# A criterion of algebraic independence of values of modular functions and an application to infinite products involving Fibonacci and Lucas numbers 

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In memoriam Eduard Wirsing (1931-2022)


#### Abstract

The aim of this paper is to give a criterion of algebraic independence for the values at the same point of two modular functions under certain conditions. As an application, we show that any two infinite products in $$
\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{n}}\right), \quad \prod_{n=3}^{\infty}\left(1-\frac{1}{F_{n}}\right), \quad \prod_{n=1}^{\infty}\left(1+\frac{1}{L_{n}}\right), \quad \prod_{n=2}^{\infty}\left(1-\frac{1}{L_{n}}\right)
$$ are algebraically independent over $\mathbb{Q}$, where $\left\{F_{n}\right\}$ and $\left\{L_{n}\right\}$ are the Fibonacci and Lucas sequences, respectively. The proof of our main theorem is based on the properties of the field of all modular functions for the principal congruence subgroup, together with a deep result of Yu. V. Nesterenko on algebraic independence of the values of the Eisenstein series.


Keywords algebraic independence, modular functions, Dedekind eta function, infinite products, Fibonacci numbers, Lucas numbers
2020 Mathematics Subject Classification 11J85, 11F03, 11F20, 11B39.

## 1 Introduction and main results

Throughout the paper, $\mathbb{H}$ denotes the upper-half plane. For an integer $k \geq 2$, the normalized Eisenstein series is defined by

$$
\begin{equation*}
E_{2 k}(\tau)=\frac{1}{2 \zeta(2 k)} \sum_{(m, n) \neq(0,0)} \frac{1}{(m \tau+n)^{2 k}}, \quad \tau \in \mathbb{H}, \tag{1}
\end{equation*}
$$

[^0]where $\zeta(s)$ is the Riemann zeta function. It is well known that the function $E_{2 k}(\tau)$ is a modular form of weight $2 k$ for the modular group $S L_{2}(\mathbb{Z})$ and has the Fourier expansion
\[

$$
\begin{equation*}
E_{2 k}(\tau)=1-\frac{4 k}{B_{2 k}} \sum_{n=1}^{\infty} \sigma_{2 k-1}(n) e^{2 \pi i n \tau}, \tag{2}
\end{equation*}
$$

\]

where $\sigma_{2 k-1}(n)=\sum_{d \mid n} d^{2 k-1}$ and $B_{2 k}$ is the $2 k$-th Bernoulli number (cf. [1]). For $k=1$, we define the function $E_{2}(\tau)$ by using the expression (2):

$$
E_{2}(\tau)=1-24 \sum_{n=1}^{\infty} \sigma_{1}(n) e^{2 \pi i n \tau}, \quad \tau \in \mathbb{H} .
$$

This is a "quasimodular" form of weight 2 (cf. [11]). In 1996, Yu. V. Nesterenko [15] made a breakthrough in transcendence theory by showing the algebraic independence of the values of the Eisenstein series.

Theorem A ([15, Theorem 8]). If $\tau \in \mathbb{H}$, then at least three of the numbers $e^{\pi i \tau}, E_{2}(\tau), E_{4}(\tau)$, $E_{6}(\tau)$ are algebraically independent over $\mathbb{Q}$.

In particular, if $e^{\pi i \tau}$ is algebraic, the three numbers $E_{2}(\tau), E_{4}(\tau), E_{6}(\tau)$ are algebraically independent over $\mathbb{Q}$. Theorem A has a number of remarkable consequences on transcendence and algebraic independence of the values of the modular forms. D. Bertrand [3] translated Theorem A in terms of the theta-constants defined in $\mathbb{H}$ by

$$
\begin{equation*}
\vartheta_{2}(\tau):=2 \sum_{n=0}^{\infty} e^{\pi i(n+1 / 2)^{2} \tau}, \quad \vartheta_{3}(\tau):=1+2 \sum_{n=1}^{\infty} e^{\pi i n^{2} \tau}, \quad \vartheta_{4}(\tau):=1+2 \sum_{n=1}^{\infty}(-1)^{n} e^{\pi i n^{2} \tau} \tag{3}
\end{equation*}
$$

Theorem B ([3, Theorem 4]). Let $\alpha, \beta, \gamma \in\{2,3,4\}$ with $\alpha \neq \beta$. Then for any $\tau \in \mathbb{H}$, at least three of the numbers $e^{\pi i \tau}, \vartheta_{\alpha}(\tau), \vartheta_{\beta}(\tau), D \vartheta_{\gamma}(\tau)$ are algebraically independent over $\mathbb{Q}$, where $D:=\frac{1}{\pi i} \frac{d}{d \tau}$ is a differential operator.
It should be noted that the sum $\sum_{n=1}^{\infty} q^{n^{2}}$ is transcendental for any algebraic number $q$ with $0<|q|<1$ (see also [6]). Recently, the second, third and fourth author [10] extended Theorem B to a more general form:

Theorem C ([10, Theorem 1.1]). Let $m, n, \ell \geq 1$ be integers and $\alpha, \beta, \gamma \in\{2,3,4\}$ with $(m, \alpha) \neq$ $(n, \beta)$. Then for any $\tau \in \mathbb{H}$, at least three of the numbers $e^{\pi i \tau}, \vartheta_{\alpha}(m \tau), \vartheta_{\beta}(n \tau), D \vartheta_{\gamma}(\ell \tau)$ are algebraically independent over $\mathbb{Q}$.

As a corollary of Theorem C, we obtain the algebraic independence over $\mathbb{Q}$ of the three numbers

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left(\frac{n}{3}\right) \frac{q^{n}}{1-q^{n}}, \quad \sum_{n=1}^{\infty}(-1)^{n}\left(\frac{n}{3}\right) \frac{q^{n}}{1-q^{n}}, \quad \sum_{n=1}^{\infty}\left(\frac{n}{3}\right) \frac{q^{n}}{1-q^{2 n}} \tag{4}
\end{equation*}
$$

for any algebraic number $q$ with $0<|q|<1$, where $\left(\frac{n}{3}\right)$ is the Legendre symbol ([10, Corollary 1.1]). On the other hand, the first author, Ke. Nishioka, Ku. Nishioka and I. Shiokawa [7] applied Theorem A to derive transcendence results for certain infinite series involving the Fibonacci and Lucas sequences; e.g.

