$$\mathscr{E}^N\langle g_N\rangle = \mathscr{E}_{r_N}^N\langle g_N\rangle. \tag{5.48}$$

Combining (5.47) and (5.48), we obtain

$$\mathscr{E}\langle f\rangle = \lim_{N\to\infty} \mathscr{E}^N\langle g_N\rangle = \lim_{N\to\infty} \mathscr{E}_{r_N}^N\langle g_N\rangle.$$

Hence, we obtain Definition 4.4 (2) for $(\mathscr{E}, \mathscr{D})$ and $\{(\mathscr{E}_{r_N}^N, \underline{\widetilde{\mathscr{D}}}_{r_N}^N)\}_{N\in\mathbb{N}}$.

Thus, the Mosco convergence in Definition 4.4 holds for $(\mathscr{E}, \mathscr{D})$ and $\{(\mathscr{E}_{r_N}^N, \underline{\widetilde{\mathscr{D}}}_{r_N}^N)\}_{N\in\mathbb{N}}$. The Mosco convergence of Dirichlet forms implies the strong convergence of the associated L^2-semi-groups, which yields the convergence of finite-dimensional distributions of $\mathfrak{X}_{r_N}^N$ to $\mathfrak{X}$ (see [18, Section 7]).

6 Cut-off Dirichlet forms in infinite volumes

In this section, we construct schemes of cut-off Dirichlet forms in infinite volumes. We shall use these schemes in Sect. 7 to prove Theorems 3.2–3.5.

Let $\mathfrak{O}_{R,\nu}$ be as in (3.11). We set

$$\mathscr{E}_{R,\nu}(f, g) = \int_{\mathfrak{O}_{R,\nu}} \mathbb{D}_R^a[f, g]\, d\mu. \tag{6.1}$$

Lemma 6.1 *Assume that μ satisfies **(QG)** with $\{O_{R,\nu}^m\}_{R,m\in\mathbb{N}}$. Assume that (3.12) holds. Then, $(\mathscr{E}_{R,\nu}, \mathscr{D}_\circ^\mu)$ is closable on $L^2(\mu)$.*

Proof Using Lemma 2.10, we see that $(\mathscr{E}_R^{\mathfrak{O}_\nu}, \mathscr{D}_\circ^{\mathfrak{O}_\nu})$ is closable on $L^2(\mu)$. From (3.12) and (6.1), we have $\mathscr{E}_R^{\mathfrak{O}_\nu} = \mathscr{E}_{R,\nu}$. Hence, we find that $(\mathscr{E}_{R,\nu}, \mathscr{D}_\circ^{\mathfrak{O}_\nu})$ is closable on $L^2(\mu)$. Combining this with $\mathscr{D}_\circ^\mu \subset \mathscr{D}_\circ^{\mathfrak{O}_\nu}$, we conclude that $(\mathscr{E}_{R,\nu}, \mathscr{D}_\circ^\mu)$ is closable on $L^2(\mu)$. □

6.1 Lower schemes of Dirichlet forms in infinite volumes

We assume that (3.12)–(3.14) hold. From Lemma 6.1, we see that $(\mathscr{E}_{R,\nu}, \mathscr{D}_\circ^\mu)$ is closable on $L^2(\mu)$. Hence, we denote the closure of $(\mathscr{E}_{R,\nu}, \mathscr{D}_\circ^\mu)$ on $L^2(\mu)$ as $(\mathscr{E}_{R,\nu}, \underline{\mathscr{D}}_{R,\nu})$. Then, we deduce from (3.12) that, for each $\nu \in \mathbb{N}$,

$$(\mathscr{E}_{R,\nu}, \underline{\mathscr{D}}_{R,\nu}) \le (\mathscr{E}_{R',\nu}, \underline{\mathscr{D}}_{R',\nu}) \quad \text{for } R \le R'. \tag{6.2}$$

From (3.13), it is easy to show that, for each $R \in \mathbb{N}$,

$$(\mathscr{E}_{R,\nu}, \underline{\mathscr{D}}_{R,\nu}) \le (\mathscr{E}_{R,\nu'}, \underline{\mathscr{D}}_{R,\nu'}) \quad \text{for } \nu \le \nu'. \tag{6.3}$$

From (6.2) and (6.3), we have that, for $f \in \bigcap_{R,\nu\in\mathbb{N}} \underline{\mathscr{D}}_{R,\nu}$,

$$\lim_{R\to\infty} \left\{ \lim_{\nu\to\infty} \mathscr{E}_{R,\nu}(f,f) \right\} = \lim_{\nu\to\infty} \left\{ \lim_{R\to\infty} \mathscr{E}_{R,\nu}(f,f) \right\}. \tag{6.4}$$

For $f, g \in \mathscr{D}_\circ^\mu$, we set

$$\mathscr{E}_{\infty,\nu}(f,g) = \int_{\mathfrak{O}_\nu} \mathbb{D}^a[f,g](\mathfrak{s})\, d\mu.$$

Because $\mathscr{E}_{\infty,\nu}(f,f) = \mathscr{E}_{R,\nu}(f,f)$ for $f \in \mathscr{D}_\circ^\mu \cap \mathscr{B}_R$, we naturally extend the domain of $\mathscr{E}_{\infty,\nu}$ to $\underline{\mathscr{D}}_{R,\nu}$. By (6.2), we see that $\{(\mathscr{E}_{R,\nu}, \underline{\mathscr{D}}_{R,\nu})\}_{R\in\mathbb{N}}$ is increasing for each $\nu \in \mathbb{N}$. Hence, we set the closed form $(\mathscr{E}_{\infty,\nu}, \underline{\mathscr{D}}_{\infty,\nu})$ on $L^2(\mu)$ such that

$$\begin{aligned} &\mathscr{E}_{\infty,\nu}(f,f) = \lim_{R\to\infty} \mathscr{E}_{R,\nu}(f,f) \quad \text{for } f \in \underline{\mathscr{D}}_{\infty,\nu}, \\ &\underline{\mathscr{D}}_{\infty,\nu} = \left\{ f \in \bigcap_{R\in\mathbb{N}} \underline{\mathscr{D}}_{R,\nu} ;\ \lim_{R\to\infty} \mathscr{E}_{R,\nu}(f,f) < \infty \right\}. \end{aligned} \tag{6.5}$$

From (6.2), (6.3), and (6.5), we see that the sequence of the forms $(\mathscr{E}_{\infty,\nu}, \underline{\mathscr{D}}_{\infty,\nu})$ is increasing in $\nu \in \mathbb{N}$. Hence, we define the closed form $(\mathscr{E}_{\infty,\infty}, \underline{\mathscr{D}}_{\infty,\infty})$ as

$$\begin{aligned} &\mathscr{E}_{\infty,\infty}(f,f) = \lim_{\nu\to\infty} \mathscr{E}_{\infty,\nu}(f,f) \quad \text{for } f \in \bigcap_{\nu=1}^{\infty} \underline{\mathscr{D}}_{\infty,\nu}, \\ &\underline{\mathscr{D}}_{\infty,\infty} = \left\{ f \in \bigcap_{\nu=1}^{\infty} \underline{\mathscr{D}}_{\infty,\nu} ;\ \lim_{\nu\to\infty} \mathscr{E}_{\infty,\nu}(f,f) < \infty \right\}. \end{aligned} \tag{6.6}$$

Lemma 6.2 *Assume that μ satisfies **(QG)** with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$ and that $\{O^m_{R,\nu}\}_{R,\nu,m\in\mathbb{N}}$ satisfies (3.12)–(3.16). Assume that **(ZC)** holds. Then,*

$$(\mathscr{E}_{\infty,\infty}, \underline{\mathscr{D}}_{\infty,\infty}) = (\mathscr{E}, \underline{\mathscr{D}}). \tag{6.7}$$

Proof From (6.3), we set the closed form $(\mathscr{E}_{R,\infty}, \underline{\mathscr{D}}_{R,\infty})$ by

$$\begin{aligned} &\mathscr{E}_{R,\infty}(f,f) = \lim_{\nu\to\infty} \mathscr{E}_{R,\nu}(f,f), \\ &\underline{\mathscr{D}}_{R,\infty} = \left\{ f \in \bigcap_{\nu=1}^{\infty} \underline{\mathscr{D}}_{R,\nu} ;\ \lim_{\nu\to\infty} \mathscr{E}_{R,\nu}(f,f) < \infty \right\}. \end{aligned} \tag{6.8}$$

Obviously, we have

$$(\mathscr{E}_{R,\infty}, \underline{\mathscr{D}}_{R,\infty}) \le (\mathscr{E}_R, \underline{\mathscr{D}}_R) \le (\mathscr{E}_R, \mathscr{D}_R). \tag{6.9}$$

It is easy to show that $(\mathscr{E}_R, \underline{\mathscr{D}}_R)$ is a quasi-regular Dirichlet form. From (6.9), we have that $\underline{\mathrm{Cap}}_R \le \mathrm{Cap}_R$, where $\underline{\mathrm{Cap}}_R$ is the capacity given by the Dirichlet form $(\mathscr{E}_R, \underline{\mathscr{D}}_R)$ on $L^2(\mu)$. Using **(ZC)**, (3.15), and the inequality of the capacities as above, we obtain

$$\underline{\mathrm{Cap}}_R\Big(\mathfrak{S} \setminus \bigcup_{\nu=1}^{\infty} \mathfrak{O}_{R,\nu}\Big) = 0. \tag{6.10}$$

Using (6.8) and (6.10) and the definition of $(\mathscr{E}_R, \underline{\mathscr{D}}_R)$, we deduce

$$(\mathscr{E}_{R,\infty}, \underline{\mathscr{D}}_{R,\infty}) = (\mathscr{E}_R, \underline{\mathscr{D}}_R). \tag{6.11}$$

Using (2.7), (6.4)–(6.6), (6.8), and (6.11), we obtain

$$\begin{aligned}
(\mathscr{E}_{\infty,\infty}, \underline{\mathscr{D}}_{\infty,\infty}) &= \lim_{\nu\to\infty} (\mathscr{E}_{\infty,\nu}, \underline{\mathscr{D}}_{\infty,\nu}) && \text{by (6.6)}\\
&= \lim_{\nu\to\infty} \lim_{R\to\infty} (\mathscr{E}_{R,\nu}, \underline{\mathscr{D}}_{R,\nu}) && \text{by (6.5)}\\
&= \lim_{R\to\infty} \lim_{\nu\to\infty} (\mathscr{E}_{R,\nu}, \underline{\mathscr{D}}_{R,\nu}) && \text{by (6.4)}\\
&= \lim_{R\to\infty} (\mathscr{E}_{R,\infty}, \underline{\mathscr{D}}_{R,\infty}) && \text{by (6.8)}\\
&= \lim_{R\to\infty} (\mathscr{E}_R, \underline{\mathscr{D}}_R) && \text{by (6.11)}\\
&= (\mathscr{E}, \underline{\mathscr{D}}) && \text{by (2.7)}.
\end{aligned}$$

