Vector-valued modular forms and quasimodular forms with applications to the theory of vertex operator algebras

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Abstract

In this note we show that every vector-valued holomorphic modular form that has a quasimodular form of weight \( k + n \) and depth at most \( n \) on \( \Gamma = SL_2(\mathbb{Z}) \) as an entry defines an \( (n + 1) \)-dimensional symmetric tensor representation of the group \( \Gamma \). We also prove that conversely every \( (n + 1) \)-dimensional symmetric tensor representation of \( \Gamma \) gives a vector-valued modular form of which an entry is a quasimodular form with depth \( n \). As applications we propose several criteria that a vertex operator algebra is not rational.

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Introduction

In this note we study vector-valued modular forms on \( \Gamma = SL_2(\mathbb{Z}) \) associated with a representation \( \rho : \Gamma \to GL_n(\mathbb{C}) \) that has a quasimodular form as an entry. Each entry of a vector-valued modular form is a modular form on the kernel of \( \rho \) if the kernel is non-trivial. Then the question is what kind of entries appear if the kernel is trivial. We demonstrate that a quasimodular form on \( \Gamma \) naturally appears as an entry of a vector-valued modular form associated with the \( (n + 1) \)-dimensional symmetric tensor representation \( \rho^{(n)} \) of \( \Gamma \) (The kernel of \( \rho^{(n)} \) is trivial.).

The concept of vector-valued modular forms of integral weight (in particular, weight 0) naturally appears in 2-dimensional conformal field theories (2DCFT) as well as the theory of vertex operator algebras. Suppose that our 2DCFT is rational with a finiteness condition (\( C_2 \)-cofiniteness, for instance). Then, it is known (by [7]) that the space of one-point functions
on a family of elliptic curves evaluated at a primary vector with weight \( k \) is finite-dimensional, and is invariant with respect to the slash operator \( |_k \gamma \) for each \( \gamma \in \Gamma \). In particular, every set of formal characters of simple modules for a rational vertex operator algebra gives rise to a vector-valued modular form with weight 0.

One of significant differences between rational theory and non-rational theory is that formal characters of a rational theory do not involve logarithmic terms of \( q = e^{2\pi i \tau} (\tau \in \mathbb{H}) \), where \( \mathbb{H} \) is the complex upper half-plane, while there is a non-rational theory of which formal characters (called pseudo-characters) have logarithmic terms (cf. [1], [2]). At the present moment, a non-rational theory without logarithmic terms is not known.

We now roughly explain our main theorems. Let \( \mathcal{F} \) be the space of holomorphic functions on \( \mathbb{H} \). Let \( V \) be an \((n+1)\)-dimensional vector subspace of \( \mathcal{F} \), which is invariant under the \(|_k\)-action of \( \Gamma \) and contains a quasimodular form of weight \( n + k \) and depth at most \( n \). Then we show that there is a vector-valued modular form of weight \( k \) such that the associated representation is a symmetric tensor representation \((V, \rho^{(n)})\) of \( \Gamma \) (Theorem 4). Conversely, let \( V \) be a \( \Gamma \)-invariant \((n+1)\)-dimensional vector space of \( \mathcal{F} \). Let \( \mathbb{F} = \begin{smallmatrix} f_0 & f_1 & \ldots & f_n \end{smallmatrix} \) be a vector-valued modular form on \( \Gamma \) with respect to the symmetric tensor representation \((V, \rho^{(n)})\), of which entries are bounded as \( \text{Im}(\tau) \to \infty \). Then it is shown (as one of assertions) that \( f_0 \) is a quasimodular form (Theorem 5). In consequences we give several criteria if a \( C_2 \)-cofinite (see §3) vertex operator algebra is not rational.

In §1 the concepts of vector-valued modular forms and quasimodular forms are reviewed. Main theorems (Theorem 4 and Theorem 5) are stated and proved in §2. Properties of several vector-valued modular forms are also discussed here. The final section (§3) gives several applications of the theorems to the theory of \( C_2 \)-cofinite vertex operator algebras. One of main issues here is to give criteria if a vertex operator algebra is not rational.

### 1 Vector-valued modular forms and quasimodular forms

In this section we recall the concepts of vector-valued modular forms (according to [5]) and quasimodular forms ([4]).

Let \( \mathcal{F} \) be the space of holomorphic functions on the complex upper half-plane \( \mathbb{H} \), and let \( \mathbb{F} = \begin{smallmatrix} f_1 & \ldots & f_n \end{smallmatrix} \) be a column vector of \( n \) holomorphic functions on \( \mathbb{H} \). The action of the group \( \Gamma = SL_2(\mathbb{Z}) \) on each \( \mathbb{F} \) is given by

\[
\mathbb{F}|_{k\gamma} = \begin{pmatrix} f_1 & \ldots & f_n \end{pmatrix}_{k\gamma}
\]

for every \( \gamma \in \Gamma \) and \( k \in \mathbb{Z} \). Here the action \( |_k \) of \( \Gamma \) on \( \mathcal{F} \) is defined by

\[
f|_{k\gamma} = (c\tau + d)^{-k} f(\gamma(\tau)), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma
\]

for each \( f \in \mathcal{F} \). Let \( \rho : \Gamma \to GL_n(\mathbb{C}) \) be an \( n \)-dimensional complex representation of the group \( \Gamma \). Then there is a right action of \( \Gamma \) on \( \mathcal{F}^n \)

\[
\gamma : \mathcal{F}^n \mapsto \rho(\gamma)^{-1}\mathbb{F}|_{k\gamma}, \quad \gamma \in \Gamma.
\]
Definition. Suppose that $F = t(f_1, \ldots, f_n)$ is $\Gamma$-invariant with respect to the action (1). Then $F$ is called a meromorphic vector-valued modular form of weight $k$ if each entry $f_j$ has a $q$-expansion, convergent in a neighborhood of the infinity ($i\infty$):

$$f_j(\tau) = q^{\lambda_j} \sum_{n=0}^{\infty} a_{n,j} q^n \ (\lambda_j \in \mathbb{R}), \quad q = e^{2\pi i \tau} \ (\tau \in \mathbb{H}).$$

