Generalized Pitman's transform and discrete integrable systems

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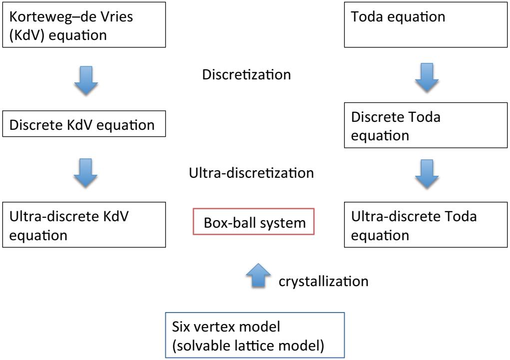
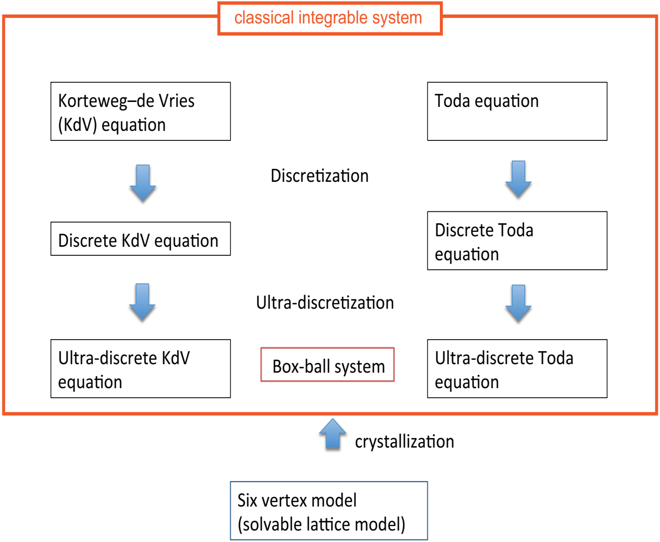
Integrable systems around box-ball system



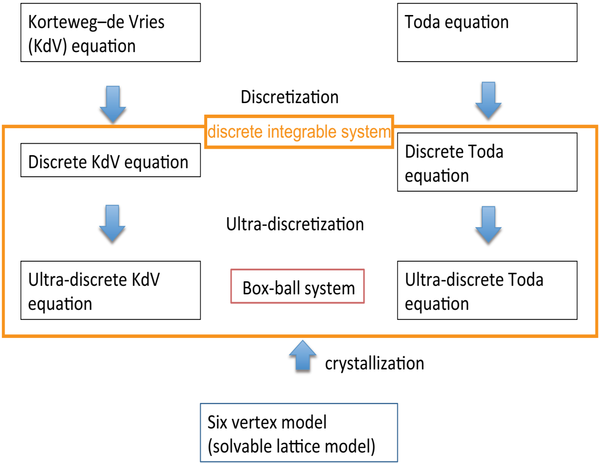
Ud-KDV equation : Euler representation of BBS

Ud-Toda equation : Lagrange representation of BBS

Classical integrable systems

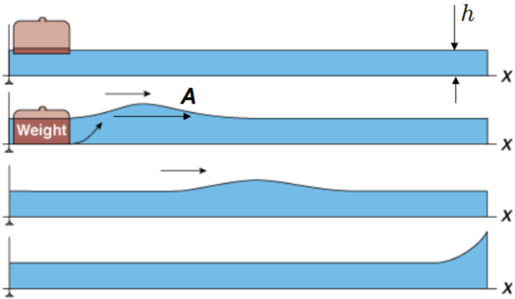


Discrete integrable systems



Korteweg-de Vries (KdV) equation

In the 1830s, John Scott Russell observed the formation of solitary waves with constant shape and constant speed, or 'solitons', in canals:



Source: Shnir

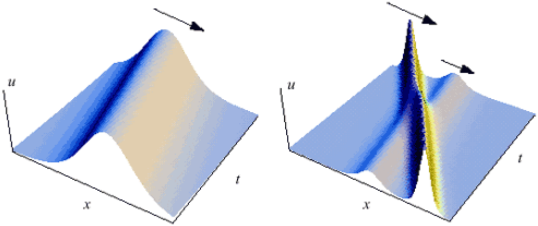
To model such shallow water waves, Korteweg and de Vries introduced the following equation:

