Braid groups and Steinberg groups

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Conference "Low-dimensional topology and number theory XIII"
in memory of Professor Toshie Takata
Kyushu University

15 March 2022

Dedication

I dedicate this lecture to the memory of our colleague Toshie Takata

In Strasbourg

- After her PhD under the guidance of Professor Kohno, Takata-san went for a postdoctoral stay to the University of Strasbourg. This is where I met her.

She stayed in Strasbourg from October 2000 to March 2001. Initially she was to stay until Summer 2001, but she obtained a position at Niigata University starting from April 2001.

Talk at Topology Seminar, Kyushu University

- In April 2010 she obtained a position at Kyushu University.

In the same year Professor Akira Masuoka (Tsukuba) invited me to the conference *Quantum groups and quantum topology* he organized at RIMS, Kyoto.
When Professor Takata heard of this, she suggested I visit her at Kyushu University, which I gladly did together with my wife.

She arranged with Professors Norio Iwase and Osamu Saeki for me to give a talk on 9 April 2010 at the Topology Seminar.

Visiting Dazaifu

- The next day Takata-san took us to Dazaifu. We visited Dazaifu Tenman-gū Shrine and the Kyushu National Museum.

In the garden of Dazaifu Tenman-gū (2010/4/10)

At the Kohno Fest (2015)

- I met Professor Takata for the last time at the conference for the 60th birthday of Prof. Toshitake Kohno (Tokyo University, September 2015).

From Kyudai (2010) to Kyudai (2022)

• As mentioned, I gave a talk at the Topology Seminar in Kyudai on 9 April 2010.
It was entitled *"On some braid group actions on free groups"*.

• In the present talk I shall resume the same subject and report on progress made in the last ten years, namely on the following two papers, one by myself, the other joint with Professor François Digne (Amiens):

∗ C. Kassel, *A braid-like presentation of the integral Steinberg group of type C_2*, Journal of Algebra; DOI: 10.1016/j.jalgebra.2020.09.015 (online); arXiv:2006.13574.

∗ F. Digne, C. Kassel, *Braid groups and symplectic Steinberg groups*, arXiv:2201.07153.

• There are now new players in the game, namely the Steinberg groups of type C_n (symplectic).

Outline

- Here is the plan of my talk.
  - I first recall what Steinberg groups are and give a presentation of the Steinberg group $\mathrm{St}(C_n, \mathbb{Z})$ of type C_n and with integral coefficients. It is a central extension of the symplectic modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$.
  - Next, I construct a homomorphism from braid groups to Steinberg groups:

    $$f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$$

    (joint work with F. Digne, arXiv:2201.07153).

    I will also explain where the defining formulas for $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$ come from. Geometry of surfaces is involved.
  - Thirdly, I concentrate on the case $n = 2$, which is the focus of the first paper I mentioned (arXiv:2006.13574).
  - Finally I state the results obtained with Digne on the image and the kernel of $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$ for $n \geq 3$.

Steinberg groups - Generalities

• **R. Steinberg (1962):** For any irreducible root system Φ he defined the now-called Steinberg group with a presentation by generators and relations. The generators and the relations are those holding in the simple complex algebraic group G of type Φ.

• **M. Stein (1971)** extended Steinberg's construction over any commutative ring R, leading to the Steinberg group $\mathrm{St}(\Phi, R)$.

• For $R = \mathbb{Z}$ (the ring of integers) $\mathrm{St}(\Phi, \mathbb{Z})$ has the following presentation:

- **Generators:** x_γ $(\gamma \in \Phi)$.
- **Relations:** if $\gamma, \delta \in \Phi$ such that $\gamma + \delta \neq 0$, then

$$[x_\gamma, x_\delta] = \prod x_{i\gamma+j\delta}^{c_{i,j}^{\gamma,\delta}},$$

where the product is taken over all positive integers i, j such that $i\gamma + j\delta \in \Phi$. The exponents $c_{i,j}^{\gamma,\delta}$ are integers depending only on the structure of the Chevalley group $G(\mathbb{Z})$.

(We have used the notation $[x_\gamma, x_\delta]$ for the commutator $x_\gamma x_\delta x_\gamma^{-1} x_\delta^{-1}$.)

• The Steinberg group comes with a natural projection $\pi : \mathrm{St}(\Phi, \mathbb{Z}) \twoheadrightarrow G(\mathbb{Z})$.