If $\lambda_j$ is nonnegative we say that $f_j$ is holomorphic at infinity, and if each $f_j$ satisfies this condition then $F$ is called a (holomorphic) vector-valued modular form.

Let $\rho$ be a representation of $\Gamma$ on $GL_n(\mathbb{C})$. Let $M_k(\rho)$ and $H_k(\rho)$ be the associated spaces of meromorphic and holomorphic modular forms of weight $k$ respectively, and

$$M(\rho) = \bigoplus_{k \in \mathbb{Z}} M_k(\rho), \quad H(\rho) = \bigoplus_{k \in \mathbb{Z}} H_k(\rho).$$

We now recall the concept of quasimodular forms and its several properties. A quasimodular form is a “nearly modular” holomorphic function on $\mathbb{H}$; one of basic examples is the (quasi) Eisenstein series $E_2$ of weight 2;

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \left( \sum_{d|n} d \right) q^n$$

on which $\Gamma$ acts as

$$(c\tau + d)^{-2} E_2(\gamma(\tau)) = E_2(\tau) + \frac{\mu c}{c\tau + d} \text{ for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where $\mu = 6/\pi i$. We do not give the precise definition of the quasimodular form here (since it is not necessary in this note), and may refer the reader to [4]. For the modular group $\Gamma$, the graded ring of quasimodular forms is identified with $\mathbb{C}[E_2, E_4, E_6]$, where $E_4$ and $E_6$ are the (normalized) Eisenstein series with weight 4 and 6 respectively;

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \left( \sum_{d|n} d^3 \right) q^n, \quad E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \left( \sum_{d|n} d^5 \right) q^n.$$ 

Let $f$ be a quasimodular form of weight $k$. We call $f$ has depth $n$ if $f$ is written as $f = \sum_{n=0}^{\infty} f_i \cdot (E_2)^i$ where each $f_i$ is a (classical) holomorphic modular form of weight $k - 2i$ and $f_n \neq 0$. We denote the derivation $\partial_{E_2}$ on $\mathbb{C}[E_2, E_4, E_6]$ simply by $\partial$, and often use notation $\partial^{(n)} = \partial^n / n!$ for short.

2 Vector-valued modular forms arising from quasimodular forms

2.1 Pleminaries

In this section we first recall the concept of the symmetric tensor representation of $\Gamma = SL_2(\mathbb{Z})$ and establish several relations between vector-valued modular forms and quasimodular forms.
**Definition.** Let \( n \) be a nonnegative integer. For every \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) we define a square matrix \( \rho^{(n)}(\gamma) = \left( \rho^{(n)}_{ij} \right) \) of degree \( n + 1 \) by

\[
\begin{pmatrix}
(c \tau + d)^n \\
(a \tau + b)(c \tau + d)^{n-1} \\
(a \tau + b)^2(c \tau + d)^{n-2} \\
\vdots \\
(a \tau + b)^n
\end{pmatrix} = \rho^{(n)}(\gamma) \begin{pmatrix} 1 \\ \tau \\ \tau^2 \\ \vdots \\ \tau^n \end{pmatrix}.
\]

The \( \Gamma \)-module \( (V_n, \rho^{(n)}) \), where \( V_n \) is the space of polynomials of \( \tau \) of degree at most \( n \), is called the symmetric tensor representation of the group \( \Gamma \).

It readily follows that the kernel of the symmetric tensor representation of \( \Gamma \) is trivial for an odd \( n \) and is \( I, -I \) for an even \( n \) (\( I \) is the identity matrix of degree \( n \)).

We now state several lemmas which are necessary in discussions of \( \S \) 2.2. We often omit \( \tau \) if it does not cause any confusions.

**Lemma 1.** Let \( F \) be a quasimodular form of weight \( k + n \) and depth at most \( n \). Then for any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \), the identity

\[
\partial^{(\ell)} F(\gamma(\tau)) = (c \tau + d)^{k+n-2\ell} \sum_{i=\ell}^{n} \binom{i}{\ell} \partial^{(i)} F \cdot \left( \frac{\mu c}{c \tau + d} \right)^{i-\ell}
\]

holds for every nonnegative integer \( \ell \).