This completes the proof of (6.7). □

6.2 Upper schemes of Dirichlet forms in infinite volumes

We now proceed with the upper scheme. We define the domain $\mathscr{B}_{R,\nu}$ such that

$$\mathscr{B}_{R,\nu} = \{f \in \mathscr{B}_R;\ f \text{ is constant on each connected component of } \mathfrak{S}\backslash\mathfrak{O}_{R,\nu}\}. \tag{6.12}$$

From (3.12) and (6.12), we deduce the following for each $\nu \in \mathbb{N}$:

$$\mathscr{B}_{R,\nu} \subset \mathscr{B}_{R+1,\nu} \quad \text{for all } R \in \mathbb{N}. \tag{6.13}$$

Using Lemma 2.10, we see that $(\mathscr{E}_R^{\mathfrak{O}_\nu}, \mathscr{D}_\circ^{\mathfrak{O}_\nu})$ is closable on $L^2(\mu)$ for each $R, \nu \in \mathbb{N}$. Because $\mathscr{D}_\circ^\mu \cap \mathscr{B}_{R,\nu} \subset \mathscr{D}_\circ^{\mathfrak{O}_\nu}$ and

$$\mathscr{E}(f,f) = \mathscr{E}_R^{\mathfrak{O}_\nu}(f,f) \quad \text{for } f \in \mathscr{D}_\circ^\mu \cap \mathscr{B}_{R,\nu},$$

we have that $(\mathscr{E}, \mathscr{D}_\circ^\mu \cap \mathscr{B}_{R,\nu})$ is closable on $L^2(\mu)$. Hence, we define $(\mathscr{E}_{R,\nu}, \mathscr{D}_{R,\nu})$ as the closure of $(\mathscr{E}, \mathscr{D}_\circ^\mu \cap \mathscr{B}_{R,\nu})$ on $L^2(\mu)$. Note that, by construction,

$$(\mathscr{E}_{R,\nu}, \underline{\mathscr{D}}_{R,\nu}) \le (\mathscr{E}_{R,\nu}, \mathscr{D}_{R,\nu}). \tag{6.14}$$

Lemma 6.3 *Assume that μ satisfies **(QG)** with $\{O_{R,\nu}^m\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$ and that $\{O_{R,\nu}^m\}_{R,\nu,m\in\mathbb{N}}$ satisfies (3.12)–(3.16). Then, $\{(\mathscr{E}_{R,\nu}, \mathscr{D}_{R,\nu})\}_{R\in\mathbb{N}}$ is decreasing in R for each $\nu \in \mathbb{N}$. The strong resolvent limit $(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu})$ of $\{(\mathscr{E}_{R,\nu}, \mathscr{D}_{R,\nu})\}_{R\in\mathbb{N}}$ is the closure of $(\mathscr{E}_{\infty,\nu}, \cup_R \mathscr{D}_{R,\nu})$.*

Proof From (6.13), we have the following for each $\nu \in \mathbb{N}$:

$$\mathscr{D}_\circ^\mu \cap \mathscr{B}_{R,\nu} \subset \mathscr{D}_\circ^\mu \cap \mathscr{B}_{R+1,\nu} \quad \text{for all } R \in \mathbb{N}. \tag{6.15}$$

Because $(\mathscr{E}_{R,\nu}, \mathscr{D}_{R,\nu})$ is the closure of $(\mathscr{E}_{R,\nu}, \mathscr{D}_\circ^\mu \cap \mathscr{B}_{R,\nu})$, we see that $\{(\mathscr{E}_{R,\nu}, \mathscr{D}_{R,\nu})\}_{R\in\mathbb{N}}$ is decreasing in R for each $\nu \in \mathbb{N}$ from (6.15). This implies the first claim.

The strong resolvent limit $(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu})$ is the closure of the largest closable part of $(\mathscr{E}_{\infty,\nu}, \cup_R \mathscr{D}_{R,\nu})$. Hence, it only remains to prove that $(\mathscr{E}_{\infty,\nu}, \cup_R \mathscr{D}_{R,\nu})$ is closable on $L^2(\mu)$.

We set $\mathscr{D}^{\mu}_{\circ\nu} = \cup_R(\mathscr{D}^{\mu}_{\circ} \cap \mathscr{B}_{R,\nu})$. Then, $\mathscr{D}^{\mu}_{\circ\nu} \subset \mathscr{D}^{\mu}_{\circ}$. Recall that $(\mathscr{E}, \mathscr{D}^{\mu}_{\circ})$ is closable on $L^2(\mu)$. Hence, $(\mathscr{E}, \mathscr{D}^{\mu}_{\circ\nu})$ is closable on $L^2(\mu)$. Note that $\mathscr{E}(f, f) = \mathscr{E}_{\infty,\nu}(f, f)$ for $f \in \mathscr{D}^{\mu}_{\circ\nu}$. Hence, the closability of $(\mathscr{E}_{\infty,\nu}, \mathscr{D}^{\mu}_{\circ\nu})$ on $L^2(\mu)$ follows from that of $(\mathscr{E}, \mathscr{D}^{\mu}_{\circ\nu})$ on $L^2(\mu)$.

Note that $\mathscr{D}^{\mu}_{\circ\nu} \subset \cup_R \mathscr{D}_{R,\nu}$ and that $\mathscr{D}^{\mu}_{\circ\nu}$ is dense in $\cup_R \mathscr{D}_{R,\nu}$ with respect to the inner product $\mathscr{E}_{\infty,\nu}(\cdot, *) + (\cdot, *)_{L^2(\mu)}$. Combining these facts with the closability of $(\mathscr{E}_{\infty,\nu}, \mathscr{D}^{\mu}_{\circ\nu})$ on $L^2(\mu)$, we deduce that $(\mathscr{E}_{\infty,\nu}, \cup_R \mathscr{D}_{R,\nu})$ is closable on $L^2(\mu)$. This completes the proof of the second claim. □

Lemma 6.4 *Consider the same assumptions as for Lemma 6.3.*

(1) $\{(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu})\}_{\nu\in\mathbb{N}}$ *is decreasing in* $\nu \in \mathbb{N}$. *For each* $\nu \in \mathbb{N}$, *we have that*

$$(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu}) \geq (\mathscr{E}, \mathscr{D}), \quad \mathscr{E}_{\infty,\nu}(f, f) = \mathscr{E}(f, f) \quad \text{for all } f \in \mathscr{D}_{\infty,\nu}. \tag{6.16}$$

Furthermore, $(\mathscr{E}, \cup_{\nu=1}^{\infty} \mathscr{D}_{\infty,\nu})$ *is closable on* $L^2(\mu)$.

(2) *The closure* $(\mathscr{E}_{\infty}, \mathscr{D}_{\infty})$ *of* $(\mathscr{E}, \cup_{\nu=1}^{\infty} \mathscr{D}_{\infty,\nu})$ *satisfies*

$$(\mathscr{E}_{\infty}, \mathscr{D}_{\infty}) = \lim_{\nu\to\infty} (\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu}) \text{ in the strong resolvent sense}, \tag{6.17}$$

$$(\mathscr{E}_{\infty}, \mathscr{D}_{\infty}) \geq (\mathscr{E}, \mathscr{D}), \quad \mathscr{E}_{\infty}(f, f) = \mathscr{E}(f, f) \quad \text{for all } f \in \mathscr{D}_{\infty}. \tag{6.18}$$

Proof From (3.13), we have $\mathfrak{O}_{R,\nu} \subset \mathfrak{O}_{R,\nu+1}$. Hence, $\mathscr{B}_{R,\nu} \subset \mathscr{B}_{R,\nu+1}$ by (6.12). Then,

$$(\mathscr{E}, \mathscr{D}^{\mu}_{\circ} \cap \mathscr{B}_{R,\nu}) \geq (\mathscr{E}, \mathscr{D}^{\mu}_{\circ} \cap \mathscr{B}_{R,\nu+1}) \quad \text{for each } R, \nu \in \mathbb{N}. \tag{6.19}$$

Taking the closures of both sides of (6.19) and using $\mathscr{D}_{R,\nu+1} \subset \mathscr{D}_R$, we obtain

$$(\mathscr{E}_{R,\nu}, \mathscr{D}_{R,\nu}) \geq (\mathscr{E}_{R,\nu+1}, \mathscr{D}_{R,\nu+1}) \geq (\mathscr{E}_R, \mathscr{D}_R) \quad \text{for each } R, \nu \in \mathbb{N}. \tag{6.20}$$

By Lemma 6.3, $(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu})$ is the strong resolvent limit of $\{(\mathscr{E}_{R,\nu}, \mathscr{D}_{R,\nu})\}_{R\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$. From this and the first inequality in (6.20), $\{(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu})\}_{\nu\in\mathbb{N}}$ is decreasing.

We see that $(\mathscr{E}, \mathscr{D}) = \lim_{R\to\infty}(\mathscr{E}_R, \mathscr{D}_R)$ in the strong resolvent sense from Lemma 2.4 (1). We can deduce $(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu}) = \lim_{R\to\infty}(\mathscr{E}_{R,\nu}, \mathscr{D}_{R,\nu})$ in the strong resolvent sense from Lemma 6.3. Hence, taking $R \to \infty$ in (6.20), we obtain the inequality in (6.16).

From $\mathscr{E}(f, f) = \mathscr{E}_{R,\nu}(f, f)$ for $f \in \mathscr{D}_{R,\nu}$, we have the equality in (6.16).

From (6.16), we have $(\mathscr{E}, \mathscr{D}) \leq (\mathscr{E}, \cup_{\nu=1}^{\infty} \mathscr{D}_{\infty,\nu})$. Recall that $(\mathscr{E}, \mathscr{D})$ is a closed form on $L^2(\mu)$. Hence, $(\mathscr{E}, \cup_{\nu=1}^{\infty} \mathscr{D}_{\infty,\nu})$ is closable $L^2(\mu)$. Thus, we obtain (1).