• We now concentrate on the root system $\Phi = C_n$ for $n \geq 2$. (Recall $C_1 = A_1$.)

The root system C_n

- Consider the Euclidean real vector space $\mathbb{R}^n$ and an orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$.

The roots of the root system C_n are $\pm\varepsilon_i \pm \varepsilon_j$ (short roots) and $\pm 2\varepsilon_i$ (long roots), where $1 \leq i \neq j \leq n$.

- The corresponding Chevalley group $G(\mathbb{Z})$ is the symplectic modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$, which consists of all matrices $M \in \mathrm{GL}_{2n}(\mathbb{Z})$ such that

$$M^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where M^T is the transpose of M and I_n is the identity matrix of size n.

- **Generators of $\mathrm{Sp}_{2n}(\mathbb{Z})$:**
  - ⋆ $X_{i,j} = I_{2n} + E_{i,j} - E_{j+n,i+n}$ $(1 \leq i \neq j \leq n)$, corresponding to the root $\varepsilon_i - \varepsilon_j$,
  - ⋆ $Y_{i,j} = I_{2n} + E_{i,j+n} + E_{j,i+n}$ $(1 \leq i \neq j \leq n)$, corresponding to $\varepsilon_i + \varepsilon_j$,
  - ⋆ $Y'_{i,j} = Y^T_{i,j}$ $(1 \leq i \neq j \leq n)$, corresponding to $-\varepsilon_i - \varepsilon_j$,
  - ⋆ $Z_i = I_{2n} + E_{i,i+n}$ $(1 \leq i \leq n)$, corresponding to $2\varepsilon_i$,
  - ⋆ $Z'_i = Z^T_i$ $(1 \leq i \leq n)$, corresponding to $-2\varepsilon_i$.

Here $E_{i,j}$ is the $2n \times 2n$ matrix which has all entries equal to 0 except the (i,j)-entry which is equal to 1.

(These generators are obtained as follows: consider the Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{C})$ with its root space decomposition; in each root space take a generator and exponentiate it.)

The Steinberg group of type C_n

• Computing the commutators of all pairs of the generators $X_{i,j}$, $Y_{i,j}$, $Y'_{i,j}$, Z_i, Z'_i of $\mathrm{Sp}_{2n}(\mathbb{Z})$, we obtain the following presentation for the Steinberg group $\mathrm{St}(C_n, \mathbb{Z})$:

⋆ **Generators:** $x_{i,j}$, $y_{i,j}$, $y'_{i,j}$, z_i, z'_i $\quad (1 \leq i \neq j \leq n)$.

⋆ **Relations:** (the subscripts $i, j, k \in \{1, \dots, n\}$ are pairwise distinct)

$$y_{i,j} = y_{j,i}\,, \qquad y'_{i,j} = y'_{j,i}\,,$$

$$[x_{i,j}, x_{j,k}] = x_{i,k}\,, \qquad [x_{i,j}, y_{j,k}] = y_{i,k}\,, \quad [x_{i,j}, y'_{i,k}] = y'^{-1}_{j,k},$$

$$[x_{i,j}, y_{i,j}] = z_i^2\,, \qquad [x_{i,j}, y'_{i,j}] = z'^{-2}_j,$$

$$[x_{i,j}, z_j] = z_i y_{i,j} = y_{i,j} z_i\,, \qquad [x_{i,j}, z'_i] = z'_j y'^{-1}_{i,j} = y'^{-1}_{i,j} z'_j\,,$$

$$[y_{i,j}, z'_i] = x_{j,i} z_j^{-1} = z_j^{-1} x_{j,i}\,, \qquad [y'_{i,j}, z_i] = x_{i,j}^{-1} z'^{-1}_j = z'^{-1}_j x_{i,j}^{-1}.$$

All remaining pairs of generators commute, except $(x_{i,j}, x_{j,i})$, $(y_{i,j}, y'_{i,j})$ and (z_i, z'_i) for which we do not prescribe any relation.