**Proof.** The quasimodular form \( F \) takes the form \( F = \sum_{i=0}^{n} f_i \cdot (E_2)^i \) as a polynomial in \( E_2 \) with modular form coefficients. It then follows from the Taylor series expansion that

\[
\sum_{i=0}^{n} f_i \cdot (E_2 + x)^i = \sum_{i=0}^{n} \partial^{(i)} F \cdot x^i,
\]

where \( x \) is a formal variable. By applying the operator \( (\partial/\partial x)^{\ell}/\ell! \) to both sides of (2) we have

\[
\sum_{i=\ell}^{n} \binom{i}{\ell} f_i \cdot (E_2 + x)^{i-\ell} = \sum_{i=\ell}^{n} \binom{i}{\ell} \partial^{(i)} F \cdot x^{i-\ell}.
\]

Since \( \partial^{(\ell)} F = \sum_{i=\ell}^{n} f_i \cdot \binom{i}{\ell} (E_2)^{i-\ell} \) it follows that

\[
(\partial^{(\ell)} F)(\gamma(\tau)) = \sum_{i=\ell}^{n} \binom{i}{\ell} f_i(\gamma(\tau)) E_2(\gamma(\tau))^{i-\ell}
\]

\[
= (c \tau + d)^{k+n-2\ell} \sum_{i=\ell}^{n} \binom{i}{\ell} f_i(\tau) \left( E_2(\tau) + \frac{\mu c}{c \tau + d} \right)^{i-\ell}
\]

\[
= (c \tau + d)^{k+n-2\ell} \sum_{i=\ell}^{n} \binom{i}{\ell} \partial^{(i)} F \cdot \left( \frac{\mu c}{c \tau + d} \right)^{i-\ell},
\]

where the last equality follows from (3). This completes the proof of the lemma. \( \square \)
Lemma 2. For any nonnegative integers \( i, m \) and \( n \), we have
\[
\sum_{\ell=0}^{m} \binom{n}{\ell}^{-1} \binom{i}{\ell} \binom{m}{\ell} e^{m-\ell}(a\tau + b)^{i-\ell}(c\tau + d)^{n-i-m} = \frac{1}{m!} \binom{n}{m}^{-1} \frac{\partial}{\partial \tau}^m \{(a\tau + b)^i(c\tau + d)^{n-i}\}.
\]

Proof. Since \( \binom{n}{m}^{-1} \binom{m}{\ell} \) for each \( m \geq \ell \geq 0 \) it is enough to show
\[
\sum_{\ell=0}^{m} \binom{i}{\ell} \binom{n-\ell}{m-\ell} e^{m-\ell}(a\tau + b)^{i-\ell}(c\tau + d)^{n-i-m} = \frac{1}{m!} \frac{\partial}{\partial \tau}^m \{(a\tau + b)^i(c\tau + d)^{n-i}\}. \tag{4}
\]
By Leibniz’s rule applied to the right-hand side of (4) it takes the form
\[
\sum_{j=0}^{m} \binom{i}{j} \binom{n-i}{m-j} a^j e^{m-j}(a\tau + b)^{i-j}(c\tau + d)^{n-i-m-j}.
\]
Therefore, by multiplying \( (a\tau + b)^{m-i}(c\tau + d)^{m+i-n} \) to both sides of (4), we see that proving (4) is equivalent to showing
\[
\sum_{\ell=0}^{m} \binom{i}{\ell} \binom{n-\ell}{m-\ell} e^{m-\ell}(a\tau + b)^{m-\ell} = \sum_{j=0}^{m} \binom{i}{j} \binom{n-i}{m-j} a^j e^{m-j}(a\tau + b)^{m-j}(c\tau + d)^j. \tag{5}
\]
Since \( a(c\tau + d) = 1 + c(a\tau + b) \), the right-hand side of (5) becomes
\[
\sum_{j=0}^{m} \binom{i}{j} \binom{n-i}{m-j} e^{m-j}(a\tau + b)^{m-j}(1 + c(a\tau + b))^j = \sum_{j=0}^{m} \binom{i}{j} \binom{n-i}{m-j} e^{m-j}(a\tau + b)^{m-j} \sum_{\ell=0}^{j} \binom{j}{\ell} c^{j-\ell}(a\tau + b)^{j-\ell} = \sum_{\ell=0}^{m} \sum_{j=\ell}^{m} \binom{i}{j} \binom{n-i}{m-j} \binom{j}{\ell} e^{m-\ell}(a\tau + b)^{m-\ell} = \sum_{\ell=0}^{m} \sum_{j=\ell}^{m} \binom{i}{j} \binom{n-i}{m-j} \binom{i}{\ell} e^{m-\ell}(a\tau + b)^{m-\ell}.
\]
Now the relation \( (1 + x)^{n-\ell} = (1 + x)^{i-\ell}(1 + x)^{n-i} \) implies
\[
\sum_{j=\ell}^{m} \binom{i}{j-\ell} \binom{n-i}{m-j} = \binom{n-\ell}{m-\ell}.
\]
This completes the proof of the lemma. \( \Box \)

The following lemma shows the linear independence of a set \( \{\tau^i\} \) over the ring of 1-periodic functions.
Lemma 3. Suppose that each element $f_i \in \mathcal{F} \ (0 \leq i \leq n)$ satisfies $f_i(\tau + 1) = f_i(\tau)$ and $\sum_{i=0}^n f_i(\tau)\tau^i = 0$. Then $f_i = 0$ for each $i$.