By (1), $\{(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu})\}_{\nu\in\mathbb{N}}$ is decreasing. Therefore, $(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu})$ converges to the closure of the largest closable part of $(\mathscr{E}_{\infty}, \cup_{\nu=1}^{\infty} \mathscr{D}_{\infty,\nu})$ in the strong resolvent sense. By (1), we see that $(\mathscr{E}_{\infty}, \cup_{\nu=1}^{\infty} \mathscr{D}_{\infty,\nu})$ is closable. Hence, we obtain (6.17). Equation (6.18) is clear because of (6.16) and the fact that $(\mathscr{E}_{\infty}, \mathscr{D}_{\infty})$ is the closure of $(\mathscr{E}, \cup_{\nu=1}^{\infty} \mathscr{D}_{\infty,\nu})$. □

Proposition 6.1 *Consider the same assumptions as for Lemma 6.3, and furthermore assume that* ***(ZC)*** *holds. Let* $(\mathscr{E}_{\infty}, \mathscr{D}_{\infty})$ *be as in Lemma 6.4. Then,* $\mathscr{D}_{\infty}$ *is dense in* $\mathscr{D}$ *with respect to the inner product* $\mathscr{E}_{(1)} := \mathscr{E}(\cdot, *) + (\cdot, *)_{L^2(\mu)}$. *Furthermore,*

$$(\mathscr{E}, \mathscr{D}) = (\mathscr{E}_{\infty}, \mathscr{D}_{\infty}), \tag{6.21}$$

$$(\mathscr{E}, \mathscr{D}) = \lim_{\nu\to\infty} (\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu}) \text{ in the strong resolvent sense}. \tag{6.22}$$

Proof We regard $\mathfrak{O}^{c}_{R,\nu}$ as a subset of $\mathrm{Cf}(S_R)$. Then,

$$\bigcap_{\nu=1}^{\infty} \mathfrak{O}^{c}_{R,\nu} = \bigcup_{m=1}^{\infty} \{\mathfrak{s} \in \mathrm{Cf}^m(S_R)\,;\ \sigma^m_R(\mathfrak{s}) = 0\}. \tag{6.23}$$

From **(ZC)**, we see that

$$\mathrm{Cap}_R\left(\bigcup_{m=1}^{\infty}\{\mathfrak{s}\in \mathrm{Cf}^m(S_R)\,;\ \sigma_R^m(\mathfrak{s})=0\}\right)=0. \tag{6.24}$$

Using (3.13), (6.23), and (6.24), we obtain

$$\lim_{\nu\to\infty}\mathrm{Cap}_R(\mathfrak{O}_{R,\nu}^c)=0 \quad \text{for each } R\in\mathbb{N}. \tag{6.25}$$

Hence, for each $R\in\mathbb{N}$, there exist a decreasing sequence of open sets $\{\mathfrak{M}_{R,\nu}\}_{\nu\in\mathbb{N}}$ and a sequence of functions $\{\varphi_{R,\nu}\}_{\nu\in\mathbb{N}}$ satisfying

$$\mathfrak{O}_{R,\nu}^c\subset\mathfrak{M}_{R,\nu},\quad \lim_{\nu\to\infty}\mathrm{Cap}_R(\mathfrak{M}_{R,\nu})=0, \tag{6.26}$$

$$\varphi_{R,\nu}=1 \ \text{ on } \mathfrak{M}_{R,\nu},\quad 0\le\varphi_{R,\nu}(\mathfrak{s})\le 1 \quad \text{for all } \mathfrak{s}\in\mathfrak{S}, \tag{6.27}$$

$$\varphi_{R,\nu}\in\mathscr{D}_R,\quad \lim_{\nu\to 0}\mathscr{E}_{R,(1)}(\varphi_{R,\nu})=0. \tag{6.28}$$

Here, $\mathscr{E}_{R,(1)}:=\mathscr{E}_R(\cdot,*)+(\cdot,*)_{L^2(\mu_R)}$. For $Q\le R$, the set $\mathfrak{O}_{Q,\nu}$ can be regarded as an open set in $\mathrm{Cf}(S_R)$, and the function $\varphi_{Q,\nu}$ can be considered as an element in $\mathscr{D}_R$. Recall the definition of $\mathscr{D}_{R,\nu}$ given before (6.14). Then, because of (6.26)–(6.28), we have the following for each $R\in\mathbb{N}$:

$$\psi_{R,\nu}:=\prod_{Q=1}^{R}(1-\varphi_{Q,\nu})\in\mathscr{D}_{R,\nu}\cap L^\infty(\mu). \tag{6.29}$$

From (6.27) and (6.29), we have $\psi_{R,\nu}=0$ on $\mathfrak{M}_{R,\nu}$. Hence, from (6.27)–(6.29) with a straightforward calculation, we have that

$$\lim_{\nu\to\infty}\mathscr{E}_{(1)}(1-\psi_{R,\nu})=\lim_{\nu\to\infty}\mathscr{E}_{R,(1)}(1-\psi_{R,\nu})=0 \tag{6.30}$$

and a subsequential limit such that

$$\lim_{\nu\to 0}\psi_{R,\nu}(\mathfrak{s})=\prod_{Q=1}^{R}\lim_{\nu\to\infty}(1-\varphi_{Q,\nu}(\mathfrak{s}))=1 \quad \text{for } \mu\text{-a.s. } \mathfrak{s}. \tag{6.31}$$

Combining (6.30), (6.31), $f\in\mathscr{D}\cap L^\infty(\mu)$, and $|1-\psi_{R,\nu}|\le 1$, we obtain

$$\begin{aligned}
&\mathscr{E}_{(1)}(f-f\psi_{R,\nu})=\mathscr{E}_{(1)}(f(1-\psi_{R,\nu}))\\
&=\int_{\mathfrak{S}}\mathbb{D}^a[f]|1-\psi_{R,\nu}|^2+|f|^2\mathbb{D}^a[1-\psi_{R,\nu}]+2\mathbb{D}^a[f,1-\psi_{R,\nu}]+|f|^2|1-\psi_{R,\nu}|^2d\mu\\
&\le\int_{\mathfrak{S}}2\left\{\mathbb{D}^a[f]|1-\psi_{R,\nu}|^2+|f|^2\mathbb{D}^a[1-\psi_{R,\nu}]\right\}+|f|^2|1-\psi_{R,\nu}|^2d\mu\\
&\to 0,\quad \nu\to\infty.
\end{aligned} \tag{6.32}$$

For $f\in\mathscr{D}\cap L^\infty(\mu)$, we have that $f\psi_{R,\nu}=0$ on $\mathfrak{M}_{R,\nu}$. For $f\in\mathscr{D}\cap L^\infty(\mu)$, this yields

$$f\psi_{R,\nu}\in\mathscr{D}_{R,\nu}\cap L^\infty(\mu)\subset\mathscr{D}_{\infty,\nu}\cap L^\infty(\mu). \tag{6.33}$$

From (6.32), (6.33), and $\cup_{\nu=1}^{\infty}\mathscr{D}_{\infty,\nu}\cap L^\infty(\mu)=\mathscr{D}_\infty\cap L^\infty(\mu)$, we deduce that $\mathscr{D}_\infty\cap L^\infty(\mu)$ is dense in $\mathscr{D}\cap L^\infty(\mu)$ with respect to $\mathscr{E}_{(1)}$. Furthermore, it is not difficult to see that $\mathscr{D}\cap L^\infty(\mu)$ is dense in $\mathscr{D}$ with respect to $\mathscr{E}_{(1)}$. Collecting these results, we see that $\mathscr{D}_\infty$ is dense in $\mathscr{D}$ with respect to $\mathscr{E}_{(1)}$, which completes the proof of the first claim.

The second claim (6.21) follows immediately from the first claim and (6.18). The third claim (6.22) follows from (6.17) and (6.21). □

6.3 Construction of $\{O^m_{R,\nu}\}_{R,\nu,m\in\mathbb{N}}$

In this subsection, we construct $\{O^m_{R,\nu}\}_{R,\nu,m\in\mathbb{N}}$ satisfying (3.12)–(3.16).

Lemma 6.5 *Assume that μ satisfies **(QG)** with $\{S^m_R\}_{R,m\in\mathbb{N}}$ and **(ZC)**. Assume that σ^m_R is uniformly continuous on S^m_R for each $R, m \in \mathbb{N}$. Then, we have a sequence of symmetric open sets $\{O^m_{R,\nu}\}_{R,\nu,m\in\mathbb{N}}$ such that μ satisfies **(QG)** with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$ and that $\{O^m_{R,\nu}\}_{R,\nu,m\in\mathbb{N}}$ satisfies (3.12)–(3.16).*

Proof Let (Φ_0, Ψ_0) be as in (2.18). Let $R, \nu, m \in \mathbb{N}$ be fixed. Let $T_1 = S_1$ and $T_Q = S_Q \setminus S_{Q-1}$ for $2 \le Q \le R$. For $\mathbf{x} = (x_i) \in S^m_R$, we set

$$\mathscr{I}_{Q,R}(\mathbf{x}) = \sum_{x_i\in T_Q} \Phi_0(x_i) + \sum_{\substack{x_i,\, x_j\in T_Q\\ i<j}} \Psi_0(x_i, x_j) + \sum_{Q'=Q+1}^{R} \sum_{x_i\in T_Q,\, x_j\in T_{Q'}} \Psi_0(x_i, x_j).$$

Let $T^l_Q = \{\mathbf{x} = (x_i) \in S^m_R\,;\, \sharp\{i\,;\, x_i \in T_Q\} = l\}$. We set $\mathbf{A}^0_{Q,R} = \emptyset$. Furthermore, we set for $Q \in \mathbb{N}$ and $l \in \mathbb{N}$

$$\mathbf{A}^l_{Q,R} = \{\mathbf{x} \in T^l_Q\,;\, \mathscr{I}_{Q,R}(\mathbf{x}) = \infty\}. \tag{6.34}$$

Let $\mathbf{L}(m) = \{(l_Q) \in \{0, \dots, m\}^R\,;\, \sum_{Q=1}^R l_Q = m\}$. We set

$$\mathbf{B}^m_R = \sum_{(l_Q)\in\mathbf{L}(m)} \sum_{Q=1}^{R} \mathbf{A}^{l_Q}_{Q,R}. \tag{6.35}$$

Let $\mathscr{H}^{\Phi,\Psi}_R$ be as in (2.16). We set

$$\mathscr{H}_R(\mathbf{x}) = \mathscr{H}^{\Phi_0,\Psi_0}_R(\mathfrak{u}(\mathbf{x})).$$

Recall that (Φ_0, Ψ_0) is bounded from below because μ satisfies **(QG)** with $\{S^m_R\}_{R,m\in\mathbb{N}}$. Let $c_6(R, k, m, \pi^c_R(\mathfrak{s}))$ be as in (2.17). We set

$$\Omega(n) = \{\mathfrak{s} \in \mathfrak{S}\,;\, n^{-1} \le c_6(R, k, m, \pi^c_R(\mathfrak{s})) \le n\}.$$

Let $\sigma^m_{R,n}$ be the density function of $\mu(\cdot \cap \Omega(n))$ on S^m_R. Suppose that $\mu(\Omega(n)) > 0$. Then, from (2.17) and (2.18), we have a positive constant c_{13} such that

$$c_{13}^{-1} \exp\{-c_7 \mathscr{H}_R(\mathbf{x})\} \le \sigma^m_{R,n}(\mathbf{x}) \le c_{13} \exp\{-c_7^{-1} \mathscr{H}_R(\mathbf{x})\} \quad \text{for } \mathbf{x} \in S^m_R. \tag{6.36}$$

Thus, for $\mathbf{x} \in S^m_R$, we deduce that $\sigma^m_{R,n}(\mathbf{x}) = 0$ if and only if $\mathscr{H}_R(\mathbf{x}) = \infty$. It is clear that $\mu(\cup_n \Omega(n)) = 1$. Hence, $\{\mathbf{x} \in S^m_R;\, \sigma^m_R(\mathbf{x}) = 0\}$ if and only if $\mathbf{x} \in \mathbf{B}^m_R$.

Let ε_ν, $\nu \in \mathbb{N}$, be functions defined on $\mathbb{N}^2$ such that, for each $Q, l \in \mathbb{N}$,

$$\varepsilon_\nu(Q, l) > \varepsilon_{\nu+1}(Q, l) > 0, \quad \lim_{\nu\to\infty} \varepsilon_\nu(Q, l) = 0. \tag{6.37}$$

We set

$$N(Q, l) = \left\{ \mathbf{x} \in T^l_Q\,;\, \inf_{\mathbf{y}\in\mathbf{A}^l_{Q,R}} |\mathbf{x} - \mathbf{y}| > \varepsilon_\nu(Q, l) \right\},$$

$$O^m_{R,\nu} = \sum_{(l_Q)\in\mathbf{L}(m)} \sum_{Q=1}^{R} N(Q, l_Q).$$

Then, it is easy to show that $O^m_{R,\nu}$ is a symmetric open set satisfying (3.12) and (3.13).