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{1}{F_{2 n-1}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2 n}} \tag{5}
\end{equation*}
$$

where $\left\{F_{n}\right\}_{n \geq 0}$ and $\left\{L_{n}\right\}_{n \geq 0}$ are the Fibonacci and Lucas sequences defined, respectively, by

$$
F_{n+2}=F_{n+1}+F_{n}, \quad(n \geq 0), \quad F_{0}=0, \quad F_{1}=1
$$

and

$$
L_{n+2}=L_{n+1}+L_{n}, \quad(n \geq 0), \quad L_{0}=2, \quad L_{1}=1
$$

We are still unaware of the transcendence of the infinite sums $\sum_{n=1}^{\infty} 1 / F_{n}, \sum_{n=1}^{\infty} 1 / F_{2 n}, \sum_{n=1}^{\infty} 1 / L_{n}$, and $\sum_{n=1}^{\infty} 1 / L_{2 n-1}$. In the direction of infinite products, the first and fourth author showed in the recent paper [9] that the two numbers

$$
\begin{equation*}
\xi_{1}:=\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{n}}\right), \quad \xi_{2}:=\prod_{n=3}^{\infty}\left(1-\frac{1}{F_{n}}\right) \tag{6}
\end{equation*}
$$

are algebraically independent over $\mathbb{Q}$. This has been done first by expressing the numbers $\xi_{1}$ and $\xi_{2}$ by means of the theta-constants and then applying Nesterenko's Theorem A. For example, the number $\xi_{1}$ has the expression

$$
\begin{equation*}
\xi_{1}^{2}=8 \beta^{-5 / 2} \frac{\vartheta_{2}\left(2 \tau_{0}\right)}{\vartheta_{4}\left(2 \tau_{0}\right)} \tag{7}
\end{equation*}
$$

where $\beta:=(\sqrt{5}-1) / 2$ and $\tau_{0} \in \mathbb{H}$ satisfies $\beta=e^{\pi i \tau_{0}}$ (cf. [9]). Note that the transcendence of $\xi_{1}$ follows immediately from a result of K. Barré-Sirieix, G. Diaz, F. Gramain and G. Philibert [2] (see also [3, Theorem 3]) on the transcendence of the values of the elliptic modular $j$-function

$$
\begin{equation*}
j(\tau):=1728 \frac{E_{4}(\tau)^{3}}{E_{4}(\tau)^{3}-E_{6}(\tau)^{2}}, \quad \tau \in \mathbb{H} \tag{8}
\end{equation*}
$$

The main purpose of the present paper is to give a criterion of algebraic independence for the values of two modular functions (Theorem 1 below). Then we will derive new results of algebraic independence for certain infinite sums and products involving the Fibonacci and Lucas sequences (Theorems 2 and 3) by expressing our target numbers by values of the Dedekind eta function. Before stating our main theorem, we prepare some notations. Let $N$ be a positive integer and $\Gamma(N)$ be the principal congruence subgroup of $S L_{2}(\mathbb{Z})$ of level $N$, which is defined by

$$
\Gamma(N):=\left\{\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in S L_{2}(\mathbb{Z}) \left\lvert\,\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \equiv\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)(\bmod N)\right.\right\}
$$

A modular function of weight $k \in \mathbb{Z}$ for $\Gamma(N)$ is a meromorphic function $f$ on $\mathbb{H}$, which satisfies the following two conditions (cf. [12, p. 125]):
(i) $f \mid[\gamma]_{k}=f$ for all $\gamma \in \Gamma(N)$, where $f \mid[\gamma]_{k}$ is the function whose value at $\tau$ is defined by

$$
f(\tau) \mid[\gamma]_{k}:=(c \tau+d)^{-k} f(\gamma \tau)
$$

(ii) For any $\gamma \in S L_{2}(\mathbb{Z})$, the Fourier expansion of $f \mid[\gamma]_{k}$ has the form

$$
\begin{equation*}
f(\tau) \mid[\gamma]_{k}=\sum_{n=-m}^{\infty} a_{n} e^{2 \pi i n \tau / N} \tag{9}
\end{equation*}
$$

where the integer $m$ and the coefficients $a_{n}$ depend on $\gamma \in S L_{2}(\mathbb{Z})$. We call such an $f$ a modular form of weight $k$ for $\Gamma(N)$ if it is holomorphic on $\mathbb{H}$ and if for any $\gamma \in S L_{2}(\mathbb{Z})$ we have $a_{n}=0$ for
all $n<0$ in (9). In what follows, let $\mathfrak{F}_{N}$ denote the field of all modular functions of weight zero for $\Gamma(N)$ whose Fourier expansions with respect to $e^{2 \pi i \tau / N}$ of the form

$$
f(\tau)=\sum_{n=-m}^{\infty} a_{n} e^{2 \pi i n \tau / N}
$$

have coefficients in $\mathbb{Q}\left(e^{2 \pi i / N}\right)$. It should be noted that $\mathfrak{F}_{1}=\mathbb{Q}(j(\tau))$ and $\mathfrak{F}_{N}$ is a Galois extension of $\mathfrak{F}_{1}$, where $j(\tau)$ is the elliptic modular $j$-function defined in (8) (cf. [5, Chapter 15, A]). Our results are the following.

Theorem 1 Let $f_{1}(\tau)$ and $f_{2}(\tau)$ be non-zero modular functions of weights $k_{1}$ and $k_{2}$, respectively, for $\Gamma(N)$ whose Fourier expansions with respect to $e^{2 \pi i \tau / N}$ have coefficients in $\mathbb{Q}\left(e^{2 \pi i / N}\right)$. Let $\tau_{0} \in \mathbb{H}$ be neither a zero nor a pole of the functions $f_{1}$ and $f_{2}$ such that the number $e^{\pi i \tau_{0}}$ is algebraic. Then the two numbers $f_{1}\left(\tau_{0}\right)$ and $f_{2}\left(\tau_{0}\right)$ are algebraically independent over $\mathbb{Q}$ if and only if the function $f_{1}^{k_{2}} / f_{2}^{k_{1}}$ is not constant.

As an application of Theorem 1, we obtain the following algebraic independence results.
Theorem 2 Any two infinite products in the set

$$
\left\{\xi_{1}:=\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{n}}\right), \quad \xi_{2}:=\prod_{n=3}^{\infty}\left(1-\frac{1}{F_{n}}\right), \quad \nu_{1}:=\prod_{n=1}^{\infty}\left(1+\frac{1}{L_{n}}\right), \quad \nu_{2}:=\prod_{n=2}^{\infty}\left(1-\frac{1}{L_{n}}\right)\right\}
$$

are algebraically independent over $\mathbb{Q}$, while any three are not.
Note that the same holds for the set of the four alternating infinite products

$$
\begin{aligned}
& \xi_{3}:=\prod_{n=2}^{\infty}\left(1+\frac{(-1)^{n}}{F_{n}}\right), \quad \xi_{4}:=\prod_{n=3}^{\infty}\left(1-\frac{(-1)^{n}}{F_{n}}\right), \\
& \nu_{3}:=\prod_{n=2}^{\infty}\left(1+\frac{(-1)^{n}}{L_{n}}\right), \quad \nu_{4}:=\prod_{n=1}^{\infty}\left(1-\frac{(-1)^{n}}{L_{n}}\right),
\end{aligned}
$$

since we will prove in Section 3 that

$$
\begin{equation*}
\xi_{3}=6 \xi_{2}, \quad 12 \xi_{4}=\xi_{1}, \quad 2 \sqrt{5} \nu_{3}=\nu_{1}, \quad \nu_{4}=2 \sqrt{5} \nu_{2} \tag{10}
\end{equation*}
$$