Proof. Suppose that $\sum_{i=0}^n f_i(\tau)\tau^i = 0$. Since $f_i(\tau + 1) = f_i(\tau)$ it follows that

$$\sum_{i=0}^n f_i(\tau)\tau^i = 0$$

for every nonnegative integer $\ell$. Then (6) with $\ell = 0, 1, \ldots, n$ imply

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\
\tau & \tau + 1 & \ldots & \tau + n \\
\tau^2 & (\tau + 1)^2 & \ldots & (\tau + n)^2 \\
\vdots & \vdots & \ddots & \vdots \\
\tau^n & (\tau + 1)^n & \ldots & (\tau + n)^n \end{pmatrix}(f_0, f_1, \ldots, f_n) = (0, 0, \ldots, 0).$$

The matrix $A = (a_{ij})$ appeared in the left-hand side of (7) is now the Vandermonde matrix with entries $a_{ij} = (\tau + j - 1)^{i-1}$ and its determinant is

$$\prod_{0 \leq i < j \leq n} \left\{ (\tau + j) - (\tau + i) \right\} = \prod_{j=1}^n j! \neq 0,$$

which shows $(f_0, f_1, \ldots, f_n) = 0$. This completes the proof of the lemma. \qed

2.2 Theorems

Let $V$ be a linear subspace of $\mathcal{F}$ that is invariant with respect to the $|k|$-action of the group $\Gamma = SL_2(\mathbb{Z})$ and $F \in V$ a quasimodular form of weight $n + k$ and depth at most $n$. Then we can obtain a vector-valued modular form of weight $(n + 1)$ with respect to a symmetric tensor representation.

Theorem 4. Let $V$ be an $(n + 1)$-dimensional subspace of $\mathcal{F}$. Suppose that $V$ is invariant with respect to the $|k|$-action of the group $\Gamma = SL_2(\mathbb{Z})$ and that there is a nonzero quasimodular form $F \in V$ of weight $k + n$ and depth at most $n$. Then the function

$$F_i(\tau) = \sum_{\ell=0}^i \binom{n}{\ell}^{-1} i^\ell \mu^{-\ell - \ell \ell} \partial^{(\ell)} F \quad (\mu = \frac{6}{\pi i})$$

is an element of $V$ for each $0 \leq i \leq n$ and the set $\{F_0 = F, F_1, \ldots, F_n\}$ is a basis of $V$. Further the $|k|$-action of $\Gamma$ with respect to this basis is given by

$$F|_k \gamma = \rho^{(n)}(\gamma) F \quad \text{for every } \gamma \in \Gamma,$$

where $F = {t}(F_0, F_1, \ldots, F_n)$.  

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Proof. By definition, the element $F$ takes the form $F = \sum_{i=0}^{n} f_i \cdot (E_2)^i$ where $f_i$ is a modular form of weight $k + n - 2i$ on $\Gamma$. The action $|k\gamma$ of $\gamma \in \Gamma$ on $F$ gives

$$(c\tau + d)^{-k} F(\gamma(\tau)) = (c\tau + d)^{n} \sum_{i=0}^{n} f_i \cdot \left( E_2 + \frac{\mu c}{c\tau + d} \right)^i, \quad \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$$

since $f_i(\gamma(\tau)) = (c\tau + d)^{k+n-2i} f_i(\tau)$ and $(c\tau + d)^{-2} E_2(\gamma(\tau)) = E_2(\tau) + \mu c/(c\tau + d)$. Then it follows by (2) and (9) that

$$F|_{k\gamma} = (c\tau + d)^{n} \sum_{\ell=0}^{n} \left( \frac{\mu c}{c\tau + d} \right)^{\ell} \partial^{(\ell)} F$$

$$= \sum_{\ell=0}^{n} \mu^{\ell} (c\tau + d)^{n-\ell} \partial^{(\ell)} F$$

$$= \sum_{\ell=0}^{n} \mu^{\ell} \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} c^{\ell+j} d^{n-\ell-j} \tau^{j} \partial^{(\ell)} F$$

$$= \sum_{i=0}^{n} \sum_{\ell=0}^{i} \mu^{\ell} \binom{n-\ell}{i-\ell} c^{\ell} d^{n-i} \tau^{i-\ell} \partial^{(\ell)} F$$

$$= \sum_{i=0}^{n} \frac{n}{i} c^{i} d^{n-i} \sum_{\ell=0}^{i} \binom{n-\ell}{\ell}^{-1} \binom{i}{\ell} \mu^{\ell} \tau^{i-\ell} \partial^{(\ell)} F$$

$$= \sum_{i=0}^{n} \frac{n}{i} c^{i} d^{n-i} \tau^{i} F_i(\tau)$$

$$= d^{n} \sum_{i=0}^{n} \frac{n}{i} \left( \frac{c}{d} \right)^{i} F_i(\tau).$$

Now, since $c$ and $d$ are any coprime integers, a rational number $c/d$ can be arbitrary. It then follows that

$$\sum_{i=0}^{n} \binom{n}{i} r^{i} F_i(\tau) \in V$$

(10)

for each rational number $r$. We now take mutually different $n + 1$ rational numbers. Then (10) (by using the Vandermonde determinant) yields that $F_i \in V$ for each $0 \leq i \leq n$.