Recall that $\mu(\pi_R^{-1}(\mathfrak{u}(\{\mathbf{x};\ \sigma^m_R(\mathbf{x}) = 0\}))) = 0$ from **(ZC)**. Then, $\mu(\pi_R^{-1}(\mathbf{B}^m_R)) = 0$ from (6.34) and (6.36). Hence, using (6.37), we obtain (3.14). Equation (3.15) follows from (6.34) and (6.36). Equation (3.16) is clear because σ^m_R is uniformly continuous on S^m_R and $\mathbf{B}^m_R$ is relatively compact in S^m_R.

To prove that μ satisfies **(QG)** with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$, we begin by checking Definition 2.2. Definition 2.2 (1) is obvious. Using **(ZC)**, we have

$$\mu_R\left(\left\{\bigcup_{\nu\in\mathbb{N}} \mathfrak{O}_{R,\nu}\right\}^c\right) = 0.$$

Hence, taking $\mu_{R,\nu} = \mu(\cdot \cap \mathfrak{O}_{R,\nu})$, we obtain Definition 2.2 (2) (replacing $\mu_{R,k}$ by $\mu_{R,\nu}$). By assumption, μ satisfies **(QG)** with $\{S^m_R\}_{R,m\in\mathbb{N}}$. Hence, applying (2.17) to S^m_R with a simple calculation, we find that

$$c_6^{-1} e^{-\mathscr{H}^{\Phi,\Psi}_R} \Lambda^m_R(d\mathfrak{x}) \le \mu^m_{(k),R,\mathfrak{s}}(d\mathfrak{x}) \le c_6 e^{-\mathscr{H}^{\Phi,\Psi}_R} \Lambda^m_R(d\mathfrak{x}). \tag{6.38}$$

Using (6.38), $O^m_{R,\nu} \subset S^m_R$, and retaking $O^m_{R,\nu}$ if necessary, we can easily find that μ is a (Φ, Ψ)-quasi-Gibbs measure with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$. Thus, Definition 2.2 (3) holds. Collecting these results, we obtain Definition 2.2, which implies **(QG1)**. By assumption, **(QG2)** is satisfied. Hence, we find that μ satisfies **(QG)** with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$. This completes the proof of Lemma 6.5. □

7 Proof of Theorems 3.2–3.3

In this section, we shall prove Theorems 3.2–3.3.

For simplicity, we make the following assumption.

(B4)* Either **(B4)**, **(B4)**$'$, or **(B4)**$''$ holds.

We introduce condition **(B4)**$_\nu$, which is similar to **(B4)**.

(B4)$_\nu$ For each $R, \nu, m \in \mathbb{N}$,

$$\lim_{N\to\infty} \left\| \frac{\sigma^{N,m}_R}{\sigma^m_R} - 1 \right\|_{O^m_{R,\nu}} = 0. \tag{7.1}$$

Lemma 7.1 *Assume that* ***(B4)**** *is satisfied. Then,* ***(B4)***$_\nu$ *holds.*

Proof If **(B4)** holds, then we obviously have **(B4)**$_\nu$. From **(B4)**$'$, we have that

$$\lim_{N\to\infty} \left\| \sigma^{N,m}_R - \sigma^m_R \right\|_{O^m_{R,\nu}} = 0 \quad \text{for each } R, \nu, m \in \mathbb{N}. \tag{7.2}$$

From (3.16), we see that σ^m_R is uniformly positive on $O^m_{R,\nu}$. Hence, (7.2) yields (7.1). This implies **(B4)**$_\nu$. Because **(B4)**$''$ implies **(B4)**$'$, **(B4)**$''$ yields **(B4)**$_\nu$. □

7.1 Lower schemes of cut-off Dirichlet forms

The main result of this subsection is Proposition 7.1, which presents an inequality for the lower scheme.

Let $\mathfrak{O}^m_{R,\nu}$ and $\mathfrak{O}_\nu$ be as in (3.11). Note that $\mathfrak{O}^m_{R,\nu} \cap \mathfrak{O}^{m'}_{R,\nu} = \emptyset$ for $m \neq m'$ and that $\mathfrak{O}_\nu \subset \mathfrak{O}_{R,\nu} = \cup_{m=1}^\infty \mathfrak{O}^m_{R,\nu}$. Replacing μ by $1_{\mathfrak{O}^m_{R,\nu}}\mu^N$ and $1_{\mathfrak{O}_\nu}\mu^N$, we introduce the cut-off bilinear forms $\mathscr{E}^{N,m}_{R,\nu}$ and $\mathscr{E}^N_{R,\nu}$ such that

$$\mathscr{E}^{N,m}_{R,\nu}(f,g) = \int_{\mathfrak{O}^m_{R,\nu}} \mathbb{D}^a_R[f,g]\, d\mu^N,$$

$$\mathscr{E}^N_{R,\nu}(f,g) = \sum_{m=1}^\infty \mathscr{E}^{N,m}_{R,\nu}(f,g) = \int_{\mathfrak{O}_\nu} \mathbb{D}^a_R[f,g] d\mu^N.$$

Lemma 7.2 *Assume that μ^N satisfies **(QG)** with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$. Then, $(\mathscr{E}^N_{R,\nu}, \mathscr{D}^N_\circ)$ is closable on $L^2(\mu^N)$.*

Proof We obtain Lemma 7.2 from Lemma 6.1 by replacing μ with μ^N. □

We denote the closure of $(\mathscr{E}^N_{R,\nu}, \mathscr{D}^N_\circ)$ on $L^2(\mu^N)$ as $(\mathscr{E}^N_{R,\nu}, \underline{\mathscr{D}}^N_{R,\nu})$.

Lemma 7.3 *Assume that μ^N satisfies **(QG)** with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$. Let $\underline{\widetilde{\mathscr{D}}}^N_R$ be as in (3.2). Then, for all $R, \nu, N \in \mathbb{N}$, $(\mathscr{E}^N_{R,\nu}, \underline{\mathscr{D}}^N_{R,\nu})$ satisfies*

$$(\mathscr{E}^N_{R,\nu}, \underline{\mathscr{D}}^N_{R,\nu}) \leq (\mathscr{E}^N_R, \underline{\widetilde{\mathscr{D}}}^N_R). \tag{7.3}$$

Proof We have $(\mathscr{E}^N_{R,\nu}, \mathscr{D}^N_\circ \cap \mathscr{B}_R) \leq (\mathscr{E}^N_R, \mathscr{D}^N_\circ \cap \mathscr{B}_R)$. Hence, $(\mathscr{E}^N_{R,\nu}, \underline{\mathscr{D}}^N_{R,\nu}) \leq (\mathscr{E}^N_R, \underline{\mathscr{D}}^N_R)$. Combining this with (3.3), we obtain (7.3). □

Let r_N be as in (5.7). We set

$$\underline{\widetilde{\mathscr{D}}}^N_{r_N,\nu} = \{f \in \underline{\mathscr{D}}^N_{r_N,\nu}\,;\ f \text{ is } \sigma[\pi_{r_N}] - \text{measurable}\}.$$

Lemma 7.4 *Assume that μ and μ^N satisfy **(QG)** with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$ such that (3.12)–(3.15) hold. Assume that **(B4)*** holds. Assume that $\{f_N\}_{N\in\mathbb{N}}$ with $f_N \in L^2(\mu^N_{r_N})$ weakly converges to $f \in L^2(\mu)$ in the sense of Definition 4.2. Then, for each $\nu \in \mathbb{N}$,*

$$\mathscr{E}_{\infty,\nu}\langle f\rangle \leq \liminf_{N\to\infty} \mathscr{E}^N_{r_N,\nu}\langle f_N\rangle \tag{7.4}$$

holds for $(\mathscr{E}_{\infty,\nu}, \underline{\mathscr{D}}_{\infty,\nu})$ and $\{(\mathscr{E}^N_{r_N,\nu}, \underline{\widetilde{\mathscr{D}}}^N_{r_N,\nu})\}_{N\in\mathbb{N}}$.

Proof Using Lemma 7.1, we deduce $\mathbf{(B4)}_\nu$ from **(B4)***. Then, we obtain (7.4) from $\mathbf{(B4)}_\nu$ in the same fashion as in Proposition 5.1. Indeed, we replace $(\mathscr{E}^N_{r_N}, \underline{\widetilde{\mathscr{D}}}^N_{r_N})$ and $(\mathscr{E}, \mathscr{D})$ in Proposition 5.1 by $(\mathscr{E}^N_{r_N,\nu}, \underline{\widetilde{\mathscr{D}}}^N_{r_N,\nu})$ and $(\mathscr{E}_{\infty,\nu}, \underline{\mathscr{D}}_{\infty,\nu})$, respectively. The remainder of the proof is the same as that of Proposition 5.1, and so we omit the details. □

Proposition 7.1 *Consider the same assumptions as for Lemma 6.5 regarding μ and μ^N. Assume that **(B4)*** holds. Assume that $\{f_N\}_{N\in\mathbb{N}}$ with $f_N \in L^2(\mu^N_{r_N})$ weakly converges to $f \in L^2(\mu)$ in the sense of Definition 4.2. Then, $(\mathscr{E}, \underline{\mathscr{D}})$ and $\{(\mathscr{E}^N_{r_N}, \underline{\widetilde{\mathscr{D}}}^N_{r_N})\}_{N\in\mathbb{N}}$ satisfy*

$$\mathscr{E}\langle f\rangle \leq \liminf_{N\to\infty} \mathscr{E}^N_{r_N}\langle f_N\rangle. \tag{7.5}$$

Proof From the definition of $\mathscr{E}\langle\cdot\rangle$, (6.7), and (6.6), we have the following for $f \in L^2(\mu)$

$$\mathscr{E}\langle f\rangle = \mathscr{E}_{\infty,\infty}\langle f\rangle = \lim_{\nu\to\infty} \mathscr{E}_{\infty,\nu}\langle f\rangle. \tag{7.6}$$

From (7.4) and (7.3), for $f \in L^2(\mu)$, we have that

$$\mathscr{E}_{\infty,\nu}\langle f\rangle \le \liminf_{N\to\infty} \mathscr{E}^N_{r_N,\nu}\langle f_N\rangle \le \liminf_{N\to\infty} \mathscr{E}^N_{r_N}\langle f_N\rangle. \tag{7.7}$$

Combining (7.6) and (7.7), we obtain (7.5). □

7.2 Upper schemes of cut-off Dirichlet forms

The main result of this section is Proposition 7.2, which presents the convergence of the upper scheme of the cut-off Dirichlet forms. The argument is similar to that in Sect. 5.2, with the replacement of (3.4) by **(B4)**$_\nu$.