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Solitons in KdV

The KdV admits soliton solutions. For example, with $u(x,t) \to 0$ as $x \to \infty$, 1- soliton solutions are given by:

$$u(x,t) = \frac{1}{2}c^2 \mathrm{sech}^2\left(\frac{1}{2}c(x + c^2 t)\right),$$



KdV 1-soliton KdV 2-soliton

Source: Brunelli

Toda equation (Toda lattice)

Chain of oscillators with potential function $V(r) = \exp(-r) + r - 1$.

$$\begin{cases} \frac{dq_n}{dt} & = \frac{\partial H}{\partial p_n} = p_n \\ \frac{dp_n}{dt} & = -\frac{\partial H}{\partial q_n} = -\exp(-(q_{n+1} - q_n)) + \exp(-(q_n - q_{n-1})) \end{cases}$$

Change of variables: $I_n := p_n$, $V_n := -\exp(-(q_{n+1} - q_n))$

$$\begin{cases} \frac{dI_n}{dt} & = V_n - V_{n-1} \\ \frac{dV_n}{dt} & = V_n(I_n - I_{n+1}) \end{cases}$$

- Infinitely many conserved quantities
- Integrable system (Lax pair expression exists)

Motivation

- The discrete integrable systems are well studied as classical integrable systems
- Explicit special solutions (soliton solutions, tau functions)
- Relation to special orthogonal polynomials
- Relation to crystals (which relate to solvable lattice models)
- Initial value problem with periodic or zero boundary condition

Motivation

- The discrete integrable systems are well studied as classical integrable systems
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- Initial value problem with periodic or zero boundary condition

Questions

- Start the dynamics from random initial condition
- Well-posedness of the initial value problem with non-zero boundary condition (or simply, definition of infinite system. Non-trivial also for higher spin stochastic vertex models)
- Invariant measures? Generalized Gibbs ensembles? (Cf. GGE and random matrix for Toda lattice (Spohn, 2019), White-noise is invariant for KdV (Killip-Murphy-Visan, 2019))
- Various scaling limit (between different models, generalized HDL, integrated current, tagged particle...)
- Relation to other models (vertex models, LPP, random polymer)

Today's talk

- Well-posedness of the initial value problem with non-zero boundary condition
- Some class of invariant measures

"Key idea : Pitman's type transform"

Remark

Relation between (some versions of) Pitman's transform and several important integrable systems, and its application to stochastic models have been studied by O'Connell and his collaborators.
(Quantum Toda lattice, random polymers, random matrices, KPZ equation...)

Pitman's transform

One-sided version $\quad S : \mathbb{R}_{\geq 0} \to \mathbb{R}$: a continuous path, $S_0 = 0$

$$M_x := \max_{0 \leq y \leq x} S_y, \qquad x \in \mathbb{R}_{\geq 0}$$

$$TS_x := 2M_x - S_x$$

Theorem (Pitman)

S : BM (+drift) $\to$ TS : 3-dimensional Bessel process (+drift)

Two-sided version $\quad S : \mathbb{R} \to \mathbb{R}$: a continuous path, $S_0 = 0$

$$M_x := \max_{y \leq x} S_y, \qquad x \in \mathbb{R}$$

$$TS_x := 2M_x - S_x - 2M_0$$

Theorem (Harrison-Williams)

S : BM + positive drift $\to$ $S \overset{d}{=} TS$

Exponential version of Pitman's transform

One-sided version $S : \mathbb{R}_{\geq 0} \to \mathbb{R}$: a continuous path, $S_0 = 0$

$$M_x := \log \int_0^x \exp(S_y) dy, \qquad x \in \mathbb{R}_{\geq 0}$$

$$TS_x := 2M_x - S_x$$

Theorem (Matsumoto-Yor)

S : BM $\to$ TS : BM conditioned to survive in the potential e^{-x}

Two-sided version $S : \mathbb{R} \to \mathbb{R}$: a continuous path, $S_0 = 0$

$$M_x := \log \int_{-\infty}^x \exp(S_y) dy, \qquad x \in \mathbb{R}$$

$$TS_x := 2M_x - S_x - 2M_0$$

Theorem (O'Connell-Yor)