The projection $\pi : \mathrm{St}(C_n, \mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z})$

- The projection $\pi : \mathrm{St}(C_n, \mathbb{Z}) \twoheadrightarrow \mathrm{Sp}_{2n}(\mathbb{Z})$ is given by

$$\pi(x_{i,j}) = X_{i,j}\,, \qquad \pi(y_{i,j}) = Y_{i,j}\,, \qquad \pi(y'_{i,j}) = Y'_{i,j}\,,$$

$$\pi(z_i) = Z_i\,, \qquad \pi(z'_i) = Z'_i\,,$$

where

$$\begin{aligned} X_{i,j} &= I_{2n} + E_{i,j} - E_{j+n,i+n}\,, & \\ Y_{i,j} &= I_{2n} + E_{i,j+n} + E_{j,i+n}\,, & Y'_{i,j} &= Y_{i,j}^T\,, \\ Z_i &= I_{2n} + E_{i,i+n}\,, & Z'_i &= Z_i^T \end{aligned}$$

defined above.

- **Matsumoto (1969):** Set $w_i = z_i z_i'^{-1} z_i \in \mathrm{St}(C_n, \mathbb{Z})$ corresponding to long root $\pm 2\varepsilon_i$.

The kernel of the projection $\pi : \mathrm{St}(C_n, \mathbb{Z}) \twoheadrightarrow \mathrm{Sp}_{2n}(\mathbb{Z})$ is an infinite cyclic group generated by w_i^4.

Actually, $w_i^4 = w_1^4$ for all $i \in \{1, \ldots, n\}$.

Braid groups

Let $n \geq 2$ be a fixed integer.

• The braid group B_{2n+2} on $2n+2$ strands can be defined algebraically as the group generated by $2n+1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{2n+1}$ subject to the following relations $(1 \leq i, j \leq 2n+1)$:

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i-j| = 1,$$

and

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{otherwise.}$$

• The center Z_{2n+2} of B_{2n+2} is infinite cyclic; it is generated by

$$(\sigma_1 \sigma_2 \cdots \sigma_{2n+1})^{2n+2}.$$

• **The pure braid groups.** There is an epimorphism $p : B_{2n+2} \twoheadrightarrow \mathfrak{S}_{2n+2}$ from the braid group to the symmetric group $\mathfrak{S}_{2n+2}$, which is the group of all permutations of the set $\{1, \ldots, 2n+2\}$.

It sends each generator σ_i of B_{2n+2} to the simple transposition $s_i = (i, i+1)$ permuting i and $i+1$ and leaving the remaining elements of $\{1, \ldots, 2n+2\}$ fixed.

The pure braid group P_{2n+2} is the kernel of $p : B_{2n+2} \twoheadrightarrow \mathfrak{S}_{2n+2}$.

From the braid groups to the Steinberg groups

We now connect the braid groups and the symplectic Steinberg groups $\mathrm{St}(C_n, \mathbb{Z})$.

- **Theorem 1 (joint with François Digne).**

(a) There exists a unique homomorphism $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$ *such that*

$$f(\sigma_1) = z_1\,, \qquad f(\sigma_{2n+1}) = z_n\,,$$

$$f(\sigma_{2i}) = z_i'^{-1}, \qquad (i = 1, \ldots, n)$$

$$f(\sigma_{2i+1}) = z_i z_{i+1} y_{i,i+1}^{-1}\,. \qquad (i = 1, \ldots, n-1)$$

(b) The homomorphism f *is surjective if and only if* $n = 2$*.*

- In the sequel we shall describe the image and the kernel of $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$.

- Beforehand, let us explain where the formulas for f come from.

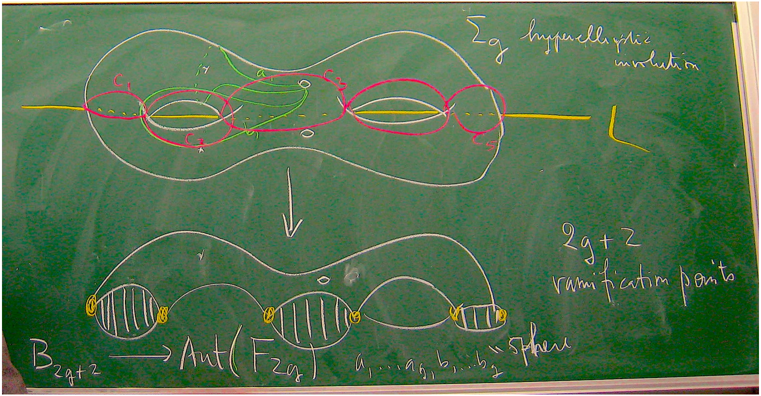
A ramified double covering of a disk

The idea goes back to work of Arnold, Magnus & Peluso, Birman... around 1968–69.