We now determine the action $|k\gamma$ ($\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$) on each $F_i$. By Lemma 1, it follows
that
\[ F_i \gamma = (c \tau + d)^{-k} F_i (\gamma(\tau)) \]
\[ = (c \tau + d)^{-k} \sum_{\ell=0}^{n} \binom{n}{\ell}^{-1} \binom{i}{\ell} \mu^\ell \gamma(\tau)^{i-\ell} (c \tau + d)^{k+n-2\ell} \sum_{m=\ell}^{n} \binom{m}{\ell} \left( \frac{\mu c}{c \tau + d} \right)^{m-\ell} \partial^{(m)} F \]
\[ = \sum_{m=0}^{n} \mu^m \partial^{(m)} F \sum_{\ell=0}^{i} \binom{n}{\ell}^{-1} \binom{i}{\ell} \binom{m}{\ell} c^{m-\ell} (a \tau + b)^{i-\ell} (c \tau + d)^{n-i-m} . \]

Because the factor \( \binom{i}{\ell} \binom{m}{\ell} \) vanishes if \( \ell > m \) we can take the sum for \( 0 \leq \ell \leq m \). Hence Lemma 2 implies
\[ F_i \gamma = \sum_{m=0}^{n} \mu^m \partial^{(m)} F \sum_{\ell=0}^{i} \binom{n}{\ell}^{-1} \binom{i}{\ell} \binom{m}{\ell} c^{m-\ell} (a \tau + b)^{i-\ell} (c \tau + d)^{n-i-m} \cdot \]

By definition, we know that
\[ (a \tau + b)^{i} (c \tau + d)^{n-i} = \sum_{j=0}^{n} \rho^{(n)}_{ij} (\gamma) \tau^j \]
which implies
\[ \frac{1}{m!} \left( \frac{\partial}{\partial \tau} \right)^m \{ (a \tau + b)^{i} (c \tau + d)^{n-i} \} = \sum_{j=m}^{n} \rho^{(n)}_{ij} (\gamma) \binom{j}{m} \tau^{j-m} . \]

It then follows that
\[ F_i \gamma = \sum_{m=0}^{n} \mu^m \partial^{(m)} F \sum_{\ell=0}^{i} \binom{n}{\ell}^{-1} \sum_{j=m}^{n} \rho^{(n)}_{ij} (\gamma) \binom{j}{m} \tau^{j-m} \]
\[ = \sum_{j=0}^{n} \rho^{(n)}_{ij} (\gamma) \left( \sum_{m=0}^{j} \binom{n}{m}^{-1} \binom{j}{m} \mu^m \tau^{j-m} \partial^{(m)} F \right) \]
\[ = \sum_{j=0}^{n} \rho^{(n)}_{ij} (\gamma) F_j . \]

Equation (11) shows that the action of \( |k\gamma \rangle (\gamma \in \Gamma) \) on \( V \) coincides with the symmetric tensor representation \( \rho^{(n)}(\gamma) \).

Finally we show that \( \{ F_i \} \) is a linearly independent set. Recall from (8) that \( F_i \) takes the form
\[ F_i(\tau) = \sum_{\ell=0}^{i} \binom{n}{\ell}^{-1} \binom{i}{\ell} \mu^{i-\ell} \partial^{(\ell)} F = \tau^i F + \frac{i}{n} \mu^{-i} \partial F + \cdots \]
and that every \( \partial^{(\ell)} F \) is \( 1 \)-periodic. Now suppose that \( \sum_{i=0}^{n} c_i F_i = 0 \) for some complex numbers \( c_i \). Then by Lemma 3, it follows that \( c_i = 0 \) for each \( i \) since the coefficients of \( F \) is \( \tau^i \). This completes the proof of the theorem.
We here give two examples of vector-valued modular forms of which representation is the symmetric tensor representation \( \rho^{(n)} \).

**Examples.** (1) Recall from [3, Theorem 2.1 (1) \( n = 1, k = 6 \)] that \( F = E_2E_4 - E_6 \) is a quasimodular solution of the Kaneko-Zagier equation:

\[
f'' - \frac{k}{6}E_2(\tau)f' + \frac{k(k-1)}{12}E_4(\tau)f = 0, \quad \tau = \frac{d}{dq}.
\]

Let \( F_0(\tau) = F(\tau) \) and \( F_1(\tau) = \tau F_0 + \mu E_4(\tau) \) where \( \mu = 6/\pi i \). Then a short calculation shows that

\[
F_0|_{5\gamma} = dF_0 + cF_1, \quad F_1|_{5\gamma} = bF_0 + aF_1, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})
\]

and hence

\[
\left( \begin{array}{c} F_0 \\ F_1 \end{array} \right) |_{5\gamma} = \left( \begin{array}{cc} d & c \\ b & a \end{array} \right) \left( \begin{array}{c} F_0 \\ F_1 \end{array} \right).
\]

(2) By \( n = 2 \) and \( k = 12 \) in [3, Theorem 2.1 (1)] one has \( F = 12E_2E_4E_6 - 5E_1^3 - 7E_6^2 \) and hence \( F_0 = F, \ F_1 = \tau F_0 + 12\mu E_4 E_6 \). Since this is the case \( n = 1 \) and \( k = 11 \) in Theorem 4 it follows that

\[
F_0|_{11\gamma} = dF_0 + cF_1, \quad F_1|_{11\gamma} = bF_0 + aF_1, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})
\]

which is equivalent to

\[
\left( \begin{array}{c} F_0 \\ F_1 \end{array} \right) |_{11\gamma} = \left( \begin{array}{cc} d & c \\ b & a \end{array} \right) \left( \begin{array}{c} F_0 \\ F_1 \end{array} \right).
\]

**Remark.** One of important aspects of a symmetric tensor representation is that its kernel is trivial. Then one might expect that any vector-valued modular forms on \( \Gamma \) with respect to a representation \( \Gamma \to GL_n(\mathbb{C}) \) that has a trivial kernel contains a quasimodular form as an entry. However, it is not true: In fact, we have a counter example arising from symplectic fermionic vertex algebras ([1]).