S : BM + positive drift $\to$ $S \overset{d}{=} TS$

Pitman's type transform for two-sided path

Path spaces

- $\mathcal{F} := \{f : \mathbb{Z} \to \mathbb{R}\}$ or $\{f : \mathbb{R} \to \mathbb{R} \ ; \ f : \text{continuous}\}$
- $\mathcal{F}_0 := \{f \in \mathcal{F} \ ; f_0 = 0\}$

Let $M : \mathcal{S} \to \mathcal{F}$ for some $\mathcal{S} \subset \mathcal{F}_0$ and $TS \in \mathcal{F}_0$

$$TS := 2M - S - 2M_0 \ (= 2M(S) - S - 2M(S)_0)$$

When $S \stackrel{d}{=} TS$ holds?

Reflections and invariance

Reflection operators:

- $R : \mathcal{F} \to \mathcal{F} : Rf_x := -f_{-x}$
- $\bar{R} : \mathcal{F} \to \mathcal{F} : \bar{R}f_x := f_{-x}$

Let $W(S) = M(S) - S$.

Theorem

Suppose S is a random process supported on $\mathcal{S}$. If

$$(i)\ S \stackrel{d}{=} RS, \quad (ii)\ (W, M - M_0) \stackrel{d}{=} (\bar{R}W, R(M - M_0))$$

then, $S \stackrel{d}{=} TS$.
Moreover, if $M - M_0 = L(W)$a.s. for some functional L satisfying $L(\bar{R}W) = RL(W)$, (ii) is equivalent to

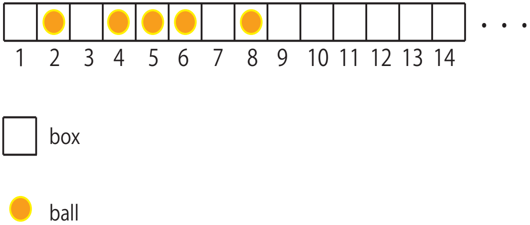
$$(ii)'\ W \stackrel{d}{=} \bar{R}W$$

* For the original Pitman's transform, if S is a two-sided BM, W is a reflected BM and $L(W)$ is a local time of W at 0

Box-Ball System (BBS)

Introduced in 1990 by Takahashi-Satsuma

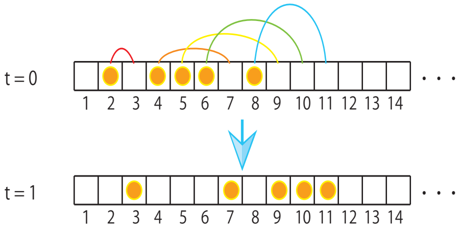
- Discrete time deterministic dynamics (Cellular-Automaton)
- Finite number of balls



Box-Ball System (BBS)

<u>Def 1</u>

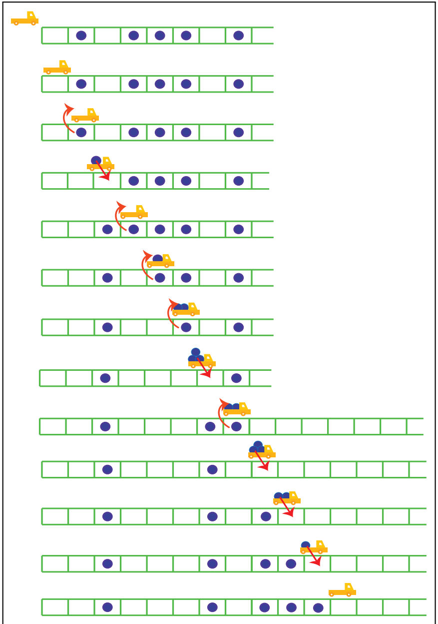
- Every ball moves exactly once in each evolution time step
- The leftmost ball moves first and the next leftmost ball moves next and so on...
- Each ball moves to its nearest right vacant box



Box-Ball System

<u>Def 2</u>

- A carrier moves from left to right
- The carrier picks up a ball if it finds a ball (The carrier can load any number of balls)
- The carrier puts down a ball if it comes to an empty box when it carries at least one ball