Theorem 3 Any two numbers in the set

$$
\begin{aligned}
\left\{\lambda_{1}:=\sum_{n=1}^{\infty} \frac{1}{F_{2 n-1}}, \quad \xi_{5}:=\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2 n-1}}\right), \quad \xi_{6}:=\prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2 n-1}}\right),\right. \\
\left.\lambda_{2}:=\sum_{n=1}^{\infty} \frac{1}{L_{2 n}}, \quad \nu_{5}:=\prod_{n=1}^{\infty}\left(1+\frac{1}{L_{2 n}}\right), \quad \nu_{6}:=\prod_{n=1}^{\infty}\left(1-\frac{1}{L_{2 n}}\right)\right\}
\end{aligned}
$$

are algebraically independent over $\mathbb{Q}$ except only three algebraically dependent cases $\left\{\xi_{5}, \nu_{5}\right\},\left\{\xi_{5}, \nu_{6}\right\}$ and $\left\{\nu_{5}, \nu_{6}\right\}$, while any three are not.

Theorems 2 and 3 give generalizations of the algebraic independence and transcendence results for the numbers (6) and (5), respectively.

The present paper is organized as follows. In Section 2, we prove Theorem 1 by using the theory of modular functions and Nesterenko's Theorem A. In Section 3, we show that the infinite products in Theorems 2 and 3 can be expressed by values of the Dedekind eta function. Section 4 is devoted to the proofs of Theorems 2 and 3. In the last Section 5, we give explicit algebraic relations for certain eta products, which yield similar relations for any two numbers in the set $\left\{\xi_{5}, \nu_{5}, \nu_{6}\right\}$ excepted in Theorem 3.

## 2 Proof of Theorem 1

We first show the following Lemma 1. The proof is almost the same as that of [10, Theorem 1.1], but we recall it for the convenience of the readers.

Lemma 1 Let $N \geq 1$ be an integer and $x(\tau), y(\tau) \in \mathfrak{F}_{N}$. Let $\tau_{0} \in \mathbb{H}$ be not a pole of the functions $x(\tau)$ and $y(\tau)$. If the function $y(\tau)$ is not constant, then the number $x\left(\tau_{0}\right)$ is algebraic over the field $\mathbb{Q}\left(y\left(\tau_{0}\right)\right)$.

Proof. Let $j(\tau)$ be the elliptic modular $j$-function. As previously stated, the field $\mathfrak{F}_{N}$ is a Galois extension of $\mathbb{Q}(j(\tau))$ and so the functions $x(\tau), y(\tau) \in \mathfrak{F}_{N}$ are algebraic over $\mathbb{Q}(j(\tau))$. Since $y(\tau)$ is not constant, the function $x(\tau)$ is algebraic over $\mathbb{Q}(y(\tau))$; namely, there exists a polynomial

$$
f(X, Y):=b_{0}(Y) X^{n}+b_{1}(Y) X^{n-1}+\cdots+b_{n}(Y), \quad b_{0}(Y) \not \equiv 0
$$

with $b_{0}(Y), b_{1}(Y), \ldots, b_{n}(Y) \in \mathbb{Q}[Y]$, such that $f(x(\tau), y(\tau))$ is identically zero, where the polynomials $b_{0}(Y), b_{1}(Y), \ldots, b_{n}(Y)$ have no common factors in $\mathbb{Q}[Y]$.

Let $\tau_{0} \in \mathbb{H}$ be as in Lemma 1. Suppose that $b_{\mu}\left(y\left(\tau_{0}\right)\right)=0$ for all $\mu=0,1, \ldots, n$. Then $y\left(\tau_{0}\right)$ is an algebraic number, since $b_{0}(Y)$ is a non-zero polynomial. Hence, all polynomials $b_{\mu}(Y)$ are divisible by the minimal polynomial of $y\left(\tau_{0}\right)$ over $\mathbb{Q}$, which is impossible. Thus, the polynomial $g(X):=f\left(X, y\left(\tau_{0}\right)\right)$ over $\mathbb{Q}\left(y\left(\tau_{0}\right)\right)$ is non-zero and satisfies $g\left(x\left(\tau_{0}\right)\right)=f\left(x\left(\tau_{0}\right), y\left(\tau_{0}\right)\right)=0$, and therefore, $x\left(\tau_{0}\right)$ is algebraic over $\mathbb{Q}\left(y\left(\tau_{0}\right)\right)$.

Now we prove Theorem 1.
Proof of Theorem 1. Let $\tau_{0} \in \mathbb{H}$ be as in Theorem 1. We first assume that $f_{1}^{k_{2}} / f_{2}^{k_{1}}$ is constant, and we show that $f_{1}\left(\tau_{0}\right)$ and $f_{2}\left(\tau_{0}\right)$ are algebraically dependent over $\mathbb{Q}$. If $f_{2}$ is constant, then $f_{2}$ is an algebraic number by the condition on the coefficients of the Fourier expansions, and hence, the assertion clearly holds. Let $f_{2}$ be non-constant. If the integers $k_{1}$ and $k_{2}$ are not both zero, the assertion is also clear, since $f_{1}^{k_{2}} / f_{2}^{k_{1}}$ is an algebraic number. In the case where $k_{1}$ and $k_{2}$ are both zero, the functions $f_{1}$ and $f_{2}$ belong to $\mathfrak{F}_{N}$ and the assertion holds by Lemma 1.

Next we assume that $f_{1}^{k_{2}} / f_{2}^{k_{1}}$ is not constant. Then the integers $k_{1}$ and $k_{2}$ are not both zero. We may assume that $k_{1} \neq 0$ (If $k_{1}=0$ and $k_{2} \neq 0$, consider the inverse $f_{2}^{k_{1}} / f_{1}^{k_{2}}$ instead of $f_{1}^{k_{2}} / f_{2}^{k_{1}}$ ). Since the Eisenstein series $E_{4}(\tau)$ is a modular form of weight 4 for $S L_{2}(\mathbb{Z})$, the functions

$$
\begin{equation*}
x(\tau):=\frac{E_{4}^{k_{1}}}{f_{1}^{4}}, \quad y(\tau):=\frac{f_{1}^{k_{2}}}{f_{2}^{k_{1}}} \tag{11}
\end{equation*}
$$

are both modular functions of weight zero for $\Gamma(N)$ and belong to the field $\mathfrak{F}_{N}$. Let $\tau_{0} \in \mathbb{H}$ be as in Theorem 1. Then by Lemma 1 the number $x\left(\tau_{0}\right)$ is algebraic over the field $\mathbb{Q}\left(y\left(\tau_{0}\right)\right)$, so that by (11) the number