Let us define holomorphic functions \( \phi_1, \phi_2, \phi_3 \) on \( \mathbb{H} \) by

\[
\phi_1(\tau) = \frac{\eta(\tau)^2}{\eta(2\tau)\eta(\tau/2)}, \quad \phi_2(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)}, \quad \phi_3(\tau) = \sqrt{2} \frac{\eta(2\tau)}{\eta(\tau)}.
\]

Then formal characters of simple modules for \( d = 1 \) symplectic fermionic vertex algebra \( \chi_0, \chi_1, \chi_2 \) and \( \chi_3 \) are

\[
\chi_0 = \frac{1}{2} \left( \frac{\phi_3(\tau)^2}{2} + \eta(\tau)^2 \right), \quad \chi_1 = \frac{1}{2} \left( \frac{\phi_3(\tau)^2}{2} - \eta(\tau)^2 \right),
\]

\[
\chi_2 = \frac{1}{2} \left( \phi_1(\tau)^2 + \phi_2(\tau)^2 \right), \quad \chi_3 = \frac{1}{2} \left( \phi_1(\tau)^2 - \phi_2(\tau)^2 \right).
\]

Then it is not difficult to see that

\[
\chi_0(\tau + 1) = e^{\pi i/6} \chi_0(\tau), \quad \chi_1(\tau + 1) = e^{\pi i/6} \chi_1(\tau),
\]

\[
\chi_2(\tau + 1) = e^{\pi i/12} \chi_2(\tau), \quad \chi_3(\tau + 1) = e^{\pi i/12} \chi_3(\tau)
\]
and

\[
\begin{align*}
\chi_0(-1/\tau) &= \frac{1}{4}(\chi_2(\tau) - \chi_3(\tau)) + \frac{(-i\tau)}{2}(\chi_0(\tau) - \chi_1(\tau)), \\
\chi_1(-1/\tau) &= \frac{1}{4}(\chi_2(\tau) - \chi_3(\tau)) - \frac{(-i\tau)}{2}(\chi_0(\tau) - \chi_1(\tau)), \\
\chi_2(-1/\tau) &= \frac{1}{2}(\chi_2(\tau) + \chi_3(\tau)) + (\chi_0(\tau) - \chi_1(\tau)), \\
\chi_3(-1/\tau) &= \frac{1}{2}(\chi_2(\tau) + \chi_3(\tau)) - (\chi_0(\tau) - \chi_1(\tau)).
\end{align*}
\]

Further \(\chi_0(-1/\tau) - \chi_1(-1/\tau)\) gives \(\tau(\chi_0(\tau) - \chi_1(\tau))\). The general theory of vertex operator algebras says that the following 5 holomorphic functions on \(\mathbb{H}\) form a vector-valued modular form on \(\Gamma\) (cf. [7]). Let

\[
\begin{align*}
f_0 &:= \chi_0 + \chi_1 = \frac{\eta(2\tau)^2}{\eta(\tau)^2}, \quad f_2 := \chi_0 - \chi_1 = \eta(\tau)^2, \\
f_3 &:= \chi_2 + \chi_3 = \frac{\eta(\tau)^4}{\eta(2\tau)^2\eta(\tau/2)^2}, \quad f_4 := \chi_2 - \chi_3 = \frac{\eta(\tau/2)^2}{\eta(\tau)^2}, \\
f_5 &:= \tau(\chi_0 - \chi_1) = \tau\eta(\tau)^2.
\end{align*}
\]

Then \(\mathbb{F} = \{f_1, f_2, f_3, f_4, f_5\}\) is a vector-valued modular form with weight 0 on \(\Gamma\) with respect to a representation \(\rho : \Gamma \rightarrow \text{GL}_5(\mathbb{C})\). Since

\[
\eta\left(\frac{a\tau + b}{c\tau + d}\right)^2 = \varepsilon(\gamma)(c\tau + d)\eta(\tau)^2, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,
\]

where \(\varepsilon(\gamma)^{24} = 1\). It follows that

\[
f_2(\gamma(\tau)) = d\varepsilon(\gamma)f_2(\tau) - c\varepsilon(\gamma)f_3(\tau), \quad f_5(\gamma(\tau)) = -b\varepsilon(\gamma)f_2(\tau) - a\varepsilon(\gamma)f_5(\tau)
\]

and that each \(\gamma(\tau) (\gamma \in \text{Ker}(\rho))\) satisfies \(b = c = 0\). Therefore we see that \(\text{Ker}(\rho) = \{I, -I\}\).

We show that every vector-valued modular form with weight \(k\) on \(\Gamma\) with respect to a symmetric tensor representation has a quasimodular form as an entry (It is the inverse of the implied statement of Theorem 4.).