Box-Ball System (BBS)

<u>Def 2</u>

- $\eta = (\eta_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}, \sum_{n\in\mathbb{Z}} \eta_n < \infty$
- BBS operator $T : \eta \to T\eta$
- W_n : the number of balls on the carrier as it passes location n
- $T\eta_n = 0$ if $\eta_n = 1$
- $T\eta_n = 1$ if $\eta_n = 0$ and $W_{n-1} \geq 1$
- $W_n = \sum_{m=-\infty}^{n} (\eta_m - T\eta_m)$
- $T\eta_n = \min\{1 - \eta_n, W_{n-1}\}$

What is known for BBS

Key properties

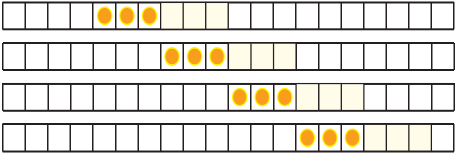
- Solitonic behavior
- Integrable system (infinitely many conserved quantities)
- Initial value problem is solved by various methods
- Reversible as a dynamical system (skew-symmetry)

Connections to many physical models

- Ultra-discretization of discrete KdV equation
- Crystallization of an integrable lattice model (six vertex model)
- Ultra-discretization of discrete Toda equation
- Many variations of BBS have been also studied and known to have connections to variants of above models

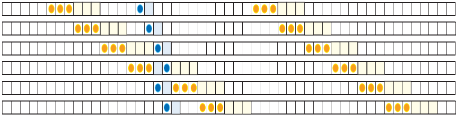
Solitons in BBS

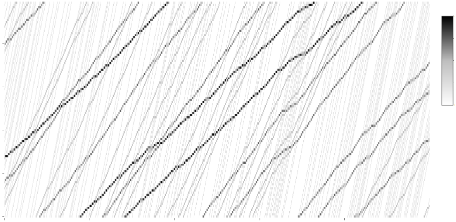
- $(1,0),(1,1,0,0),(1,1,1,0,0,0)$... are "Solitons"
- $(1,1,1,\ldots,1,0,0,0,\ldots,0)$: soliton of size K
- soliton of size K moves with speed K



Solitons in BBS

- Number of each type of solitons is conserved $\Rightarrow \exists$ Infinite number of conserved quantities
- The interaction between solitons are nonlinear
- Integrable system





Our previous result

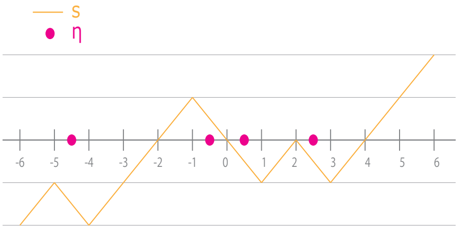
arXiv:1806.02147 (Croydon-Kato-S-Tsujimoto)

- Define the dynamics for configuration with infinitely many particles for both direction (cf. Ferrari-Nguyen-Rolla-Wang)
- Study general properties of invariant measures for BBS
- Find some explicit invariant measures
- Ergodicity of BBS (= Pitman's transform) for the above examples (cf. For BM, the ergodicity of Pitman's transform is open. For the white-noise, the ergodicity of the KdV flow is open.)
- LLN, CLT and LDP for integrated currents of particles at origin for the above examples
- LLN, CLT and LDP for a tagged particle for some of the above examples

Key Observation: BBS is Pitman's transform

Path encoding

- $\eta = (\eta_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$
- $S = (S_n)_{n\in\mathbb{Z}} \in \mathfrak{S}^0$, $\mathfrak{S}^0 := \{S : \mathbb{Z} \to \mathbb{Z};\ S_0 = 0, |S_n - S_{n-1}| = 1\}$
- $\eta \leftrightarrow S$: $S_n - S_{n-1} = 1 - 2\eta_n$: One-to-one
- $\eta_n = 1$: particle $\leftrightarrow$ $S_n - S_{n-1} = -1$: down jump
- $\eta_n = 0$: empty $\leftrightarrow$ $S_n - S_{n-1} = 1$: up jump