• Consider a surface Σ of genus $n \geq 1$, which is invariant under the hyperelliptic involution, which is the reflection in the line L. This line intersects Σ in $2n+2$ points.

• The quotient of Σ by the hyperelliptic involution is a sphere. We thus obtain a double covering $p : \Sigma \to S^2$ of the sphere with $2n+2$ ramification points.

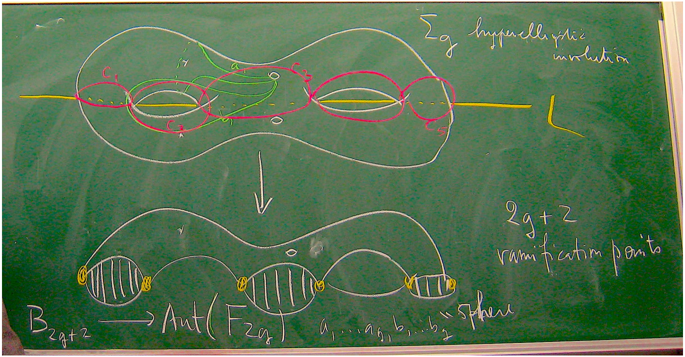
• Removing two symmetric small discs ("holes") from Σ, we obtain a double covering $p : \Sigma_0 \to D$ of a disk D with $2n+2$ ramification points.



An action of the braid group B_{2n+2} on the free group F_{2n}

• The braid group B_{2n+2} can be realized as the mapping class group of the disk D with $2n+2$ distinguished points. Each element of B_{2n+2} can be represented as an orientation-preserving homeomorphism fixing each point of the boundary of D and permuting the distinguished points.

• Lifting each such homeomorphism to a homeomorphism of Σ_0 (fixing the two holes) induces a group homomorphism from B_{2n+2} to the mapping class group of Σ_0, hence to the automorphism group of the fundamental group $\pi_1(\Sigma_0)$. The latter is the free group generated by the loops $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ of the figure.



The symplectic representation $B_{2n+2} \to \mathrm{Sp}_{2n}(\mathbb{Z})$

• Representing each generator $\sigma_1, \ldots, \sigma_{2n+1}$ of B_{2n+2} by a homeomorphism of D, lifting the latter to Σ_0 and computing the action of each lift on the loops $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ yields a homomorphism $\widetilde{f} : B_{2n+2} \to \mathrm{Aut}(F_{2n})$.

• Composing the homomorphism $\widetilde{f} : B_{2n+2} \to \mathrm{Aut}(F_{2n})$ with the linearization map $\mathrm{ab} : \mathrm{Aut}(F_{2n}) \to \mathrm{GL}_{2n}(\mathbb{Z})$, we obtain a homomorphism

$$\bar{f} = \mathrm{ab} \circ \widetilde{f} : B_{2n+2} \to \mathrm{GL}_{2n}(\mathbb{Z}).$$

One checks that

$$\bar{f}(\sigma_1) = Z_1 \,, \qquad \bar{f}(\sigma_{2n+1}) = Z_n \,,$$

$$\bar{f}(\sigma_{2i}) = Z_i'^{-1}, \qquad (i = 1, \ldots, n)$$

$$\bar{f}(\sigma_{2i+1}) = Z_i Z_{i+1} Y_{i,i+1}^{-1} \,. \qquad (i = 1, \ldots, n-1)$$

where Z_i, Z_i' and $Y_{i,i+1}$ are the symplectic matrices defined above.

• **Conclusion.** The homomorphism $\bar{f} = \mathrm{ab} \circ \widetilde{f}$ takes values in the symplectic modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$. Moreover,

$$\bar{f} = \pi \circ f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z}).$$

In other words, the homomorphism $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$ of Theorem 1 is a lifting of the symplectic representation $\bar{f} : B_{2n+2} \to \mathrm{Sp}_{2n}(\mathbb{Z})$ of the braid group.

The case $n = 2$

We now consider the case $n = 2$ for the homomorphism $f : B_6 \to \mathrm{St}(C_2, \mathbb{Z})$.

In a special issue of the Journal of Algebra in honor of the late Patrick Dehornoy (2020) I published the following result.