$$
E_{4}\left(\tau_{0}\right)^{k_{1}}=x\left(\tau_{0}\right) f_{1}\left(\tau_{0}\right)^{4}
$$

is algebraic over the field $\mathbb{Q}\left(y\left(\tau_{0}\right), f_{1}\left(\tau_{0}\right)\right) \subset \mathbb{K}:=\mathbb{Q}\left(f_{1}\left(\tau_{0}\right), f_{2}\left(\tau_{0}\right)\right)$. This implies that the number $E_{4}\left(\tau_{0}\right)$ is algebraic over $\mathbb{K}$, since $k_{1}$ is non-zero. Similarly, replacing $E_{4}$ by $E_{6}$ and $f_{1}^{4}$ by $f_{1}^{6}$ in (11) yields that the number $E_{6}\left(\tau_{0}\right)$ is algebraic over $\mathbb{K}$. Thus, we have

$$
2 \geq \text { trans.deg } \mathbb{Q}_{\mathbb{Q}} \mathbb{K}=\operatorname{trans} \cdot \operatorname{deg}_{\mathbb{Q}} \mathbb{K}\left(E_{4}\left(\tau_{0}\right), E_{6}\left(\tau_{0}\right)\right) \geq \operatorname{trans} \cdot \operatorname{deg}_{\mathbb{Q}} \mathbb{Q}\left(E_{4}\left(\tau_{0}\right), E_{6}\left(\tau_{0}\right)\right)=2,
$$

where the last equality follows from Nesterenko's Theorem A in Section 1, since $e^{\pi i \tau_{0}}$ is algebraic. Therefore, we obtain trans. $\operatorname{deg}_{\mathbb{Q}} \mathbb{K}=2$, namely, the numbers $f_{1}\left(\tau_{0}\right)$ and $f_{2}\left(\tau_{0}\right)$ are algebraically independent over $\mathbb{Q}$. The proof of Theorem 1 is completed.

## 3 Infinite products and Dedekind eta function

In this section, we show that certain infinite products involving the Fibonacci and Lucas numbers can be expressed by means of values of the Dedekind eta function

$$
\begin{equation*}
\eta(\tau):=q^{1 / 24} \prod_{n=1}^{\infty}\left(1-q^{n}\right), \quad q:=e^{2 \pi i \tau} \tag{12}
\end{equation*}
$$

which defines a holomorphic function on $\mathbb{H}$. The Dedekind eta function $\eta(\tau)$ has no zeros or poles in $\mathbb{H}$, and moreover, the function $\eta(\tau)^{24}$ is a modular form of weight 12 for $S L_{2}(\mathbb{Z})$ and has the expression

$$
\eta(\tau)^{24}=\frac{1}{1728}\left(E_{4}(\tau)^{3}-E_{6}(\tau)^{2}\right),
$$

where $E_{4}$ and $E_{6}$ are the normalized Eisenstein series defined in (1) (cf. [1]). From (12) we deduce at once

$$
\begin{array}{ll}
\prod_{n=1}^{\infty}\left(1-q^{n}\right)=q^{-1 / 24} \eta(\tau), & \prod_{n=1}^{\infty}\left(1-q^{2 n-1}\right)=q^{1 / 24} \frac{\eta(\tau)}{\eta(2 \tau)} \\
\prod_{n=1}^{\infty}\left(1+q^{n}\right)=q^{-1 / 24} \frac{\eta(2 \tau)}{\eta(\tau)}, & \prod_{n=1}^{\infty}\left(1+q^{2 n-1}\right)=q^{1 / 24} \frac{\eta(2 \tau)^{2}}{\eta(\tau) \eta(4 \tau)} . \tag{14}
\end{array}
$$

Let $\left\{F_{n}\right\}$ and $\left\{L_{n}\right\}$ be the Fibonacci and Lucas sequences defined in Section 1. In what follows, let $\beta:=(\sqrt{5}-1) / 2$ and $\tau_{0} \in \mathbb{H}$ be a fixed complex number such that

$$
\begin{equation*}
e^{2 \pi i \tau_{0}}=\beta^{2} . \tag{15}
\end{equation*}
$$

Then we obtain

$$
\begin{equation*}
\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2 n}}\right)=2 \beta^{-1}, \quad \prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2 n}}\right)=\frac{1}{3} \beta^{-1} \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2 n-1}}\right)=2 \beta^{-1 / 4} \frac{\eta\left(4 \tau_{0}\right)}{\eta\left(\tau_{0}\right)}, \quad \prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2 n-1}}\right)=\sqrt{5} \beta^{-1 / 4} \frac{\eta\left(\tau_{0}\right)^{3} \eta\left(4 \tau_{0}\right)}{\eta\left(2 \tau_{0}\right)^{2}} \tag{17}
\end{equation*}
$$

(cf. [9], see also [8]). Similar expressions are obtained for the case of the Lucas numbers:

Lemma 2 Let $\tau_{0} \in \mathbb{H}$ satisfy (15). Then we have

$$
\begin{align*}
\prod_{n=1}^{\infty}\left(1+\frac{1}{L_{2 n}}\right) & =\frac{\eta\left(2 \tau_{0}\right) \eta\left(3 \tau_{0}\right)}{\eta\left(\tau_{0}\right) \eta\left(4 \tau_{0}\right)}, & \prod_{n=1}^{\infty}\left(1-\frac{1}{L_{2 n}}\right) & =\frac{\eta\left(\tau_{0}\right) \eta\left(6 \tau_{0}\right)}{\eta\left(3 \tau_{0}\right) \eta\left(4 \tau_{0}\right)}  \tag{18}\\
\prod_{n=1}^{\infty}\left(1+\frac{1}{L_{2 n-1}}\right) & =2 \beta^{-1 / 4} \frac{\eta\left(2 \tau_{0}\right)^{2}}{\eta\left(\tau_{0}\right)}, & \prod_{n=2}^{\infty}\left(1-\frac{1}{L_{2 n-1}}\right) & =\frac{1}{\sqrt{5}} \beta^{-1 / 4} \frac{\eta\left(2 \tau_{0}\right)^{2}}{\eta\left(\tau_{0}\right)} . \tag{19}
\end{align*}
$$

Proof. By Binet's formula, we have for all $n \geq 1$

$$
\begin{equation*}
L_{2 n}=\beta^{-2 n}+\beta^{2 n}, \quad L_{2 n-1}=\beta^{-2 n+1}-\beta^{2 n-1} \tag{20}
\end{equation*}
$$

Let $\varepsilon= \pm 1$. Then by (20)

$$
\begin{aligned}
\prod_{n=1}^{\infty}\left(1+\frac{\varepsilon}{L_{2 n}}\right) & =\prod_{n=1}^{\infty}\left(1+\frac{\varepsilon}{\beta^{-2 n}+\beta^{2 n}}\right)=\prod_{n=1}^{\infty}\left(1+\frac{\varepsilon \beta^{2 n}}{1+\beta^{4 n}}\right) \\
& =\prod_{n=1}^{\infty} \frac{1+\varepsilon \beta^{2 n}+\beta^{4 n}}{1+\beta^{4 n}}=\prod_{n=1}^{\infty} \frac{1-\varepsilon \beta^{6 n}}{\left(1-\varepsilon \beta^{2 n}\right)\left(1+\beta^{4 n}\right)}
\end{aligned}
$$