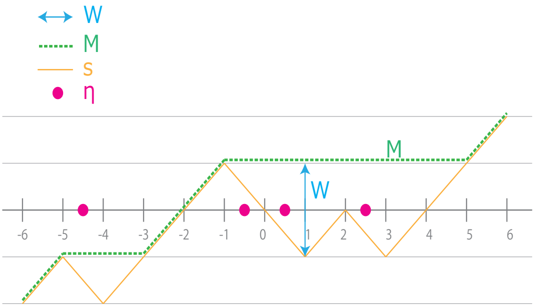
S: Path encoding of η

Past maximum and the carrier via path encoding

Suppose $\sum_{n\in\mathbb{Z}} \eta_n < \infty$

Lemma

$W_n = M_n - S_n$ *where* $M_n = \sup_{m \le n} S_m$.



W_n: the number of particles on the carrier

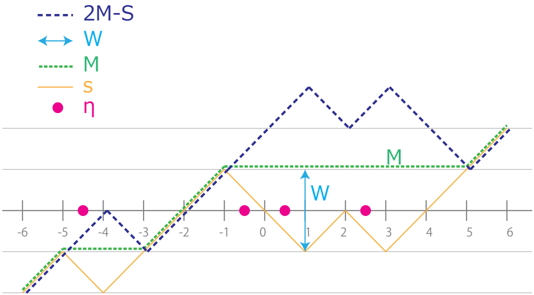
BBS is the reflection w.r.t. the past maximum

Suppose $\sum_{n\in\mathbb{Z}} \eta_n < \infty$.

$S \to TS$ is the reflection with respect to the past maximum :

Lemma

$TS_n = 2M_n - S_n - 2M_0$ *where* TS *is the path encoding of* $T\eta$.



Dynamics with infinitely many particles on both sides

The definition $S \mapsto TS = 2M - S - 2M_0$ can be extended to two-sided functions in the domain:

$$\mathcal{S}^T := \{S \,:\, M_0 < \infty\}.$$

Moreover, can check:

- $\eta_n^{[k]} := \eta_n \mathbf{1}_{\{n \geq k\}}$: truncated configuration
- $T\eta_n^{[k]}$: well-defined for any k
- $T\eta_n = \lim_{k \to -\infty} T\eta_n^{[k]} = \min\{1 - \eta_n, W_{n-1}\}$, where $W = M - S$

For $\eta \in \mathcal{S}^T$, we define the box ball dynamics by $S \mapsto TS = 2M - S - 2M_0$.

A sufficient condition to be invariant under T

Theorem (Three conditions theorem)

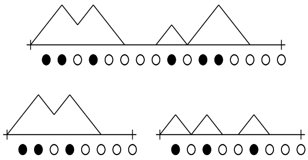
Any two of the three following conditions imply the third:

$$\overleftarrow{\eta} \stackrel{d}{=} \eta \ (\Leftrightarrow RS \stackrel{d}{=} S), \qquad \bar{R}W \stackrel{d}{=} W, \qquad T\eta \stackrel{d}{=} \eta \ (\Leftrightarrow TS \stackrel{d}{=} S)$$

where $\overleftarrow{\eta}$ *is the reversed configuration and* $\bar{R}W$ *is the reversed carrier process given as*

$$\overleftarrow{\eta}_n = \eta_{-(n-1)}, \qquad \bar{R}W_n = W_{-n}.$$

Any one of the three conditions does not imply the others.



Invariant measures for BBS

Applying the three conditions theorem, we can show that the following measures are invariant for BBS:

- Bernoulli product measure with density $p \in [0, \frac{1}{2})$ (Pitaman's theorem for SRW with drift)
- Two sided stationary Markov chain on $\{0, 1\}$ with $p_0 + p_1 < 1$ where $p_i = P(\eta_1 = 1 | \eta_0 = i)$
- Bernoulli product measure with density $p \in [0, 1)$ conditioned on the event $\sup_{n \in \mathbb{Z}} W_n \leq K$
- Periodic Gibbs measures with infinitely many parameters associated to the density of solitons

Results for other discrete integrable systems

- Introduce path encoding for Ud-KdV equation, d-KdV equation, Ud-Toda equation, d-Toda equation
- Define an infinite system of them through a Pitman's type transform
- Three conditions theorem for Ud-KdV equation, d-KdV equation, Ud-Toda equation
- Apply the three conditions theorem to obtain an explicit class of invariant measures