**Theorem 5.** Let \(V\) be an \(\text{SL}_2(\mathbb{Z})\)-invariant (under the \(|_k\)-action) \((n + 1)\)-dimensional vector space consisting of functions in \(\mathcal{F}\) which are bounded as \(\text{Im}(\tau) \rightarrow \infty\). Suppose there is a basis \(\{f_0, f_1, \ldots, f_n\}\) of which \((n + 1) \times (n + 1)\) matrix representing \(\text{SL}_2(\mathbb{Z})\) is the symmetric tensor representation \(\rho^{(n)}\) on \(V\). Define \(g_i\) and \(h_i\) (0 ≤ \(i\) ≤ \(n\)) by

\[
\begin{align*}
\ell(g_0, g_1, \ldots, g_n) &= \begin{pmatrix} (-1)^i \binom{i}{j} \tau^{i-j} \end{pmatrix}_{0 \leq i, j \leq n} \ell(f_0, f_1, \ldots, f_n), \\
\ell(h_0, h_1, \ldots, h_n) &= \begin{pmatrix} \binom{n-i}{n-j} \left(\frac{1}{\mu} E_2\right)^{j-i} \end{pmatrix}_{0 \leq i, j \leq n} \ell(g_0, g_1, \ldots, g_n).
\end{align*}
\](13)
Then every \( h_i \) \((0 \leq i \leq n)\) is a modular form of weight \( k + n - 2i \) and there is an inversion formula of (13)

\[
\ell(g_0, g_1, \ldots, g_n) = \left((-1)^{i+j} \left(\frac{n-i}{n-j}\right) \left(\frac{1}{\mu} E_2\right)^{j-i}\right)_{0 \leq i, j \leq n} \ell(h_0, h_1, \ldots, h_n),
\]

and each \( g_i \) \((0 \leq i \leq n)\) is a quasimodular form of weight \( k + n - 2i \) and depth \( n - i \). In particular, \( f_0 = g_0 \) is a quasimodular form of weight \( k + n \) and depth \( n \).

**Proof.** We first show that \( h_i(\tau + 1) = h_i(\tau) \) and \( \tau^{-k-n+2i} h_i(-1/\tau) = h_i(\tau) \) for each \( 0 \leq i \leq n \). Introduce \((n+1) \times (n+1)\) matrices

\[
A(\tau) = \left((-1)^j \left(\frac{i}{j}\right) \tau^{i-j}\right)_{0 \leq i, j \leq n} \quad \text{and} \quad B(\tau) = \left(\left(n-i\right)\left(\frac{1}{\mu} E_2\right)^{j-i}\right)_{0 \leq i, j \leq n}.
\]

Since (obviously) \( B(\tau + 1) = B(\tau) \) it follows that proving \( h_i(\tau + 1) = h_i(\tau) \) is equivalent to showing \( g_i(\tau + 1) = g_i(\tau) \). By definition,

\[
\rho^{(n)} \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right) = \left(\begin{array}{c} i \\ j \end{array}\right)_{0 \leq i, j \leq n}.
\]

It then suffices to show that

\[
A(\tau + 1) \rho^{(n)} \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right) = A(\tau).
\]

Now, the left-hand side of (14) becomes

\[
\left((-1)^j \left(\frac{i}{j}\right) (\tau + 1)^{i-j}\right) \left(\begin{array}{c} i \\ j \end{array}\right) = \sum_{\ell=0}^{n} (-1)^{\ell} \left(\frac{i}{\ell}\right) \left(\frac{\ell}{j}\right) (\tau + 1)^{i-\ell}
\]

and each entry is computed as

\[
\sum_{\ell=0}^{n} (-1)^{\ell} \left(\begin{array}{c} i \\ \ell \end{array}\right) \left(\begin{array}{c} \ell \\ j \end{array}\right) (\tau + 1)^{i-\ell} = \sum_{\ell=0}^{n} (-1)^{\ell} \left(\begin{array}{c} i \\ \ell \end{array}\right) \sum_{m=0}^{i-\ell} \left(\begin{array}{c} i-\ell \\ m \end{array}\right) \tau^{i-\ell-m}
\]

\[
= \sum_{m=0}^{i} \sum_{\ell=0}^{i} (-1)^{\ell} \left(\begin{array}{c} i \\ \ell \end{array}\right) \left(\begin{array}{c} \ell \\ m \end{array}\right) \tau^{i-m}
\]

and each entry is computed as

\[
= \sum_{m=0}^{i} \sum_{\ell=0}^{i} (-1)^{\ell} \left(\begin{array}{c} i \\ \ell \end{array}\right) \left(\begin{array}{c} \ell \\ m \end{array}\right) \tau^{i-m}.
\]

\[11\]
The coefficient of $\tau^{i-m}$ in (15) divided by $\binom{i}{m}$ is

\[
\sum_{\ell=0}^{i} (-1)^\ell \binom{\ell}{j} \binom{m}{\ell} = \begin{cases} 
0 & \text{if } i < j, \\
\sum_{\ell=j}^{i} (-1)^\ell \binom{\ell}{j} \binom{m}{\ell} & \text{if } i \geq j
\end{cases}
\]

\[
= \sum_{\ell=j}^{i} (-1)^\ell \binom{m-j}{\ell-j} \binom{m}{j} 
= \binom{m}{j} \sum_{\ell=j}^{m} (-1)^\ell \binom{m-j}{\ell-j}
= (-1)^j \binom{m}{j} \sum_{\ell=j}^{m} (-1)^{\ell-j} \binom{m-j}{\ell-j}
= (-1)^m \binom{m}{j} \delta_{mj}.
\]

Hence, the right-hand side of (15) becomes $(-1)^j \binom{m}{j} \tau^{i-j}$. Thus, we have proved (14).