Hence, we get by (13) and (14)

$$
\begin{aligned}
& \prod_{n=1}^{\infty}\left(1+\frac{1}{L_{2 n}}\right)=\prod_{n=1}^{\infty} \frac{1-\beta^{6 n}}{\left(1-\beta^{2 n}\right)\left(1+\beta^{4 n}\right)}=\frac{\eta\left(2 \tau_{0}\right) \eta\left(3 \tau_{0}\right)}{\eta\left(\tau_{0}\right) \eta\left(4 \tau_{0}\right)}, \\
& \prod_{n=1}^{\infty}\left(1-\frac{1}{L_{2 n}}\right)=\prod_{n=1}^{\infty} \frac{1+\beta^{6 n}}{\left(1+\beta^{2 n}\right)\left(1+\beta^{4 n}\right)}=\frac{\eta\left(\tau_{0}\right) \eta\left(6 \tau_{0}\right)}{\eta\left(3 \tau_{0}\right) \eta\left(4 \tau_{0}\right)},
\end{aligned}
$$

which are the formulas in (18). On the other hand, we have by (20)

$$
\begin{aligned}
\prod_{n=2}^{\infty}\left(1+\frac{\varepsilon}{L_{2 n-1}}\right) & =\prod_{n=2}^{\infty}\left(1+\frac{\varepsilon L_{1}}{L_{2 n-1}}\right)=\prod_{n=2}^{\infty}\left(1+\frac{\varepsilon \beta^{-1}-\varepsilon \beta}{\beta^{-2 n+1}-\beta^{2 n-1}}\right) \\
& =\prod_{n=2}^{\infty}\left(1+\frac{\varepsilon \beta^{2 n-2}-\varepsilon \beta^{2 n}}{1-\beta^{4 n-2}}\right)=\prod_{n=2}^{\infty} \frac{\left(1+\varepsilon \beta^{2 n-2}\right)\left(1-\varepsilon \beta^{2 n}\right)}{1-\beta^{4 n-2}} \\
& =\frac{1-\beta^{2}}{1-\varepsilon \beta^{2}} \cdot \prod_{n=1}^{\infty} \frac{1-\beta^{4 n}}{1-\beta^{4 n-2}}=\frac{1-\beta^{2}}{1-\varepsilon \beta^{2}} \cdot \beta^{-1 / 4} \frac{\eta\left(2 \tau_{0}\right)^{2}}{\eta\left(\tau_{0}\right)}
\end{aligned}
$$

which proves (19). The proof of Lemma 2 is completed.
From (16), (17), (18), and (19), we deduce immediately that

$$
\begin{array}{ll}
\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{n}}\right)=4 \beta^{-5 / 4} \frac{\eta\left(4 \tau_{0}\right)}{\eta\left(\tau_{0}\right)}, & \prod_{n=3}^{\infty}\left(1-\frac{1}{F_{n}}\right)=\frac{\sqrt{5}}{3} \beta^{-5 / 4} \frac{\eta\left(\tau_{0}\right)^{3} \eta\left(4 \tau_{0}\right)}{\eta\left(2 \tau_{0}\right)^{2}} \\
\prod_{n=1}^{\infty}\left(1+\frac{1}{L_{n}}\right)=2 \beta^{-1 / 4} \frac{\eta\left(2 \tau_{0}\right)^{3} \eta\left(3 \tau_{0}\right)}{\eta\left(\tau_{0}\right)^{2} \eta\left(4 \tau_{0}\right)}, & \prod_{n=2}^{\infty}\left(1-\frac{1}{L_{n}}\right)=\frac{1}{\sqrt{5}} \beta^{-1 / 4} \frac{\eta\left(2 \tau_{0}\right)^{2} \eta\left(6 \tau_{0}\right)}{\eta\left(3 \tau_{0}\right) \eta\left(4 \tau_{0}\right)}, \tag{22}
\end{array}
$$

and that

$$
\begin{array}{ll}
\prod_{n=2}^{\infty}\left(1+\frac{(-1)^{n}}{F_{n}}\right)=2 \sqrt{5} \beta^{-5 / 4} \frac{\eta\left(\tau_{0}\right)^{3} \eta\left(4 \tau_{0}\right)}{\eta\left(2 \tau_{0}\right)^{2}}, & \prod_{n=3}^{\infty}\left(1-\frac{(-1)^{n}}{F_{n}}\right)=\frac{1}{3} \beta^{-5 / 4} \frac{\eta\left(4 \tau_{0}\right)}{\eta\left(\tau_{0}\right)} \\
\prod_{n=2}^{\infty}\left(1+\frac{(-1)^{n}}{L_{n}}\right)=\frac{1}{\sqrt{5}} \beta^{-1 / 4} \frac{\eta\left(2 \tau_{0}\right)^{3} \eta\left(3 \tau_{0}\right)}{\eta\left(\tau_{0}\right)^{2} \eta\left(4 \tau_{0}\right)}, & \prod_{n=1}^{\infty}\left(1-\frac{(-1)^{n}}{L_{n}}\right)=2 \beta^{-1 / 4} \frac{\eta\left(2 \tau_{0}\right)^{2} \eta\left(6 \tau_{0}\right)}{\eta\left(3 \tau_{0}\right) \eta\left(4 \tau_{0}\right)}
\end{array}
$$

which shows (10).

## 4 Proofs of Theorems 2 and 3

To prove Theorems 2 and 3 , we need two lemmas.
Lemma 3 Let $f(\tau)$ be a modular function of weight $k$ for $\Gamma(N)$ whose Fourier expansion with respect to $e^{2 \pi i \tau / N}$ has coefficients in $\mathbb{Q}\left(e^{2 \pi i / N}\right)$. Let $\tau_{0} \in \mathbb{H}$ be not a pole of $f$. Then the number $f\left(\tau_{0}\right)$ is algebraic over the field $\mathbb{Q}\left(E_{4}\left(\tau_{0}\right), E_{6}\left(\tau_{0}\right)\right)$.

Proof. This follows immediately from Lemma 1 with the functions

$$
x(\tau):=\frac{f^{12}}{\eta^{24 k}}=(1728)^{k} \frac{f^{12}}{\left(E_{4}^{3}-E_{6}^{2}\right)^{k}}, \quad y(\tau):=j(\tau)=1728 \frac{E_{4}^{3}}{E_{4}^{3}-E_{6}^{2}}
$$

which belong to the field $\mathfrak{F}_{N}$.
Lemma 4 (cf. [12, Proposition 17 (a)]) Let $f(\tau)$ be a modular form of weight $k$ for $S L_{2}(\mathbb{Z})$. Then for any integer $N \geq 1$ the function $g(\tau):=f(N \tau)$ is a modular form of weight $k$ for $\Gamma(N)$.