Ultra-discrete KdV (Ud-KdV) equation

- $L > 0$: model parameter

$$\eta_n^{t+1} = \min\{L - \eta_n^t, \sum_{m=-\infty}^{n-1} (\eta_m^t - \eta_m^{t+1})\}$$

Remark

$L = 1, \eta_n^t \in \{0, 1\}$ *is BBS.* $L \in \mathbb{N}, \eta_n^t \in \{0, 1, 2, \ldots, L\}$ *is BBS with boxes of capacity* L.

Path encoding and Pitman's transform for UdKdV equation

$$S_n - S_{n-1} := L - 2\eta_n, \quad S_0 := 0, \quad M_n := \max_{m \leq n} \frac{S_m + S_{m-1}}{2}$$

Theorem

Suppose S *is the path encoding of* $(\eta_n^t)_n$. *Then,* $TS := 2M - S - 2M_0$ *is the path encoding of* $(\eta_n^{t+1})_n$.

Discrete KdV (dKdV) equation

- $0 < \delta < 1$: model parameter, $u_n^t > 0$

$$u_n^{t+1} = \frac{\delta}{u_n^t} + \prod_{m=-\infty}^{n-1} \frac{u_m^t}{u_m^{t+1}} \quad \text{“}\Leftrightarrow\text{”} \frac{1}{u_{n+1}^{t+1}} - \frac{1}{u_n^t} = \delta(u_n^{t+1} - u_{n+1}^t)$$

Remark

By a proper scaling limit, solutions for dKdV equation converges to a solution of KdV equation.

Path encoding and Pitman's transform for dKdV equation

$$S_n - S_{n-1} := -\log\delta - 2\log(u_n), \quad S_0 := 0, \quad M_n := \log(\sum_{m\le n} \exp(\frac{S_n + S_{n-1}}{2}))$$

Theorem

Suppose S is the path encoding of $(u_n^t)_n$. Then, $TS := 2M - S - 2M_0$ is the path encoding of $(u_n^{t+1})_n$.

Ultra-discretization

Ultra-discretization "=" tropicalization "=" zero-temperature limit : $(+, \times) \to (\min, +)$

- $\{u_n^t(\epsilon)\}_{n,t\in\mathbb{Z}}$: sols. of dKdV equation $\delta(\epsilon)$ with parameter $\epsilon > 0$.

Suppose the following limit exists.

$$\eta_n^t := \lim_{\epsilon\to 0} -\epsilon \log(u_n^t(\epsilon)),$$

$$-\epsilon \log \delta(\epsilon) \to L.$$

Then, $\{\eta_n^t\}$: sol of UdKdV.

d-KdV equation

$$u_n^{t+1} = \frac{\delta}{u_n^t} + \prod_{m=-\infty}^{n-1} \frac{u_m^t}{u_m^{t+1}}$$

Ud-KdV equation

$$\eta_n^{t+1} = \min\{L - \eta_n^t, \sum_{m=-\infty}^{n-1} (\eta_m^t - \eta_m^{t+1})\}$$

Ultra-discrete Toda equation

$$\begin{cases} Q_n^{t+1} & = \min\{E_n^t,\ \sum_{j=-\infty}^{n} Q_j^t - \sum_{j=-\infty}^{n-1} Q_j^{t+1}\} \\ E_n^{t+1} & = Q_{n+1}^t + E_n^t - Q_n^{t+1} \end{cases}$$

Remark

If $Q_n^t, E_n^t \in \mathbb{Z}_{\geq 0}$, the dynamics is BBS where Q_n (resp. E_n) are the number of 1's (0's) in the n-th set of consecutive 1's (0's).

