エネルギー変分型の破壊現象モデル

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Energy variational models for fracture phenomena

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In engineering, for the last few decades, various kinds of numerical simulation techniques for crack propagation and structure failure have been developed and applied to various phenomena, e.g. DEM (discrete element method)[7, 13], RBSM (rigid body spring model) [11], and PDS-FEM (FEM-β) method [10] etc. These simulation techniques are extremely powerful and can very flexibly include various kind of effects besides fracture. On the other hand, one thing common to them is that only very few mathematical analysis has been done for them because of their including many ad hoc numerical steps in their numerical models.

We propose the following mathematical model for the mode III (anti-plane shear mode) crack growth in an elastic plate [15]. Let Ω be a bounded two dimensional domain with a piecewise smooth boundary Γ, and let Γ_D be a nonempty open portion of Γ which consists of a finite number of connected components. We define $\Gamma_N := \Gamma\backslash\Gamma_D$. For $t > 0$, we consider the equations:

$$\left\{\begin{array}{rcll}
\alpha_1 \dfrac{\partial u}{\partial t} &=& \mathrm{div}\left((1-z)^2\nabla u\right) & x\in\Omega \\[2ex]
\alpha_2 \dfrac{\partial z}{\partial t} &=& \left(\varepsilon\Delta z - \dfrac{\gamma^2}{\varepsilon}z + |\nabla u|^2(1-z)\right)_+ & x\in\Omega \\[2ex]
u(x,t) &=& g(x,t) & x\in\Gamma_D \\[2ex]
\dfrac{\partial u}{\partial n} &=& 0 & x\in\Gamma_N \\[2ex]
\dfrac{\partial z}{\partial n} &=& 0 & x\in\Gamma \\[2ex]
+\ \text{I.C. (2)} & & & x\in\Omega
\end{array}\right. \tag{1}$$

where $u(x,t)$ represents the small anti-plane displacement at the position $x\in\overline{\Omega}$ and time $t\geq 0$, and $g(x,t)$ is a given anti-plane displacement on the boundary Γ_D. The variable $z(x,t)$ satisfies $0\leq z(x,t)\leq 1$ in Ω and represents the crack shape, as $z\approx 0$ in the region without crack and $z\approx 1$ near the crack. The minimum length scale of z is given as $O(\varepsilon)$ with a small regularization parameter $\varepsilon > 0$. The function $z(x,t)$ is called the phase field for the crack shape. For stable numerical simulations, we also introduce small time relaxation parameters $\alpha_1\geq 0$ and $\alpha_2 > 0$.

The initial conditions for (1) are given as follows:

$$\begin{cases} u(x,0) &= u_0(x) & x\in\Omega \quad \text{(omitted if } \alpha_1 = 0) \\ z(x,0) &= z_0(x)\in[0,1] & x\in\Omega \end{cases} \tag{2}$$

The first equation of (1) expresses the force balance in the uncracked region ($z \approx 0$), and the second equation expresses the crack evolution due to the modulus of the stress $|\nabla u|$. The material constant $\gamma > 0$ is called the fracture toughness, which prescribes the critical value of the energy release rate in the Griffith's criterion [9]. It is harder for the crack to grow, if the value of γ is larger.

A crack once generated can be no longer repaired. We put $(\ )_+$ to the right hand side of the second equation, where $(a)_+ = \max(a,0)$. It guarantees the non-repair condition for the crack: $\frac{\partial z}{\partial t} \geq 0$.

In this talk, we also propose a discrete model for crack propagation based on a spring-block system [12]. We give their derivations of our continuous and discrete models with some numerical examples. A common feature of these continuous and discrete models is the energy gradient structure.

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