Proof of Theorem 2. Define the four functions

$$
\begin{equation*}
\Xi_{1}(\tau):=\frac{\eta(4 \tau)}{\eta(\tau)}, \quad \Xi_{2}(\tau):=\frac{\eta(\tau)^{3} \eta(4 \tau)}{\eta(2 \tau)^{2}}, \quad N_{1}(\tau):=\frac{\eta(2 \tau)^{3} \eta(3 \tau)}{\eta(\tau)^{2} \eta(4 \tau)}, \quad N_{2}(\tau):=\frac{\eta(2 \tau)^{2} \eta(6 \tau)}{\eta(3 \tau) \eta(4 \tau)} \tag{23}
\end{equation*}
$$

Then by (21) and (22) we have

$$
\xi_{1}=4 \beta^{-5 / 4} \Xi_{1}\left(\tau_{0}\right), \quad \xi_{2}=\frac{\sqrt{5}}{3} \beta^{-5 / 4} \Xi_{2}\left(\tau_{0}\right), \quad \nu_{1}=2 \beta^{-1 / 4} N_{1}\left(\tau_{0}\right), \quad \nu_{2}=\frac{1}{\sqrt{5}} \beta^{-1 / 4} N_{2}\left(\tau_{0}\right)
$$

where $\tau_{0} \in \mathbb{H}$ satisfies (15). Hence, noting that $\beta$ is an algebraic number, we only have to study the algebraic independence over $\mathbb{Q}$ of the four values

$$
\begin{equation*}
\Xi_{1}\left(\tau_{0}\right), \quad \Xi_{2}\left(\tau_{0}\right), \quad N_{1}\left(\tau_{0}\right), \quad N_{2}\left(\tau_{0}\right) \tag{24}
\end{equation*}
$$

We first show the latter assertion of Theorem 2, namely, that any three numbers in (24) are algebraically dependent over $\mathbb{Q}$. Let $N \geq 1$ be an integer. Then by Lemma 4 the function $\eta(N \tau)^{24}$ is a modular form of weight 12 for $\Gamma(N)$, since $\eta(\tau)^{24}$ is a modular form of weight 12 for $S L_{2}(\mathbb{Z})$. Moreover, it is clear by definition (12) that the coefficients of the Fourier expansion with respect to $e^{2 \pi i \tau / N}$ of $\eta(N \tau)^{24}$ are all rational integers. Thus, by Lemma 3 the number $\eta\left(N \tau_{0}\right)$ is algebraic over the field $\mathbb{Q}\left(E_{4}\left(\tau_{0}\right), E_{6}\left(\tau_{0}\right)\right)$, and hence, so are the numbers $(24)$ by definition (23). Therefore,
the transcendence degree of the field generated over $\mathbb{Q}$ by any three numbers in (24) is less than three.

Next we prove the algebraic independence of any pair of numbers in (24). We find by Lemma 4 again that the 24 th powers of the functions (23) are modular forms for $\Gamma(12)$ having the following weights

|  | $\Xi_{1}^{24}$ | $\Xi_{2}^{24}$ | $N_{1}^{24}$ | $N_{2}^{24}$ |
| :---: | :---: | :---: | :---: | :---: |
| weight | 0 | 24 | 12 | 12 |

Let $f_{1}=\Xi_{1}^{24}$ and $f_{2} \in\left\{\Xi_{2}^{24}, N_{1}^{24}, N_{2}^{24}\right\}$. Then $f_{1}^{k_{2}} / f_{2}^{k_{1}}=f_{1}^{k_{2}}$ is not constant, since the weights $k_{1}=0$ and $k_{2}=12$ or 24 . Hence, by Theorem 1 the two numbers $f_{1}\left(\tau_{0}\right)$ and $f_{2}\left(\tau_{0}\right)$ are algebraically independent over $\mathbb{Q}$. Similarly, considering the functions $f_{1}, f_{2} \in\left\{\Xi_{2}^{24}, N_{1}^{24}, N_{2}^{24}\right\}$, we can deduce the same conclusion by Theorem 1 , since the ratios

$$
\frac{\Xi_{2}}{N_{1}^{2}}, \quad \frac{\Xi_{2}}{N_{2}^{2}}, \quad \frac{N_{1}}{N_{2}}
$$

are non-constant functions. The proof of Theorem 2 is completed.
Proof of Theorem 3. Let $\tau_{0} \in \mathbb{H}$ satisfy (15). Then we have

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{1}{F_{2 n-1}}=\frac{\sqrt{5}}{4} \vartheta_{2}\left(2 \tau_{0}\right)^{2}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2 n}}=\frac{1}{4}\left(\vartheta_{3}\left(2 \tau_{0}\right)^{2}-1\right) \tag{25}
\end{equation*}
$$

(cf. [4, $\S 3.7$ Comments and Exercises]), where $\vartheta_{2}$ and $\vartheta_{3}$ are the theta-constants defined in (3). Hence, using Jacobi's triple-product identities together with (13) and (14), we have

$$
\begin{align*}
& \vartheta_{2}(2 \tau)=2 q^{1 / 4} \prod_{n=1}^{\infty}\left(1-q^{2 n}\right)\left(1+q^{2 n}\right)^{2}=2 \frac{\eta(4 \tau)^{2}}{\eta(2 \tau)},  \tag{26}\\
& \vartheta_{3}(2 \tau)=\prod_{n=1}^{\infty}\left(1-q^{2 n}\right)\left(1+q^{2 n-1}\right)^{2}=\frac{\eta(2 \tau)^{5}}{\eta(\tau)^{2} \eta(4 \tau)^{2}}, \tag{27}
\end{align*}
$$

where $q:=e^{2 \pi i \tau}(\tau \in \mathbb{H})$. Thus, by (17), (18), (25), (26), and (27), for proving Theorem 3 we only have to investigate the six values

$$
\begin{equation*}
\Lambda_{1}\left(\tau_{0}\right), \quad \Xi_{5}\left(\tau_{0}\right), \quad \Xi_{6}\left(\tau_{0}\right), \quad \Lambda_{2}\left(\tau_{0}\right), \quad N_{5}\left(\tau_{0}\right), \quad N_{6}\left(\tau_{0}\right) \tag{28}
\end{equation*}
$$

where

$$
\begin{gather*}
\Lambda_{1}(\tau):=\frac{\eta(4 \tau)^{2}}{\eta(2 \tau)}, \quad \Xi_{5}(\tau):=\frac{\eta(4 \tau)}{\eta(\tau)}, \quad \Xi_{6}(\tau):=\frac{\eta(\tau)^{3} \eta(4 \tau)}{\eta(2 \tau)^{2}},  \tag{29}\\
\Lambda_{2}(\tau):=\frac{\eta(2 \tau)^{5}}{\eta(\tau)^{2} \eta(4 \tau)^{2}}, \quad N_{5}(\tau):=\frac{\eta(2 \tau) \eta(3 \tau)}{\eta(\tau) \eta(4 \tau)}, \quad N_{6}(\tau):=\frac{\eta(\tau) \eta(6 \tau)}{\eta(3 \tau) \eta(4 \tau)} . \tag{30}
\end{gather*}
$$