Path encoding and Pitman's transform for UdToda equation

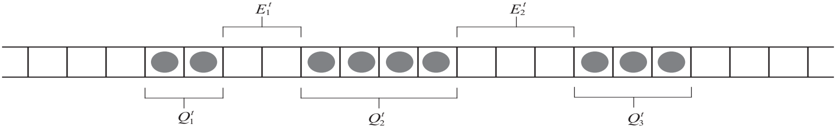
$$S_{2n+1} - S_{2n} := -Q_{n+1}, \quad S_{2n} - S_{2n-1} := E_n,$$

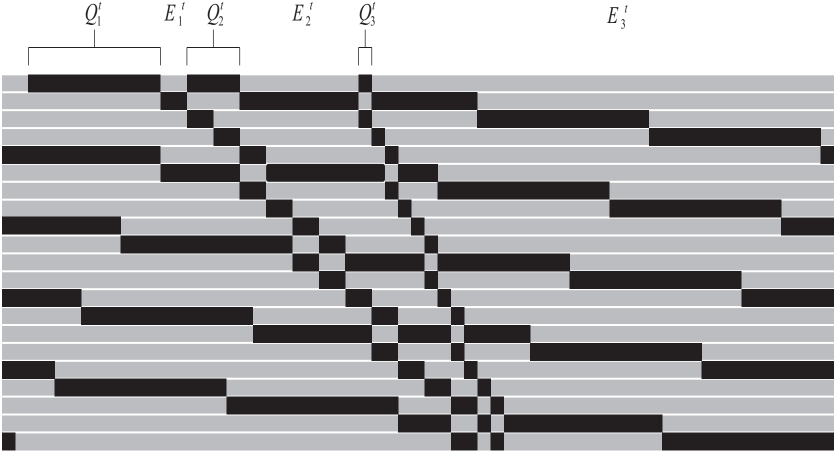
$$M_{2n+1} := \max_{m \leq n} S_{2m}, \quad M_{2n} = \frac{M_{2n-1} + M_{2n+1}}{2}$$

Theorem

Suppose S is the path encoding of $(Q_n^t, E_n^t)_n$. Then, θTS is the path encoding of $(Q_n^{t+1}, E_n^{t+1})_n$ where $\theta S_n = S_{n+1}$ and $TS := 2M - S - 2M_0$.

Dynamics of (periodic) Ultra-discrete Toda equation





Discrete Toda equation

- $I_n^t > 0, V_n^t > 0$

$$\begin{cases} I_n^{t+1} & = I_n^t + V_n^t - V_{n-1}^{t+1} \\ V_n^{t+1} & = \frac{I_{n+1}^t V_n^t}{I_n^{t+1}} \end{cases} \text{``}\Leftrightarrow\text{''} \begin{cases} I_n^{t+1} & = V_n^t + \frac{\prod_{j=-\infty}^{n} I_j^t}{\prod_{j=-\infty}^{n-1} I_j^{t+1}} \\ V_n^{t+1} & = \frac{I_{n+1}^t V_n^t}{I_n^{t+1}} \end{cases}$$

Path encoding and Pitman's transform for UdToda equation

$$S_{2n+1} - S_{2n} := \log I_{n+1}, \quad S_{2n} - S_{2n-1} := -\log V_n,$$

$$M_{2n+1} := \log(\sum_{m \le n} \exp(S_{2m})), \quad M_{2n} = \frac{M_{2n-1} + M_{2n+1}}{2}$$

Theorem

Suppose S is the path encoding of $(I_n^t, V_n^t)_n$. Then, θTS is the path encoding of $(I_n^{t+1}, V_n^{t+1})_n$ where $\theta S_n = S_{n+1}$ and $TS := 2M - S - 2M_0$.

Ud-Toda equation and D-Toda equation

- D-Toda equation shares many properties with (original) Toda equation (ex. Lax pair expression exists). Generalized Gibbs ensembles relate to random matrix theory (S, in preparation).
- Ud-Toda equation can be understood as a queueing model.
- Ud-Toda equation and the last passage percolation has a similar structure.
- D-Toda equation and the random polymer model has a similar structure.