Let $\mathfrak{O}_\nu$ be as in (3.11). We set

$$\mathscr{E}^N_{R,\nu}(f,g) = \int_{\mathfrak{O}_\nu} \mathbb{D}^a_R[f,g]d\mu^N.$$

Lemma 7.5 *Assume that μ^N satisfies **(QG)** with $\{O^m_{R,\nu}\}_{R,m\in\mathbb{N}}$ for each $\nu \in \mathbb{N}$. Then, $(\mathscr{E}^N_{R,\nu}, \mathscr{D}^\mu_\circ \cap \mathscr{B}_R)$ is closable on $L^2(\mu^N)$.*

Proof Using **(B2)**, we have $\mathscr{D}^\mu_\circ \subset \cap_{N\in\mathbb{N}}\mathscr{D}^N_\circ$. Then,

$$\mathscr{D}^\mu_\circ \cap \mathscr{B}_R \subset \mathscr{D}^\mu_\circ \subset \cap_{N\in\mathbb{N}}\mathscr{D}^N_\circ \subset \mathscr{D}^N_\circ. \tag{7.8}$$

From Lemma 7.2, we see that $(\mathscr{E}^N_{R,\nu}, \mathscr{D}^N_\circ)$ is closable on $L^2(\mu^N)$. Combining this with (7.8) completes the proof. □

We denote the closure of $(\mathscr{E}^N_{R,\nu}, \mathscr{D}^\mu_\circ \cap \mathscr{B}_R)$ on $L^2(\mu^N)$ as $(\mathscr{E}^N_{R,\nu}, \mathscr{D}^N_{R,\nu})$. We set

$$\mathscr{E}^N_{\infty,\nu}(f,g) = \int_{\mathfrak{O}_\nu} \mathbb{D}^a[f,g]d\mu^N.$$

Similar to Lemma 2.3, we see that $(\mathscr{E}^N_{\infty,\nu}, \cup^\infty_{R=1}\mathscr{D}^\mu_\circ \cap \mathscr{B}_R)$ is closable on $L^2(\mu^N)$. Let $(\mathscr{E}^N_{\infty,\nu}, \mathscr{D}^N_{\infty,\nu})$ be the closure of $(\mathscr{E}^N_{\infty,\nu}, \cup^\infty_{R=1}\mathscr{D}^\mu_\circ \cap \mathscr{B}_R)$ on $L^2(\mu^N)$. Then, similar to Lemma 2.4, we see that $\{(\mathscr{E}^N_{R,\nu}, \mathscr{D}^N_{R,\nu})\}$ converges to $(\mathscr{E}^N_{\infty,\nu}, \mathscr{D}^N_{\infty,\nu})$ on $L^2(\mu^N)$ as $R \to \infty$ in the strong resolvent sense. Let $(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu})$ be as in Lemma 6.3.

Lemma 7.6 *Assume that **(B4)*** holds. Then, $(\mathscr{E}^N_{\infty,\nu}, \mathscr{D}^N_{\infty,\nu})$ and $(\mathscr{E}_{\infty,\nu}, \mathscr{D}_{\infty,\nu})$ satisfy the following. For each $f \in L^2(\mu)$, there exists a sequence $\{g_{N,\nu}\}_{N\in\mathbb{N}}$ satisfying*

$$g_{N,\nu} \text{ is } \sigma[\pi_{r_N}] - \text{measurable}, \ g_{N,\nu} \in L^2(\mu^N_{r_N}), \tag{7.9}$$

$$\lim_{N\to\infty} g_{N,\nu} = f \text{ strongly in the sense of Definition 4.2}, \tag{7.10}$$

$$\lim_{N\to\infty} \mathscr{E}^N_{\infty,\nu}\langle g_{N,\nu}\rangle = \mathscr{E}_{\infty,\nu}\langle f\rangle. \tag{7.11}$$

Proof From Lemma 7.1, we deduce **(B4)**$_\nu$ from **(B4)***. We then obtain (7.9)–(7.11) from **(B4)**$_\nu$ in the same fashion as in Proposition 5.2. We omit the details of the proof. □

Next, we check Definition 4.4 (2).

Proposition 7.2 *Consider the same assumptions as for Lemma 6.5. Assume that* ***(B4)***** holds. Then,* $(\mathscr{E}^N, \mathscr{D}^N)$ *and* $(\mathscr{E}, \mathscr{D})$ *satisfy the following. For any* $f \in L^2(\mu)$*, there exists a sequence* $\{g_N\}_{N\in\mathbb{N}}$ *such that*

$$g_N \text{ is } \sigma[\pi_{r_N}] - \text{measurable}, g_N \in L^2(\mu^N_{r_N}), \tag{7.12}$$

$$\lim_{N\to\infty} g_N = f \text{ strongly in the sense of Definition 4.2}, \tag{7.13}$$

$$\lim_{N\to\infty} \mathscr{E}^N\langle g_N\rangle = \mathscr{E}\langle f\rangle. \tag{7.14}$$

Proof From Lemma 6.5, we see that the assumptions of Proposition 6.1 are fulfilled. Using Proposition 6.1, we find that $\cup_{\nu=1}^{\infty}\mathscr{D}^N_{\infty,\nu}$ is dense in $\mathscr{D}$. Hence, without loss of generality, we can assume that $f \in \cup_{\nu=1}^{\infty}\mathscr{D}^N_{\infty,\nu}$. From Proposition 6.1, we obtain

$$\mathscr{E}(f) = \lim_{\nu\to\infty} \mathscr{E}_{\infty,\nu}(f) \tag{7.15}$$

Let $g_{N,\nu}$ denote the sequence in Lemma 7.6. Then, $g_{N,\nu}$ satisfies (7.12). Combining (7.9)–(7.11) and (7.15), we can take g_N satisfying (7.13) and (7.14) by choosing a subsequence of $\{g_{N,\nu}\}_{N,\nu\in\mathbb{N}}$. □

7.3 Proof of Theorems 3.2–3.3

Proof of Theorem 3.2 We first assume that **(B4)**$''$ holds. Assume that $\{f_N\}_{N\in\mathbb{N}}$ with $f_N \in L^2(\mu^N_{r_N})$ weakly converges to $f \in L^2(\mu)$ in the sense of Definition 4.2. Let $(\mathscr{E}^N_{r_N}, \widetilde{\mathscr{D}}^N_{r_N})$ be as in (3.3) with $R = r_N$.

From Proposition 7.1, we see that $(\mathscr{E}, \underline{\mathscr{D}})$ and $\{(\mathscr{E}^N_{r_N}, \widetilde{\underline{\mathscr{D}}}^N_{r_N})\}_{N\in\mathbb{N}}$ satisfy

$$\mathscr{E}\langle f\rangle \le \liminf_{N\to\infty} \mathscr{E}^N_{r_N}\langle f_N\rangle. \tag{7.16}$$

From **(A2)**, we have $(\mathscr{E}, \mathscr{D}) = (\mathscr{E}, \underline{\mathscr{D}})$. Combining these results, we see that $(\mathscr{E}, \mathscr{D})$ and $\{(\mathscr{E}^N_{r_N}, \widetilde{\underline{\mathscr{D}}}^N_{r_N})\}_{N\in\mathbb{N}}$ satisfy Definition 4.4 (1).

Let g_N be as in Proposition 7.2. Applying (7.16) to $\{g_N\}$, we have that

$$\mathscr{E}\langle f\rangle \le \liminf_{N\to\infty} \mathscr{E}^N_{r_N}\langle g_N\rangle. \tag{7.17}$$

Clearly, $\mathscr{E}^N_{r_N}\langle g_N\rangle \le \mathscr{E}^N\langle g_N\rangle$. Hence, from Proposition 7.2, we obtain

$$\limsup_{N\to\infty} \mathscr{E}^N_{r_N}\langle g_N\rangle \le \limsup_{N\to\infty} \mathscr{E}^N\langle g_N\rangle = \mathscr{E}\langle f\rangle. \tag{7.18}$$

Combining (7.17) and (7.18), we have

$$\mathscr{E}\langle f\rangle = \lim_{N\to\infty} \mathscr{E}^N_{r_N}\langle g_N\rangle. \tag{7.19}$$

Hence, we find that $(\mathscr{E}, \mathscr{D})$ and $\{(\mathscr{E}^N_{r_N}, \widetilde{\underline{\mathscr{D}}}^N_{r_N})\}_{N\in\mathbb{N}}$ satisfy Definition 4.4 (2).

From (7.16) and (7.19), we see that the Mosco convergence of $(\mathscr{E}^N_{r_N}, \widetilde{\underline{\mathscr{D}}}^N_{r_N})$ on $L^2(\mu^N_{r_N})$ to $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$ holds. Combining this with Lemma 4.1, we conclude that (3.6) holds.

Next, we suppose that **(B4)**$''$ holds. Combining (3.18) and (3.19), we see that

$$\sup_{\mathbf{x}_m\in S^m_R} \rho^m(\mathbf{x}_m) \le c_9^m m^{c_{10}m}. \tag{7.20}$$

Then, from (3.19) and (7.20), we obtain

$$\sigma_R^{N,m}(\mathbf{x}_m) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{S_R^n} \rho^{N,m+n}(\mathbf{x}_m, \mathbf{y}_n) d\mathbf{y}_n, \tag{7.21}$$

$$\sigma_R^{m}(\mathbf{x}_m) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{S_R^n} \rho^{m+n}(\mathbf{x}_m, \mathbf{y}_n) d\mathbf{y}_n. \tag{7.22}$$

Combining (3.18), (7.21), and (7.22) and using the Lebesgue convergence theorem, we obtain (3.17), which implies that **(B4)**′ holds. Hence, (3.6) follows from the first part of the proof. □

Proof of Theorem 3.3 We first check the Mosco convergence of $(\mathscr{E}_{r_N}^N, \widehat{\mathscr{D}}_{r_N}^N)$ on $L^2(\mu_{r_N}^N)$. Assume that $\{f_N\}_{N\in\mathbb{N}}$ with $f_N \in L^2(\mu_{r_N}^N)$ weakly converges to $f \in L^2(\mu)$ in the sense of Definition 4.2. From (7.16) and the first inequality in (3.21), $(\mathscr{E}_{r_N}^N, \widehat{\mathscr{D}}_{r_N}^N)$ satisfies

$$\mathscr{E}\langle f\rangle \le \liminf_{N\to\infty} \mathscr{E}_{r_N}^N\langle f_N\rangle. \tag{7.23}$$

Combining this with **(A2)** implies Definition 4.4 (1).

Let g_N be as in Proposition 7.2. Then, from (7.18) and the second inequality in (3.21), $(\mathscr{E}_{r_N}^N, \widehat{\mathscr{D}}_{r_N}^N)$ satisfies

$$\limsup_{N\to\infty} \mathscr{E}_{r_N}^N\langle g_N\rangle \le \mathscr{E}\langle f\rangle. \tag{7.24}$$

Taking $f_N = g_N$ in (7.23) and combining this with (7.24) and **(A2)**, we obtain Definition 4.4 (2).