Similarly as in the proof of Theorem 2, we find that the 24th powers of the functions in (29) and (30) are modular forms for $\Gamma$ (12) having the weights

|  | $\Lambda_{1}^{24}$ | $\Xi_{5}^{24}$ | $\Xi_{6}^{24}$ | $\Lambda_{2}^{24}$ | $N_{5}^{24}$ | $N_{6}^{24}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| weight | 12 | 0 | 24 | 12 | 0 | 0 |

and that any three numbers in (28) are algebraically dependent over $\mathbb{Q}$.
If $f_{1}, f_{2} \in\left\{\Xi_{5}^{24}, N_{5}^{24}, N_{6}^{24}\right\}$, then $f_{1}^{k_{2}} / f_{2}^{k_{1}}=1$ is a constant. Hence, by Theorem 1 any two numbers in $\left\{\Xi_{5}^{24}\left(\tau_{0}\right), N_{5}^{24}\left(\tau_{0}\right), N_{6}^{24}\left(\tau_{0}\right)\right\}$ are algebraically dependent over $\mathbb{Q}$, and so are any two numbers in the set $\left\{\xi_{5}, \nu_{5}, \nu_{6}\right\}$. Next, we consider the case when $f_{1} \in\left\{\Xi_{5}^{24}, N_{5}^{24}, N_{6}^{24}\right\}$ and $f_{2} \in$ $\left\{\Lambda_{1}^{24}, \Xi_{6}^{24}, \Lambda_{2}^{24}\right\}$. Then $f_{1}^{k_{2}} / f_{2}^{k_{1}}=f_{1}^{k_{2}}$ is not constant, since $k_{1}=0$ and $k_{2}=12$ or 24 . Hence, by Theorem 1 the numbers $f_{1}\left(\tau_{0}\right)$ and $f_{2}\left(\tau_{0}\right)$ are algebraically independent over $\mathbb{Q}$. Moreover, we can deduce the same conclusion for the case when $f_{1}, f_{2} \in\left\{\Lambda_{1}^{24}, \Xi_{6}^{24}, \Lambda_{2}^{24}\right\}$, since the corresponding ratios

$$
\frac{\Lambda_{1}^{2}}{\Xi_{6}}, \quad \frac{\Lambda_{1}}{\Lambda_{2}}, \quad \frac{\Xi_{6}}{\Lambda_{2}^{2}}
$$

are not constant. The proof of Theorem 3 is completed.

## 5 Algebraic relations for certain eta products

Let $\xi_{5}, \nu_{5}, \nu_{6}$ be as in Theorem 3 and let $f, g, h$ be eta products defined by

$$
\begin{equation*}
f:=\frac{\eta(4 \tau)}{\eta(\tau)}, \quad g:=\frac{\eta(2 \tau) \eta(3 \tau)}{\eta(\tau) \eta(4 \tau)}, \quad h:=\frac{\eta(\tau) \eta(6 \tau)}{\eta(3 \tau) \eta(4 \tau)} . \tag{31}
\end{equation*}
$$

Then for $\tau_{0} \in \mathbb{H}$ with $e^{2 \pi i \tau_{0}}=\beta^{2}$ we have by (17) and (18)

$$
\begin{equation*}
\xi_{5}=2 \beta^{-1 / 4} f\left(\tau_{0}\right), \quad \nu_{5}=g\left(\tau_{0}\right), \quad \nu_{6}=h\left(\tau_{0}\right) \tag{32}
\end{equation*}
$$

In this section, we give the algebraic relations for each two functions in $\{f, g, h\}$. Such relations yield the algebraic relations for each two numbers in $\left\{\xi_{5}, \nu_{5}, \nu_{6}\right\}$ through the equalities (32).
I). Algebraic relation for $f$ and $h$.

It is known that the symmetric polynomial

$$
\begin{aligned}
P(X, Y):= & X^{4}+Y^{4}-X^{3} Y^{3}-2^{3} 3^{2} X^{2} Y^{2}(X+Y)-2^{2} 3^{2} 5^{2} X Y\left(X^{2}+Y^{2}\right) \\
& +2 \cdot 3^{2} \cdot 1579 X^{2} Y^{2}-2^{15} 3^{2} X Y(X+Y)-2^{24} X Y
\end{aligned}
$$

vanishes identically for

$$
X=\left(\frac{\eta(3 \tau)}{\eta(6 \tau)}\right)^{24}, \quad Y=\left(\frac{\eta(\tau)}{\eta(2 \tau)}\right)^{24}
$$

(cf. [14, see $\Psi_{3}^{\Gamma_{0}(2)}$ in Table 2, p. 183]). On the other hand, we have the identities

$$
\begin{equation*}
f^{8}+16 f^{16}=\left(\frac{\eta(2 \tau)}{\eta(\tau)}\right)^{24} \tag{33}
\end{equation*}
$$

(cf. $\left[16, \S 34\right.$, p. 114]), which is equivalent to Jacobi's celebrated relation $\vartheta_{3}(\tau)^{4}=\vartheta_{2}(\tau)^{4}+\vartheta_{4}(\tau)^{4}$, and

$$
(f h)^{24}=\left(\frac{\eta(6 \tau)}{\eta(3 \tau)}\right)^{24}
$$

Hence, the polynomial

$$
\begin{aligned}
Q(X, Y):= & X^{4} Y^{4} P\left(X^{-1}, Y^{-1}\right) \\
= & X^{4}+Y^{4}-2^{24} X^{3} Y^{3}-2^{15} 3^{2} X^{2} Y^{2}(X+Y)-2^{2} 3^{2} 5^{2} X Y\left(X^{2}+Y^{2}\right) \\
& +2 \cdot 3^{2} \cdot 1579 X^{2} Y^{2}-2^{3} 3^{2} X Y(X+Y)-X Y
\end{aligned}
$$

vanishes identically for $X=f^{8}+16 f^{16}$ and $Y=(f h)^{24}$.
II). Algebraic relation for $f$ and $g$.