Existence of solution for discrete integrable systems

- BBS : $M_n^{\vee(1)} := \max_{m \le n} S_m$ for $S : \mathbb{Z} \to \mathbb{Z}$, $|S_n - S_{n-1}| = 1$
- UdKdV : $M_n^{\vee(2)} := \max_{m \le n} \frac{S_m + S_{m-1}}{2}$ for $S : \mathbb{Z} \to \mathbb{R}$
- dKdV : $M_n^{\int(2)} := \log(\sum_{m \le n} \exp(\frac{S_m + S_{m-1}}{2}))$ for $S : \mathbb{Z} \to \mathbb{R}$
- UdToda: $M_{2n+1}^{\vee(2),*} := \max_{m \le n} S_{2m}$, $M_{2n}^{\vee(2),*} = \frac{M_{2n-1}^{\vee(2),*} + M_{2n+1}^{\vee(2),*}}{2}$ for $S : \mathbb{Z} \to \mathbb{R}$
- dToda: $M_{2n+1}^{\int(2),*} := \log(\sum_{m \le n} \exp(S_{2m})$, $M_{2n}^{\int(2),*} = \frac{M_{2n-1}^{\int(2),*} + M_{2n+1}^{\int(2),*}}{2}$ for $S : \mathbb{Z} \to \mathbb{R}$

All transforms are well-defined and invariant on asymptotically linear functions with positive drift:

$$\mathcal{S}^{lin} := \{S : * \to \mathbb{R} \ ; \ \lim_{n \to \pm\infty} \frac{S_n}{n} = c \text{ for some } c > 0\}$$

for $* = \mathbb{Z}$ or $\mathbb{R}$. Hence, under any shift ergodic initial measure with a proper average, the solution is well-defined for all time t with prob. 1.

Invariant measures for Pitman's type transforms

We apply the last theorem to find invariant measures for Pitman's type transforms:

Proposition

$TS \stackrel{d}{=} S$ *holds for*

- $M^{\vee(1)}$: S : *SRW*, W_0 : *geometric*
- $M^{\vee(2)}$: S : *RW with truncated shifted geometric increments*, W_0 : *shifted geometric*
- $M^{\vee(2)}$: S : *RW with truncated shifted exponential increments*, W_0 : *shifted exponential*
- $M^{\vee(\mathbb{R})}$: S : *BM*, W_0 : *exponential*
- $M^{\vee(\mathbb{R})}$: S : *zig-zag process*, W_0 : *exponential + delta measure at* 0
- $M^{\int(2)}$: S: *RW with log of generalized inverse Gaussian increments*, $\exp(W_0)$: *inverse gamma*
- $M^{\int(\mathbb{R})}$: S : *BM*, $\exp(W_0)$: *inverse gamma*

where we assume all processes have a positive drift.

Invariant measures for discrete integrable systems

Corollary

- *BBS :* $\{\eta_n\}$*: i.i.d. Bernoulli is invariant*
- *BBS with capacity of box L :* $\{\eta_n\}$*: i.i.d. truncated geometric is invariant*
- *UdKdV :* $\{\eta_n\}$*: i.i.d. truncated exponential is invariant*
- *dKdV :* $\{u_n\}$ *: i.i.d. generalized inverse Gaussian is invariant*
- *UdToda :* $\{Q_n\}, \{E_n\}$ *: i.i.d. exponentials with parameters* λ_Q, λ_E *with* $\lambda_Q > \lambda_E$
- *dToda :* $\{I_n\}, \{V_n\}$ *: i.i.d. gamma with parameters* γ_I, γ_V *with* $\gamma_I > \gamma_V$

Remark on the inverse gamma distribution

Recall that if $(S_x)_{x\geq 0}$ is a BM with negative drift, then

$$\int_0^\infty \exp(S_x)dx$$

has an inverse gamma distribution.

Corollary

Suppose $\{S_n\}_{n\geq 0}$ is a RW with generalized inverse Gaussian increments with negative expectation. Then,

$$\sum_{n=0}^{\infty} \exp(\frac{S_n + S_{n+1}}{2})$$

has an inverse gamma distribution.

Remark

We do not know is there any RW $(S_n)_{n\geq 0}$ for which $\sum_{n=0}^\infty \exp(S_n)$ has an inverse gamma distribution.

On-going and future work

- Generalized Gibbs ensembles
- Multi-color BBS $\Leftrightarrow$ Multi-dimensional version of Pitman's transform (Kondo,2018)
- Scaling limit in non-equilibrium states
- Direct connections to other integrable stochastic models (random polymers, random matrices, KPZ equation,...) and classical and quantum integrable systems (KdV equation, Toda lattice, six-vertex model,...)