• **Theorem 2.** *(a) The homomorphism $f : B_6 \to \mathrm{St}(C_2, \mathbb{Z})$ is surjective.*

(b) Its kernel is the normal closure of the braid

$$\alpha_2 = (\sigma_1\sigma_2)^3(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2)^{-3}(\sigma_1\sigma_3^{-1}\sigma_5).$$

• **Corollary.** *We have the isomorphisms $\mathrm{St}(C_2, \mathbb{Z}) \cong B_6/N$ and $\mathrm{Sp}_4(\mathbb{Z}) \cong B_6/\bar{N}$, where N is the normal closure of the braid α_2 above and $\bar{N}$ is the normal closure of the set $\{\alpha_2, (\sigma_1\sigma_2)^6\}$.*

These isomorphisms yield braid-type presentations of $\mathrm{St}(C_2, \mathbb{Z})$ and of $\mathrm{Sp}_4(\mathbb{Z})$.

• The second isomorphism $\mathrm{Sp}_4(\mathbb{Z}) \cong B_6/\bar{N}$ follows from the first one and the fact that the kernel of the projection $\pi : \mathrm{St}(C_2, \mathbb{Z}) \to \mathrm{Sp}_4(\mathbb{Z})$ is generated by

$$w_1^4 = f\left((\sigma_1\sigma_2\sigma_1)^4\right) = f\left((\sigma_1\sigma_2)^6\right).$$

The image of $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$ when $n \geq 3$

When $n \geq 3$, the homomorphism f is not surjective. So what is its image?

• Consider the level 2 congruence subgroup $\mathrm{Sp}_{2n}(\mathbb{Z})[2] = \mathrm{Ker}\left(\mathrm{Sp}_{2n}(\mathbb{Z}) \twoheadrightarrow \mathrm{Sp}_{2n}(\mathbb{F}_2)\right)$ induced by reduction modulo 2. The composite map $B_{2n+2} \xrightarrow{\bar{f}} \mathrm{Sp}_{2n}(\mathbb{Z}) \twoheadrightarrow \mathrm{Sp}_{2n}(\mathbb{F}_2)$ factors through the symmetric group $\mathfrak{S}_{2n+2}$, inducing an injection (not onto for $n \geq 3$)

$$\mathfrak{S}_{2n+2} \hookrightarrow \mathrm{Sp}_{2n}(\mathbb{F}_2).$$

• We lift $\mathrm{Sp}_{2n}(\mathbb{Z})[2]$ to the Steinberg group by taking its preimage

$$\mathrm{St}(C_n, \mathbb{Z})[2] = \pi^{-1}\left(\mathrm{Sp}_{2n}(\mathbb{Z})[2]\right) \subset \mathrm{St}(C_n, \mathbb{Z})$$

under the natural projection $\pi : \mathrm{St}(C_n, \mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z})$.

Theorem 3 (joint with François Digne). *Assume $n \geq 3$.*

(a) Let P_{2n+2} be the pure braid group. Then its image under f is

$$f(P_{2n+2}) = \mathrm{St}(C_n, \mathbb{Z})[2].$$

(b) For the full braid group B_{2n+2} we have the short exact sequence

$$1 \to \mathrm{St}(C_n, \mathbb{Z})[2] \longrightarrow f(B_{2n+2}) \longrightarrow \mathfrak{S}_{2n+2} \to 1.$$

• The proof of Theorem 3 uses results by Arnold (1968) and A'Campo (1979).

Towards the kernel of $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$

• The kernels of $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$ and of $\bar{f} : B_{2n+2} \to \mathrm{Sp}_{2n}(\mathbb{Z})$ are normal subgroups of the pure braid group P_{2n+2}:

$$\mathrm{Ker}(f) \subset \mathrm{Ker}(\bar{f}) \subset P_{2n+2}\,.$$

• They fit into the short exact sequence

$$1 \to \mathrm{Ker}(f) \to \mathrm{Ker}(\bar{f}) \xrightarrow{f} \langle w_1^4 \rangle \to 1.$$

This sequence is split with splitting given by $w_1^4 \mapsto (\sigma_1\sigma_2)^6$.

• Set $\nabla_{2k+1} = (\sigma_1\sigma_2 \cdots \sigma_{2k})^{2k+1} \in P_{2n+2}$ for $1 \leq k \leq n-1$
(Dehn twists about curves in the disk surrounding exactly $2k+1$ marked points).