Set

$$
\begin{equation*}
x:=2^{12}\left(\frac{\eta(2 \tau)}{\eta(\tau)}\right)^{24}, \quad y:=3^{6}\left(\frac{\eta(3 \tau)}{\eta(\tau)}\right)^{12}, \quad z:=j(\tau) \tag{34}
\end{equation*}
$$

where $j(\tau)$ is the elliptic modular function defined by (8). We use the well-known identities

$$
\begin{equation*}
z=\frac{(x+16)^{3}}{x}, \quad z=\frac{(y+3)^{3}(y+27)}{y} \tag{35}
\end{equation*}
$$

(cf. [16, § 72, p. 250-252]). By the second identity in (35), we have

$$
z^{2} y^{2}-72 z y^{2}\left(y^{2}+21\right)-\left(y^{2}-9\right)^{3}\left(y^{2}-27^{2}\right) \equiv 0,
$$

and then together with the first identity in (35) we obtain

$$
\begin{equation*}
(x+16)^{6} y^{2}-72 x y^{2}(x+16)^{3}\left(y^{2}+21\right)-x^{2}\left(y^{2}-9\right)^{3}\left(y^{2}-3^{6}\right) \equiv 0 . \tag{36}
\end{equation*}
$$

Multiplying (36) by $x^{2}$ yields

$$
\begin{equation*}
(x+16)^{6} x \cdot x y^{2}-72 x \cdot x y^{2}(x+16)^{3}\left(x y^{2}+21 x\right)-\left(x y^{2}-9 x\right)^{3}\left(x y^{2}-3^{6} x\right) \equiv 0 . \tag{37}
\end{equation*}
$$

On the other hand, by (31), (33) and (34) we have

$$
\begin{equation*}
x=2^{12}\left(f^{8}+16 f^{16}\right), \quad x y^{2}=6^{12}(f g)^{24} . \tag{38}
\end{equation*}
$$

Therefore, substituting (38) into the polynomial in (37), we find that the polynomial

$$
R(X, Y):=\left(2^{8} X+1\right)^{6} X Y-2^{3} 3^{3} X Y\left(2^{8} X+1\right)^{3}\left(7 X+3^{11} Y\right)-\left(X-3^{10} Y\right)^{3}\left(X-3^{6} Y\right)
$$

vanishes identically for $X=f^{8}+16 f^{16}$ and $Y=(f g)^{24}$.
III). Algebraic relation for $g$ and $h$.

We use the identities

$$
\begin{align*}
3 \frac{\eta(3 \tau)^{3} \eta(4 \tau)}{\eta(\tau) \eta(12 \tau)} & =2 \frac{\eta(2 \tau)^{7} \eta(3 \tau)}{\eta(\tau)^{3} \eta(4 \tau)^{2} \eta(6 \tau)}+\frac{\eta(\tau)^{3} \eta(4 \tau) \eta(6 \tau)^{2}}{\eta(2 \tau)^{2} \eta(3 \tau) \eta(12 \tau)},  \tag{39}\\
3 \frac{\eta(\tau) \eta(4 \tau)^{2} \eta(6 \tau)^{9}}{\eta(2 \tau)^{3} \eta(3 \tau)^{3} \eta(12 \tau)^{4}} & =\frac{\eta(2 \tau)^{7} \eta(3 \tau)}{\eta(\tau)^{3} \eta(4 \tau)^{2} \eta(6 \tau)}+2 \frac{\eta(\tau)^{3} \eta(4 \tau) \eta(6 \tau)^{2}}{\eta(2 \tau)^{2} \eta(3 \tau) \eta(12 \tau)},  \tag{40}\\
2 \frac{\eta(4 \tau)^{4} \eta(6 \tau)^{2}}{\eta(2 \tau)^{2} \eta(12 \tau)^{2}} & =\frac{\eta(2 \tau)^{7} \eta(3 \tau)}{\eta(\tau)^{3} \eta(4 \tau)^{2} \eta(6 \tau)}+\frac{\eta(\tau)^{3} \eta(4 \tau) \eta(6 \tau)^{2}}{\eta(2 \tau)^{2} \eta(3 \tau) \eta(12 \tau)} \tag{41}
\end{align*}
$$

(cf. [13, Example 26.32]). Multiplying (39) by

$$
w(\tau):=\frac{\eta(2 \tau)^{2} \eta(12 \tau)}{\eta(\tau) \eta(3 \tau) \eta(4 \tau)^{3}},
$$

we have

$$
\begin{equation*}
2 \frac{\eta(2 \tau)^{9} \eta(12 \tau)}{\eta(\tau)^{4} \eta(4 \tau)^{5} \eta(6 \tau)}=3\left(\frac{\eta(2 \tau) \eta(3 \tau)}{\eta(\tau) \eta(4 \tau)}\right)^{2}-\left(\frac{\eta(\tau) \eta(6 \tau)}{\eta(3 \tau) \eta(4 \tau)}\right)^{2}=3 g^{2}-h^{2} \tag{42}
\end{equation*}
$$

Moreover, multiplying (40) by $2 w(\tau) / 3$ and using (42), we have

$$
\begin{align*}
2 \frac{\eta(6 \tau)^{9}}{\eta(2 \tau) \eta(3 \tau)^{4} \eta(4 \tau) \eta(12 \tau)^{3}} & =\frac{2}{3} \cdot \frac{\eta(2 \tau)^{9} \eta(12 \tau)}{\eta(\tau)^{4} \eta(4 \tau)^{5} \eta(6 \tau)}+\frac{4}{3}\left(\frac{\eta(\tau) \eta(6 \tau)}{\eta(3 \tau) \eta(4 \tau)}\right)^{2} \\
& =\left(g^{2}-\frac{h^{2}}{3}\right)+\frac{4}{3} h^{2}=g^{2}+h^{2} \tag{43}
\end{align*}
$$

Similarly, multiplying (41) by $2 w(\tau)$ and using (42) again, we obtain

$$
\begin{equation*}
4 \frac{\eta(4 \tau) \eta(6 \tau)^{2}}{\eta(\tau) \eta(3 \tau) \eta(12 \tau)}=3 g^{2}+h^{2} \tag{44}
\end{equation*}
$$

Thus, by (42), (43), and (44)

$$
\begin{equation*}
2^{12}\left(3 g^{2}-h^{2}\right)\left(g^{2}+h^{2}\right)^{3}=2^{16} \frac{\eta(2 \tau)^{6} \eta(6 \tau)^{26}}{\eta(\tau)^{4} \eta(3 \tau)^{12} \eta(4 \tau)^{8} \eta(12 \tau)^{8}}=g^{6} h^{10}\left(3 g^{2}+h^{2}\right)^{8} \tag{45}
\end{equation*}
$$

and therefore, the polynomial

$$
S(X, Y):=2^{12}(3 X-Y)(X+Y)^{3}-X^{3} Y^{5}(3 X+Y)^{8}
$$

vanishes identically for $X=g^{2}$ and $Y=h^{2}$.

## Acknowledgments

The authors would like to express their gratitude to the referee for his/her careful reading of our manuscript. This work was partly supported by JSPS KAKENHI Grant Numbers JP18K03201 (Y. Tachiya) and JP16H06336, JP18K18712, JP21H04430, JP21K18141 (M. Kaneko).

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