Thus, the Mosco convergence of $(\mathscr{E}_{r_N}^N, \widehat{\mathscr{D}}_{r_N}^N)$ on $L^2(\mu_{r_N}^N)$ to $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$ holds. Using this and Lemma 4.1 completes the proof of Theorem 3.3. □

8 Proof of Theorems 3.4–3.5

Let $(\mathfrak{X}_{r_N}^N, \widetilde{\mathfrak{P}}_{r_N}^N)$ be the unlabeled diffusion given by the Dirichlet form $(\mathscr{E}_{r_N}^N, \widetilde{\mathscr{D}}_{r_N}^N)$ as in Theorem 3.1. Originally, $\mathfrak{X}_{r_N}^N$ was a $\mathrm{Cf}(\overline{S}_{r_N})$-valued process. We regard $\mathfrak{X}_{r_N}^N$ as a $W(\mathfrak{S}_{\mathrm{s}})$-valued process in an obvious manner.

Let $(\mathfrak{X}, \mathfrak{P}), \mathfrak{P} = \{\mathfrak{P}_{\mathfrak{s}}\}_{\mathfrak{s}\in\mathfrak{S}}$, be the $\mathfrak{S}$-valued, μ-reversible diffusion associated with $(\mathscr{E}, \mathscr{D})$ on $L^2(\mu)$ (see Lemma 2.5). Let ξ be as in **(B3)**. Let $\mathfrak{P}_{\xi d\mu} = \int_{\mathfrak{S}} \mathfrak{P}_{\mathfrak{s}}\xi(\mathfrak{s})d\mu$.

Recall that the diffusion processes $\mathfrak{X}_{r_N}^N$ and $\mathfrak{X}$ are given by $\mathfrak{X}_{r_N}^N(t) = \mathfrak{w}(t)$ and $\mathfrak{X}(t) = \mathfrak{w}(t)$ as functions defined on $W(\mathfrak{S}_{\mathrm{s}})$, where $\mathfrak{S}_{\mathrm{s}}$ is defined by (2.8). Thus, we write $\mathfrak{l}_{\mathrm{path}}^N(\mathfrak{X}_{r_N}^N(\mathfrak{w})) = \mathfrak{l}_{\mathrm{path}}^N(\mathfrak{w})$ and $\mathfrak{l}_{\mathrm{path}}(\mathfrak{X}(\mathfrak{w})) = \mathfrak{l}_{\mathrm{path}}(\mathfrak{w})$.

We consider the discontinuity set of the sequence $\{\mathfrak{l}_{\mathrm{path}}^N(\mathfrak{w})(t)\}_{N\in\mathbb{N}}$ converging to $\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(t)$ such that

$$\mathrm{Disc}[\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(t)] = \{\mathfrak{w} \in W(\mathfrak{S})\,;\, \mathfrak{w} \notin W(\mathfrak{S}_{\mathrm{s}}) \text{ or there exists } \{\mathfrak{w}_N\}_{N\in\mathbb{N}} \text{ in } W(\mathfrak{S}_{\mathrm{s}})$$
$$\text{such that } \lim_{N\to\infty} \mathfrak{w}_N = \mathfrak{w} \text{ and } \lim_{N\to\infty} \mathfrak{l}_{\mathrm{path}}^N(\mathfrak{w})(t) \ne \mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(t)\}.$$

We set $\mathrm{Disc}[\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(u) - \mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(t)]$ similarly.

Lemma 8.1 *For each* $0 \le t < u < \infty$,

$$\mathfrak{P}_{\xi d\mu}(\mathrm{Disc}[\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(t)]) = 0, \tag{8.1}$$

$$\mathfrak{P}_{\xi d\mu}(\mathrm{Disc}[\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(u) - \mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(t)]) = 0. \tag{8.2}$$

Proof Let $W_{\mathrm{NE}}(\mathfrak{S}_{\mathrm{si}})$ and $W_{\mathrm{NE}}(\mathfrak{S}_{\mathrm{s}})$ be as in (2.9). By definition, $W_{\mathrm{NE}}(\mathfrak{S}_{\mathrm{si}}) \subset W_{\mathrm{NE}}(\mathfrak{S}_{\mathrm{s}}) \subset W(\mathfrak{S}_{\mathrm{s}})$. Recall that $\mathfrak{P}_{\mu}(W_{\mathrm{NE}}(\mathfrak{S}_{\mathrm{si}})) = 1$ by **(A4)**.

Let t be fixed. It is easy to show that $\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(t)$ restricted on $W_{\mathrm{NE}}(\mathfrak{S}_{\mathrm{s}})$ is a continuous function in $(\mathfrak{l}(\mathfrak{w}(0)), \mathfrak{w})$ in the sense that if $(\mathfrak{l}(\mathfrak{w}_N(0)), \mathfrak{w}_N)$ converge to $(\mathfrak{l}(\mathfrak{w}(0)), \mathfrak{w})$, then $\mathfrak{l}^N_{\mathrm{path}}(\mathfrak{w}_N)(t)$ converge to $\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(t)$. Hence, from **(C1)**, we have

$$\mathfrak{P}_{\mu}(\mathrm{Disc}[\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})(t)]) = 0. \tag{8.3}$$

Because $\mathfrak{P}_{\xi d\mu}$ is absolutely continuous with respect to $\mathfrak{P}_{\mu}$, we deduce (8.1) from (8.3).

From (8.1) with a simple calculation, we obtain (8.2). □

Proof of Theorem 3.4 From **(A4)**, **(C2)**, and (3.22), we can construct the labeled processes $\mathbf{X}^N_{r_N} = (\mathfrak{l}_{\mathrm{path}}(\mathfrak{X}^N_{r_N}), o, o, \ldots)$ and $\mathbf{X} = \mathfrak{l}_{\mathrm{path}}(\mathfrak{X})$. Note that the initial distribution of $\mathfrak{X}^N_{r_N}$ has a density in $L^2(\mu^N_{r_N})$ from **(B3)**. Hence, it is sufficient for the tightness to prove the case in which $\mathfrak{X}^N_{r_N}$ start from the stationary distribution $\mu^N_{r_N}$. Let

$$\mu^{N,m}_{r_N} = \mu^N \circ \pi^{-1}_{\overline{S}_{r_N}}(\cdot\,|\mathfrak{s}(\overline{S}_{r_N}) \ge m).$$

By construction, $\mu^{N,m}_{r_N}(\{\mathfrak{s};\ m \le \mathfrak{s}(\overline{S}_{r_N}) < \infty\}) = 1$.

We assume that $\mathfrak{X}^N_{r_N}(0) \overset{\mathrm{law}}{=} \mu^{N,m}_{r_N}$ in the rest of the proof. We write $\mathbf{X}^N_{r_N} = (X^{N,i}_{r_N})_{i=1}^{\infty}$. To apply the Lyons–Zheng decomposition to $X^{N,i}_{r_N}$, $1 \le i \le m$, we use the m-labeled process such that

$$\mathbf{X}^{N,[m]}_{r_N} = \left((X^{N,i}_{r_N})_{i=1}^m, \sum_{i=m+1}^{n_N} \delta_{X^{N,i}_{r_N}}\right).$$

Then, $\mathbf{X}^{N,[m]}_{r_N}$ is the diffusion process associated with the Dirichlet form $(\mathscr{E}^{N,[m]}_{r_N}, \widetilde{\mathscr{D}}^{N,[m]}_{r_N})$ on $L^2(\overline{S}^m_{r_N} \times \mathfrak{S}_{r_N}, \mu^{N,[m]}_{r_N})$. Here, $\mu^{N,[m]}_{r_N}$ is the m-Campbell measure of $\mu^{N,m}_{r_N}$ and $\mathscr{E}^{N,[m]}_{r_N}$ is the Dirichlet form such that

$$\mathscr{E}^{N,[m]}_{r_N}(f, g) = \int_{\overline{S}^m_{r_N} \times \mathfrak{S}_{r_N}} \mathbb{D}^{a,[m]}[f, g]\, d\mu^{N,[m]}_{r_N}.$$

Furthermore, $\mathbb{D}^{a,[m]}$ is the carré du champ on $\overline{S}^m_{r_N} \times \mathfrak{S}_{r_N}$ such that

$$\mathbb{D}^{a,[m]}[f, g](\mathbf{x}, \mathfrak{s}) = \frac{1}{2}\sum_{i=1}^m \left(a\left(x_i, \left\{\sum_{j\ne i}^m \delta_{x_j}\right\} + \mathfrak{s}\right)\nabla_{x_i} f(\mathbf{x}, \mathfrak{s}), \nabla_{x_i} g(\mathbf{x}, \mathfrak{s})\right)_{\mathbb{R}^d} + \mathbb{D}^a[f, g](\mathbf{x}, \mathfrak{s}),$$

where $\mathbf{x} = (x_1, \ldots, x_m) \in S^m$, and we regard $\mathbb{D}^a$ as the carré du champ on $\overline{S}^m_{r_N} \times \mathfrak{S}_{r_N}$ in an obvious fashion. The domain $\widetilde{\mathscr{D}}^{N,[m]}_{r_N}$ is taken to be the closure of

$$\left\{f \in C_0^{\infty}(S^m) \otimes \mathscr{D}_{\circ};\ \mathscr{E}^{N,[m]}_{r_N}(f, f) < \infty,\ f \in L^2(\overline{S}^m_{r_N} \times \mathfrak{S}_{r_N}, \mu^{N,[m]}_{r_N})\right\}.$$

We see that $X^{N,i}_{r_N}$, $1 \le i \le m$, is an additive functional of the $\mu^{N,[m]}_{r_N}$-symmetric, conservative diffusion $\mathbf{X}^{N,[m]}_{r_N}$. Moreover, $X^{N,i}_{r_N}$ is a Dirichlet process of $\mathbf{X}^{N,[m]}_{r_N}$. Here, a Dirichlet process is an additive functional of a Markov process associated with a Dirichlet form given by the composition of the Markov process with a function belonging to the domain of the Dirichlet form locally. Thus, we can apply the Lyons–Zheng decomposition to $X^{N,i}_{r_N}$. Note that we cannot apply the Lyons–Zheng decomposition to $X^{N,i}_{r_N}$ as an additive functional of the unlabeled diffusion $\mathfrak{X}^N_{r_N}$ directly, because $X^{N,i}_{r_N}$ is not a Dirichlet process of $\mathfrak{X}^N_{r_N}$. See Section 9 in [16] for the proof of the Lyons–Zheng decomposition.