Work by Brendle, Margalit and Putnam (2015) on the hyperelliptic Torelli group imply that $\mathrm{Ker}(\bar{f}) \cap P_{2n+1}$ is the normal closure of ∇_3^2 and ∇_5^2.

• It remains to pass from $\mathrm{Ker}(\bar{f}) \cap P_{2n+1}$ to the whole $\mathrm{Ker}(\bar{f}) \subset P_{2n+2}$.

Passing from P_{2n+1} to P_{2n+2}

• Forgetting the righmost strand yields an epimorphism $P_{2n+2} \twoheadrightarrow P_{2n+1}$ and a semi-direct decomposition $P_{2n+2} \cong P_{2n+1} \ltimes F$, where F is the free group generated by σ_{2n+1}^2 and its conjugates ($1 \le i \le 2n$)

$$(\sigma_{2n}\sigma_{2n-1}\cdots\sigma_i)^{-1}\,\sigma_{2n+1}^2\,(\sigma_{2n}\sigma_{2n-1}\cdots\sigma_i).$$

• To compute the full kernel $\mathrm{Ker}(\bar f)$ from $\mathrm{Ker}(\bar f) \cap P_{2n+1}$, it suffices to find a braid $\alpha_n \in P_{2n+1}\,\sigma_{2n+1}^2$ such that $f(\alpha_n) = 1$.

• Here comes the "miraculous" braid. Consider the following two elements of B_{2n+2}:

(a) $\beta_n = \sigma_1\sigma_3^{-1}\cdots\sigma_{2n+1}^{(-1)^n}$,

(b) $\gamma_n = \nabla_3\nabla_5\cdots\nabla_{2n-1}$, where $\nabla_{2k+1} = (\sigma_1\sigma_2\cdots\sigma_{2k})^{2k+1}$,

and set

$$\alpha_n = \gamma_n\beta_n\gamma_n^{-1}\beta_n \in B_{2n+2}\,.$$

• **Proposition (joint with François Digne).** *We have*

$$f(\alpha_n) = 1 \qquad \textit{and} \qquad \alpha_n \in P_{2n}\,(\sigma_{2n+1}^2)^{(-1)^n} \subset P_{2n+2}\,.$$

Remark. We have $\alpha_2 = (\sigma_1\sigma_2)^3(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2)^{-3}(\sigma_1\sigma_3^{-1}\sigma_5)$.

The kernel of $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$

We now state our main result on the kernel of $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$.

- **Theorem 4 (joint with François Digne).** *Assume $n \geq 3$.*

(a) The kernel of $\bar{f} : B_{2n+2} \to \mathrm{Sp}_{2n}(\mathbb{Z})$ is the normal closure of the set consisting of the three braids

$$\alpha_n\,, \quad \nabla_3^2 \quad \textit{and} \quad \nabla_5^2.$$

(b) The kernel of $f : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$ is the normal closure of the set consisting of

$$\alpha_n\,, \quad \nabla_5^2\,\nabla_3^{-8} \quad \textit{and} \quad [\sigma_3, \nabla_3^2].$$

- *Proof of (a).* Use the results of the previous slides.

- *Proof of (b).* It is consequence of (a), of the equalities

$$f(\nabla_3^2) = w_1^4, \quad f(\nabla_5^2) = w_1^{16}, \quad f(\nabla_5^2\,\nabla_3^{-8}) = 1 = f(\alpha_n)$$

and of the centrality of w_1^4 in the Steinberg group.

Question

- Recall the "miraculous" braid

$$\alpha_n = \gamma_n \beta_n \gamma_n^{-1} \beta_n \in \mathrm{Ker}(f) \subset B_{2n+2}\,,$$

where

(a) $\beta_n = \sigma_1 \sigma_3^{-1} \cdots \sigma_{2n+1}^{(-1)^n}$,

(b) $\gamma_n = \nabla_3 \nabla_5 \cdots \nabla_{2n-1}$, where $\nabla_{2k+1} = (\sigma_1 \sigma_2 \cdots \sigma_{2k})^{2k+1}$,

- **Question.** Is there a geometric interpretation for α_n (e.g. in terms of Dehn twists)?