Let $M^{[x_i]}$ be a continuous martingale additive functional of $\mathbf{X}^{N,[m]}_{r_N}$ such that

$$M^{[x_i]}_t\big(\mathbf{X}^{N,[m]}_{r_N}\big) = \int_0^t \sigma(X^{N,i}_{r_N}(u), \mathfrak{X}^{N,i\diamond}_{r_N}(u))dB^{N,i}(u). \tag{8.4}$$

For $T > 0$, we set $\mathscr{R}_T(\mathbf{w})(t) := \mathbf{w}(T-t)$. For each $0 \le t \le T$ and $1 \le i \le m$, we set

$$\begin{aligned} M^i_t &= M^{[x_i]}_t\big(\mathbf{X}^{N,[m]}_{r_N}\big), \\ M^{*i}_t &= M^{[x_i]}_{T-t}\big(\mathscr{R}_T(\mathbf{X}^{N,[m]}_{r_N})\big) - M^{[x_i]}_T\big(\mathscr{R}_T(\mathbf{X}^{N,[m]}_{r_N})\big). \end{aligned} \tag{8.5}$$

Using the Lyons–Zheng decomposition for solutions of SDE (3.24) with the function x_i, we have that, for each $0 \le t \le T$ and $1 \le i \le m$, that

$$X^{N,i}_{r_N}(t) - X^{N,i}_{r_N}(0) = \frac{1}{2}\big\{M^i_t + M^{*i}_t\big\} \tag{8.6}$$

and that under $\underline{\widetilde{\mathfrak{P}}}^{N,[m]}_{r_N,(x,\mathfrak{s})}$ for $\mu^{N,[m]}_{r_N}$-a.e. $(x,\mathfrak{s})$, M^i and M^{*i} are continuous martingales such that $M^i_0 = M^{*i}_0 = 0$. Here, $\underline{\widetilde{\mathfrak{P}}}^{N,[m]}_{r_N,(x,\mathfrak{s})}$ is the distribution of the diffusion process $\mathbf{X}^{N,[m]}_{r_N}$ associated with the Dirichlet form $(\mathscr{E}^{N,[m]}_{r_N}, \underline{\widetilde{\mathscr{D}}}^{N,[m]}_{r_N})$ on $L^2(\overline{S}^m_{r_N} \times \mathfrak{S}_{r_N}, \mu^{N,[m]}_{r_N})$ starting at $(x,\mathfrak{s})$.

From (2.1) and (8.4)–(8.6), there exists a constant c_{14} independent of i such that

$$E[|X^{N,i}_{r_N}(t) - X^{N,i}_{r_N}(u)|^4] \le c_{14}|t-u|^2 \ \text{ for all } 0 \le t, u \le T. \tag{8.7}$$

Using (3.22), we see that $\{X^{N,i}_{r_N}(0)\}_{N\in\mathbb{N}}$ is tight in S for each $1 \le i \le m$. Combining this with (8.7), we easily obtain the tightness of $\{X^{N,i}_{r_N}\}_{N\in\mathbb{N}}$ in $C([0,T];S)$ for each $1 \le i \le m$. Because T is arbitrary, this implies the tightness of $\{X^{N,i}_{r_N}\}_{N\in\mathbb{N}}$ in $C([0,\infty);S)$ for each $1 \le i \le m$.

Taking an arbitrary $m \in \mathbb{N}$, we obtain the tightness of $\{X^{N,i}_{r_N}\}_{N\in\mathbb{N}}$ in $C([0,\infty);S)$ for all $i \in \mathbb{N}$. From this, we deduce the tightness of $\mathbf{X}^N_{r_N} = (X^{N,i}_{r_N})_{i=1}^\infty$ in $C([0,\infty);S^{\mathbb{N}})$. Here, we endow $C([0,\infty);S^{\mathbb{N}}) = \prod_{i\in\mathbb{N}} C([0,\infty);S)$ with the product topology. We use the fact that, in general, the tightness of random variables with the value of a countable product of Polish spaces follows from that of each component-wise random variable.

Recall that $\mathbf{X}^N_{r_N} = (\mathfrak{l}_{\mathrm{path}}(\mathfrak{X}^N_{r_N}), o, o, \ldots)$ and $\mathbf{X} = \mathfrak{l}_{\mathrm{path}}(\mathfrak{X})$. From the tightness of $\mathfrak{X}^N_{r_N}(0)$, **(C3)**, and (8.7), it easily follows that $\mathfrak{X}^N_{r_N}$ is tight in $W(\mathfrak{S})$.

From Theorem 3.1, Lemma 8.1, **(LIN)**, and the tightness of $\mathfrak{X}^N_{r_N}$ in $W(\mathfrak{S})$, we see that the random variables $\mathbf{X}^N_{r_N}(t)$ and $\mathbf{X}^N_{r_N}(u) - \mathbf{X}^N_{r_N}(t)$ converge weakly to $\mathbf{X}(t)$ and $\mathbf{X}(u) - \mathbf{X}(t)$, respectively. Thus, we have the convergence of the finite-dimensional distributions of $\mathbf{X}^N_{r_N}$ to those of $\mathbf{X}$. From this, we deduce the convergence of the finite-dimensional distributions of $\mathbf{X}^{N,m}_{r_N}$ to $\mathbf{X}^m$. Collecting these results, we obtain Theorem 3.4. □

Proof of Theorem 3.5 From the proof of Theorem 3.4, we see that $\{\mathfrak{X}^N_{r_N}\}$ is tight in $W(\mathfrak{S})$. Hence, Theorem 3.5 follows from Theorem 3.4. □

9 A sufficient condition for (C3)

In this section, we present a sufficient condition for **(C3)** in terms of initial distributions. Let $\mathfrak{l}^N$ and $\mu^N_{r_N}$ be as in Sect. 3.2. Let $\mathscr{R}(t) = \int_t^\infty (1/\sqrt{2\pi})e^{-|x|^2/2}dx$ be a (scaled) complementary error function. We specify the following condition on the initial distributions $\mu^N_{r_N} \circ (\mathfrak{l}^N)^{-1}$, $N \in \mathbb{N}$.

(D) For any positive numbers R and T,

$$\lim_{l\to\infty}\Big\{\sup_{N\in\mathbb{N}}\int\sum_{i\geq l}\mathscr{R}\Big(\frac{|s_i|-R}{T}\Big)\mu^N_{r_N}\circ(\mathfrak{l}^N)^{-1}(d\mathbf{s})\Big\} = 0. \tag{9.1}$$

Lemma 9.1 *Assume that* ***(D)*** *holds. Then,* $(\mathfrak{X}^N_{r_N}, \underline{\widetilde{\mathfrak{P}}}^N_{r_N})$ *satisfies* ***(C3)****.*

Proof Let $(\mathfrak{X}^N_{r_N}, \underline{\widetilde{\mathfrak{P}}}^N_{r_N})$ be the diffusion defined in Sect. 3.2. Then, $(\mathfrak{X}^N_{r_N}, \underline{\widetilde{\mathfrak{P}}}^N_{r_N})$ is the diffusion associated with $(\mathscr{E}^N_{r_N}, \underline{\widetilde{\mathscr{D}}}^N_{r_N})$ on $L^2(\mu^N_{r_N})$. We denote the distribution of the diffusion with the initial distribution $\mu^N_{r_N}$ using the same symbol $\underline{\widetilde{\mathfrak{P}}}^N_{r_N}$.

From (2.15), we have that

$$\begin{aligned}\underline{\widetilde{\mathfrak{P}}}^N_{r_N}\Big(\mathfrak{m}_{R,T}(\mathfrak{l}^N_{\mathrm{path}}(\mathfrak{X}^N_{r_N})) > l\Big) &= \underline{\widetilde{\mathfrak{P}}}^N_{r_N}\Big(\bigcup_{i\geq l}\Big\{\inf_{t\in[0,T]}|X^i_t| \leq R\Big\}\Big)\\ &\leq \sum_{i\geq l}\underline{\widetilde{\mathfrak{P}}}^N_{r_N}\Big(\inf_{t\in[0,T]}|X^i_t| \leq R\Big).\end{aligned} \tag{9.2}$$

Let $(\mathscr{E}^{N,[m]}_{r_N}, \underline{\widetilde{\mathscr{D}}}^{N,[m]}_{r_N})$ be the Dirichlet form on $L^2(\overline{S}^m_{r_N}\times\mathfrak{S}_{r_N}, \mu^{N,[m]}_{r_N})$ given in the proof of Theorem 3.4 in Sect. 8. Let $\underline{\widetilde{\mathfrak{P}}}^{N,[m]}_{r_N,(x,\mathfrak{s})}$ be the distribution of the associated diffusion process starting at $(x,\mathfrak{s})$, as before.

From (8.6), we see that, under $\underline{\widetilde{\mathfrak{P}}}^{N,[m]}_{r_N,(x,\mathfrak{s})}$ for $\mu^{N,[m]}_{r_N}$-a.e. $(x,\mathfrak{s})$, the stochastic processes M^i and M^{*i} are continuous martingales with $M^i_0 = M^{*i}_0 = 0$ satisfying

$$X^{N,i}_{r_N}(t) - X^{N,i}_{r_N}(0) = \frac{1}{2}\{M^i_t + M^{*i}_t\}. \tag{9.3}$$

We set $X^i = X^{N,i}_{r_N}$, $\mathbf{X}^m = (X^1,\dots,X^m)$, and $\mathfrak{X}^{m*} = \sum_{i>m}^{n_N}\delta_{X^i}$. Here, n_N is the number of particles in $\overline{S}_{r_N}$ such that $n_N = \mathfrak{X}^N_{r_N}(\overline{S}_{r_N})(0)$. By construction, $\mathbf{X}^m = \mathfrak{l}^{N,m}(\mathfrak{X}^N_{r_N})$. Using (2.18) of Theorem 2.4 in [26], we have the identity

$$\underline{\widetilde{\mathfrak{P}}}^{N,[m]}_{r_N,(x,\mathfrak{s})} = \underline{\widetilde{\mathfrak{P}}}^N_{r_N}((\mathbf{X}^m,\mathfrak{X}^{m*}) \in \cdot\,|(\mathbf{X}^m,\mathfrak{X}^{m*})(0) = (x,\mathfrak{s})). \tag{9.4}$$

Using (9.3) and (9.4), we see that $X^i = X^{N,i}_{r_N}$ satisfies (9.3) under $\underline{\widetilde{\mathfrak{P}}}^N_{r_N}$. Hence, we have that

$$\begin{aligned}
\underline{\widetilde{\mathfrak{P}}}^N_{r_N}\left(\inf_{t\in[0,T]}|X^i_t|\le R\right) &\le \underline{\widetilde{\mathfrak{P}}}^N_{r_N}\left(\sup_{t\in[0,T]}|X^i_t - X^i_0| \ge |X^i_0| - R\right)\\
&\le \underline{\widetilde{\mathfrak{P}}}^N_{r_N}\left(\sup_{t\in[0,T]}|M^i_t| \ge |X^i_0| - R\right) + \underline{\widetilde{\mathfrak{P}}}^N_{r_N}\left(\sup_{t\in[0,T]}|M^{*i}_t| \ge |X^i_0| - R\right) \quad \text{by (9.3), (9.4)}\\
&= 2\,\underline{\widetilde{\mathfrak{P}}}^N_{r_N}\left(\sup_{t\in[0,T]}|M^i_t| \ge |X^i_0| - R\right). \qquad (9.5)
\end{aligned}$$

Let $\underline{\widetilde{\mathfrak{E}}}^N_{r_N}$ denote the expectation with respect to $\underline{\widetilde{\mathfrak{P}}}^N_{r_N}$. Then, using the martingale inequality, we see that there exists a positive constant c_{15} such that

$$\begin{aligned}
\underline{\widetilde{\mathfrak{P}}}^N_{r_N}\left(\sup_{t\in[0,T]}|M^i_t| \ge |X^i_0| - R\right) &\le \underline{\widetilde{\mathfrak{E}}}^N_{r_N}\left(\mathscr{R}\left(\frac{|X^i_0| - R}{c_{15}T}\right)\right)\\
&= \int \mathscr{R}\Big(\frac{|s_i| - R}{c_{15}T}\Big)\mu^N_{r_N}\circ(\mathfrak{l}^N)^{-1}(d\mathbf{s}). \qquad (9.6)
\end{aligned}$$

Combining (9.1), (9.2), (9.5), and (9.6), we can deduce that

$$\sup_{N\in\mathbb{N}}\underline{\widetilde{\mathfrak{P}}}^N_{r_N}(\mathfrak{m}_{R,T}(\mathfrak{l}^N_{\mathrm{path}}(\mathfrak{X}^N_{r_N})) > l) \le 2\sup_{N\in\mathbb{N}}\sum_{i\ge l}\int_{\mathfrak{S}}\mathscr{R}\left(\frac{|s_i| - R}{c_{15}T}\right)\mu^N_{r_N}\circ(\mathfrak{l}^N)^{-1}(d\mathbf{s}).$$

Hence, **(D)** implies **(C3)**. This completes the proof. □

10 Examples of dynamical universality

In this section, we give some examples of dynamical universality. We consider the sine_β random point field, $\beta = 1, 2, 4$, and the Ginibre random point field. All examples satisfy the assumptions in Theorems 3.1–3.5, and the main theorems are thus applicable to these examples.