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Thank you for your attention

The homomorphism $B_{2n+2} \to \mathrm{Aut}(F_{2n})$

For the homomorphism $\widetilde{f} : B_{2n+2} \to \mathrm{Aut}(F_{2n})$ we have $\widetilde{f}(\sigma_i) = u_i$,
where $u_1, \ldots, u_{2n+1}$ are the following automorphisms of
the free group $F_{2n} = \langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle$.

(a) The automorphism u_1 fixes all generators, except b_1 for which

$$u_1(b_1) = a_1 b_1 .$$

(b) The automorphism u_{2n+1} fixes all generators, except b_n for which

$$u_{2n+1}(b_n) = b_n a_n .$$

(c) The automorphism u_{2i} $(1 \leq i \leq n)$ fixes all generators, except a_i for which

$$u_{2i}(a_i) = b_i^{-1} a_i .$$

(d) The automorphism u_{2i+1} $(1 \leq i \leq n-1)$ fixes all generators, except b_i and b_{i+1} for which we have

$$u_{2i+1}(b_i) = b_i a_i a_{i+1}^{-1} \quad \text{and} \quad u_{2i+1}(b_{i+1}) = a_{i+1} a_i^{-1} b_{i+1} .$$

The case $n = 2$ - Proof of Theorem 2

Back to the case $n = 2$.

- **Theorem 2.** *(a) The map $f : B_6 \to \mathrm{St}(C_2, \mathbb{Z})$ is surjective.*

(b) Its kernel is the normal closure N of the braid

$$\alpha_2 = (\sigma_1\sigma_2)^3(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2)^{-3}(\sigma_1\sigma_3^{-1}\sigma_5).$$

Part (b) of Theorem 2 is a consequence of the following two lemmas.

- **Lemma 1.** *We have $f(N) = 1$.*

Proof. It suffices to check $f(\alpha_2) = 1$. Indeed,

$$f(\alpha_2) = w_1^2\, y_{1,2}\, w_1^{-2} y_{1,2} = y_{1,2}^{-1}\, y_{1,2} = 1.$$

- By Lemma 1 the map f induces a surjective homomorphism $f : B_6/N \to \mathrm{St}(C_2, \mathbb{Z})$.

Lemma 2. *There exists a homomorphism $\varphi : \mathrm{St}(C_2, \mathbb{Z}) \to B_6/N$ such that $\varphi \circ f = \mathrm{id}$.*

Hence, $f : B_6/N \to \mathrm{St}(C_2, \mathbb{Z})$ is also injective.

The case $n = 2$ - About Lemma 2

Let N be the normal subgroup of B_6 generated by

$$\alpha_2 = (\sigma_1\sigma_2)^3(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2)^{-3}(\sigma_1\sigma_3^{-1}\sigma_5).$$

• **Lemma 2.** *There exists a homomorphism* $\varphi : \mathrm{St}(C_2, \mathbb{Z}) \to B_6/N$ *such that* $\varphi \circ f = \mathrm{id}$*. It is given modulo* N *by*

$$\varphi(z_1) \equiv \sigma_1\,, \quad \varphi(z_2) \equiv \sigma_5\,, \quad \varphi(z_1') \equiv \sigma_2^{-1}, \quad \varphi(z_2') \equiv \sigma_4^{-1},$$

$$\begin{aligned}
\varphi(y_{1,2}) &\equiv \sigma_1\sigma_3^{-1}\sigma_5\,,\\
\varphi(y_{1,2}') &\equiv (\sigma_1\sigma_2\sigma_5\sigma_4)(\sigma_1\sigma_3^{-1}\sigma_5)^{-1}(\sigma_1\sigma_2\sigma_5\sigma_4)^{-1},\\
\varphi(x_{1,2}) &\equiv (\sigma_5\sigma_4)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_5\sigma_4)^{-1},\\
\varphi(x_{2,1}) &\equiv (\sigma_1\sigma_2)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2)^{-1}.
\end{aligned}$$

Proof. One checks that the image under φ of each defining relation of $\mathrm{St}(C_2, \mathbb{Z})$ is satisfied in B_6/N. For instance, for the relation $[y_{1,2}, z_1'] = x_{2,1}\, z_2^{-1}$, one has

$$\left[\varphi(y_{1,2}), \varphi(z_1')\right]^{-1} \varphi(x_{2,1})\, \varphi(z_2^{-1}) = \alpha_2 \in N.$$