For these random point fields, **(A1)** is proved in [28]. We have **(A2)** from [15]. Assumption **(A3)** obviously holds. We obtain **(A4)** from [31] based on the result in [25]. We have **(A5)** from [27]. We check **(A6)** in [15]. Both **(ZC)** and **(C2)** hold according to [25]. In [25], these are only proved for $R = \infty$. The current case can be proved in a similar fashion. The quasi-Gibbs property is checked in [28]. Assumptions **(B1)** and **(B2)** are clear because the random point fields in these conditions are supported on finite particle systems. Condition (3.19) in **(B4)**$''$ holds because the correlation functions come from determinants of matrices given by kernels that are uniformly bounded on $S_R \times S_R$ for each $R \in \mathbb{N}$ (see, for example, [28, Lemma 10.1] for the Ginibre random point field). As for **(C3)**, we present a sufficient condition in Sect. 9, which is satisfied by all the examples. Condition **(C4)** obviously holds.

10.1 The sine$_\beta$ interacting Brownian motion with $\beta = 1, 2, 4$

Let $\mu^N_{V,\beta,\theta}$ be the random point field whose density is $\mathbf{m}^N_{V,\beta,\theta}$ given by (1.8). Then, the logarithmic derivative $\mathrm{d}^N_{r_N}$ of $\mu^N_{V,\beta,\theta}$ on S_{r_N} is given by

$$\mathrm{d}^N_{r_N}(x,\mathfrak{s}) = -\frac{1}{\boldsymbol{\rho}_V(\theta)}V'_\beta\Big(\frac{x}{N\boldsymbol{\rho}_V(\theta)}+\theta\Big) + \frac{\beta}{2}\sum_{s_i\in S_{r_N}}\frac{1}{x-s_i}+\frac{\beta}{2}\int_{S\setminus S_{r_N}}\frac{1}{x-y}\rho^{N,1}(y)dy. \tag{10.1}$$

From (3.24), (3.25), and (10.1), we see that the associated SDE is given by

$$\begin{aligned} X^{N,i}_{r_N}(t) - X^{N,i}_{r_N}(0) &= B^i(t) - \int_0^t \frac{1}{\boldsymbol{\rho}_V(\theta)}V'_\beta\Big(\frac{X^{N,i}_{r_N}(u)}{N\boldsymbol{\rho}_V(\theta)}+\theta\Big)du \\ &+ \frac{\beta}{2}\int_0^t\sum_{j\neq i}^N\frac{1}{X^{N,i}_{r_N}(u)-X^{N,j}_{r_N}(u)}du + \frac{\beta}{2}\int_{S\setminus S_{r_N}}\frac{1}{X^{N,i}_{r_N}(t)-y}\rho^{N,1}(y)dy \\ &+ \frac{1}{2}\int_0^t \mathbf{n}^{r_N}(X^{N,i}_{r_N}(u))L^{N,i}_{r_N}(du). \end{aligned} \tag{10.2}$$

For $\beta = 1, 2, 4$ and V as in (1.2), condition (3.18) in **(B4)**$''$ is proved in [5]. For $\beta = 2$ with a real analytic function V satisfying (1.1), condition (3.18) in **(B4)**$''$ is proved in [6]. Hence, we apply the results in Theorems 3.1–3.5 to these models.

Proposition 10.1 *The following equation holds. In particular,* (1.16) *holds.*

$$\begin{aligned} &\lim_{N\to\infty} P\Big(\max_{0\le t\le T}\Big\{X^{N,i}(t) - X^{N,i}(0) - \frac{\beta}{2}\int_0^t\sum_{j\neq i}^{n_N}\frac{1}{X^{N,i}(u)-X^{N,j}(u)}du \\ &\quad + c_{16}(\beta)t - \frac{\beta}{2}\int_{S\setminus S_{r_N}}\frac{1}{X^{N,i}(t)-y}\rho^{N,1}(y)dy\Big\}\ge a\Big) \\ &= 2\int_{x\ge a}\frac{1}{\sqrt{2\pi T}}e^{-|x|/2T}dx. \end{aligned}$$

Here we set $c_{16}(\beta) = c_1$ for $\beta = 1, 2$ and $c_{16}(4) = 2c_1$.

Proof We write $\mu^N = \mu^N_{V,\beta,\theta}$. We set $\mu^N_{r_N} = \mu^N\circ\pi^{-1}_{r_N}$. Recall that the unlabeled dynamics $\mathfrak{X}^N_{r_N}$ are $\mu^N_{r_N}$-reversible. Let $\mathbf{X}^N_{r_N} = (\mathfrak{l}_{\mathrm{path}}(\mathfrak{X}^N_{r_N}), o, o, \ldots)$ be as in the proof of Theorem 3.4. Let $\widetilde{P}^N = P^N\circ(\mathbf{X}^N_{r_N})^{-1}$. We write $\mathbf{w} = (w^i)_{i\in\mathbb{N}}$. Applying the Lyons–Zheng decomposition to w^i, we have that

$$w^i(t) - w^i(0) = \frac{1}{2}\{B^i(\mathbf{w})(t) + B^i(\mathscr{R}_T(\mathbf{w}))(t)\}, \tag{10.3}$$

where $\mathscr{R}_T : C([0,T];\mathbb{R}^{\mathbb{N}}) \to C([0,T];\mathbb{R}^{\mathbb{N}})$ such that $\mathscr{R}_T(\mathbf{w})(t) = \mathbf{w}(T-t)$. The function B^i is a Brownian motion under $\widetilde{P}^N$.

From (1.4) and (1.15), we have that

$$\lim_{N\to\infty}\frac{1}{\boldsymbol{\rho}_V(\theta)}V'_\beta\Big(\frac{x}{N\boldsymbol{\rho}_V(\theta)}+\theta\Big) = c_{16} \tag{10.4}$$

uniformly in $x \in S_R$ for each $R \in \mathbb{N}$. Hence, from (10.3) and (10.4), we have the following for each $\epsilon > 0$

$$\lim_{N\to\infty} \widetilde{P}^N\Big(\max_{0\le t\le T} \Big| \int_0^t \frac{1}{\boldsymbol{\rho}_V(\theta)} V_\beta'\Big(\frac{w^i(u)}{N\boldsymbol{\rho}_V(\theta)} + \theta\Big)du - c_{16}t\Big| \ge \epsilon\Big) = 0. \tag{10.5}$$

Because $\lim_{N\to\infty} r_N = \infty$, (10.3) implies that

$$\lim_{N\to\infty} \widetilde{P}^N\Big(\max_{0\le t\le T} \Big|\frac{1}{2}\int_0^t \mathbf{n}^{r_N}(w^i(u))L^i_{r_N}(du)\Big| \ge \epsilon\Big) = 0. \tag{10.6}$$

Under $\widetilde{P}^N$, we can rewrite (10.2) as

$$\begin{aligned} w^i(t) - w^i(0) = B^i(t) - \int_0^t \frac{1}{\boldsymbol{\rho}_V(\theta)} V_\beta'\Big(\frac{w^i(u)}{N\boldsymbol{\rho}_V(\theta)} + \theta\Big)du + \frac{\beta}{2}\int_0^t \sum_{j\ne i}^N \frac{1}{w^i(u) - w^j(u)}du \\ + \frac{\beta}{2}\int_{S\setminus S_{r_N}} \frac{1}{w^i(t) - y}\rho^{N,1}(y)dy + \frac{1}{2}\int_0^t \mathbf{n}^{r_N}(w^i(u))L^i_{r_N}(du). \end{aligned}$$

Then, we have that

$$\begin{aligned} & w^i(t) - w^i(0) + \int_0^t \frac{1}{\boldsymbol{\rho}_V(\theta)} V_\beta'\Big(\frac{w^i(u)}{N\boldsymbol{\rho}_V(\theta)} + \theta\Big)du \\ & - \frac{\beta}{2}\int_0^t \sum_{j\ne i}^N \frac{1}{w^i(u) - w^j(u)}du - \frac{\beta}{2}\int_{S\setminus S_{r_N}} \frac{1}{w^i(t) - y}\rho^{N,1}(y)dy \\ & - \frac{1}{2}\int_0^t \mathbf{n}^{r_N}(w^i(u))L^i_{r_N}(du) = B^i(t). \end{aligned} \tag{10.7}$$

The boundary of the set $\{w \in C([0,\infty);\mathbb{R})\,;\, \max_{0\le t\le T} w(t) \ge a\}$ has Wiener measure zero. Combining this with (10.5)–(10.7), and using Theorem 3.4, we obtain

$$\begin{aligned} & \lim_{N\to\infty} \widetilde{P}^N\Big(\max_{0\le t\le T} \Big\{ w^i(t) - w^i(0) + c_{16}t - \frac{\beta}{2}\int_0^t \sum_{j\ne i}^N \frac{1}{w^i(u) - w^j(u)}du \\ & - \frac{\beta}{2}\int_{S\setminus S_{r_N}} \frac{1}{w^i(t) - y}\rho^{N,1}(y)dy\Big\} \ge a\Big) \\ & = P\big(\max_{0\le t\le T} B(t) \ge a\big) = 2\int_{x\ge a} \frac{1}{\sqrt{2\pi T}} e^{-|x|/2T}dx. \end{aligned}$$

Thus, the proof is complete. □

10.2 The Ginibre interacting Brownian motion

We apply our result to the random matrix model with strong non-Hermiticity introduced in [1]. Condition (3.18) in **(B4)**$''$ follows from Proposition 1.1. The SDE for the finite particle system is given by (1.23) with the addition of the following two terms on the right-hand side.:

$$\int_{S\setminus S_{r_N}} \frac{X^{N,i}_{r_N}(t) - y}{|X^{N,i}_{r_N}(t) - y|^2}\rho^{N,1}(y)dy + \frac{1}{2}\int_0^t \mathbf{n}^{r_N}(X^{N,i}_{r_N}(u))L^{N,i}_{r_N}(du).$$

Thus, we obtain the results of Theorems 3.1–3.5 for the Ginibre random point field.

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Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest We declare that we have no conflict of interest.

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