We now verify the transformation of each $h_i(\tau) (0 \leq i \leq n)$ under $\tau \rightarrow -1/\tau$. Since

\[
B(\tau)A(\tau) = \left(-1\right)^j \sum_{\ell=0}^{n} \left(\frac{n-i}{n-\ell}\right) \frac{1}{\mu} E_2 \binom{\ell}{j} \frac{\ell-i}{\tau^{\ell-j}},
\]

it follows that

\[
h_i(\tau) = \sum_{j=0}^{n} \sum_{\ell=0}^{n} (-1)^j \binom{n-i}{n-\ell} \binom{\ell}{j} \frac{1}{\mu} E_2 \binom{\ell-i}{j} \tau^{\ell-j} f_j(\tau)
\]

\[
= \sum_{j=0}^{n} \sum_{\ell=0}^{n} (-1)^{n-j} \binom{n-i}{n-\ell} \binom{\ell}{n-j} \frac{1}{\mu} E_2 \binom{\ell-i}{n-j} \tau^{\ell+j-n} f_{n-j}(\tau).
\]

Now, by the definition of $\Gamma$-module $(V, \rho^{(n)})$, we have $f_{n-j}(1/\tau) = (-1)^{n-j} \tau^k f_j(\tau)$ and hence

\[
h_i(-1/\tau) = \sum_{j=0}^{n} \sum_{\ell=0}^{n} (-1)^{n+\ell+j} \binom{n-i}{n-\ell} \binom{\ell}{n-j} \tau^{2\ell-2i} \left(\frac{E_2}{\mu} + \frac{1}{\tau}\right)^{\ell-i} \tau^{-\ell+n-j} \tau^k f_j,
\]

i.e.

\[
\tau^{-k-n+2i} h_i(-1/\tau) = \sum_{j=0}^{n} (-1)^j \tau^{-j} f_j \sum_{\ell=0}^{n} (-1)^{\ell+n} \binom{n-i}{n-\ell} \binom{\ell}{n-j} \left(\frac{E_2}{\mu} + \frac{1}{\tau}\right)^{\ell-i} \tau^\ell, \quad (16)
\]
where the second sum on the right-hand side becomes
\[\sum_{\ell=0}^{n} (-1)^{\ell-n} \binom{n-\ell}{n-\ell} \sum_{m=0}^{\ell-i} \left( \frac{1}{\mu E_2} \right)^{\ell-i-m} \tau^{\ell-m}\]
\[= \sum_{\ell=0}^{n} (-1)^{\ell-n} \binom{n-\ell}{n-\ell} \sum_{m=1}^{\ell-i} \left( \frac{1}{\mu E_2} \right)^{m-i} \tau^{m-\ell-m}\]
\[= \sum_{m=1}^{n} \tau^{m} \left( \frac{1}{\mu E_2} \right)^{m-i} \sum_{\ell=0}^{n-1} (-1)^{\ell-n} \binom{n-\ell}{n-\ell} \binom{\ell}{n-m} \binom{n-m}{n-j} \binom{\ell}{n-j} \binom{\ell}{n-j}.\]

The second sum on the right of the last equality is
\[\sum_{\ell=0}^{n} (-1)^{\ell-n} \binom{n-m}{n-\ell} \binom{\ell}{n-j} = \sum_{\ell=0}^{n} (-1)^{\ell-n} \binom{n-m}{n-\ell} \binom{\ell}{n-j}\]
\[= \sum_{\ell=0}^{j} (-1)^{\ell} \binom{n-m}{\ell} \binom{n-\ell}{j}\]
\[= \binom{m}{j},\]

where the second sum of first equality may run from \(\ell = 0\) to \(n\) because if \(\ell < i\) and \(n - i \geq n - m\), we have \(n - \ell > n - m\) and then \(\binom{n-m}{n-\ell} = 0\). Therefore, it follows from (16) that
\[\tau^{-k-n+2i} h_i(-1/\tau) = \sum_{j=0}^{n} (-1)^{j} \tau^{-j} f_j(\tau) \sum_{m=1}^{n} \tau^{m} \left( \frac{1}{\mu E_2} \right)^{m-i} \binom{n-i}{n-m} \binom{m}{j}\]
\[= \sum_{j=0}^{n} \sum_{m=0}^{n} (-1)^{j} \binom{n-i}{n-m} \binom{m}{j} \left( \frac{1}{\mu E_2} \right)^{m-i} \tau^{m-j} f_j(\tau)\]
\[= h_i(\tau).\]

Finally we show the inversion formula. It suffices to show that
\[\left( -1 \right)^{i+j} \binom{n-i}{n-j} \left( \frac{1}{\mu E_2} \right)^{j-i} \binom{n-i}{n-j} \left( \frac{1}{\mu E_2} \right)^{j-i} = I_{n+1}.\]

The \((i, j)\) entry of the left hand-side of the last equation is
\[\left( \frac{1}{\mu E_2} \right)^{j-i} \sum_{k=0}^{n} (-1)^{i+k} \binom{n-i}{n-k} \binom{n-k}{n-j} = \left( \frac{1}{\mu E_2} \right)^{j-i} \sum_{k=0}^{n} (-1)^{i+n-k} \binom{n-i}{n-k} \binom{k}{n-j}\]
\[= \left( \frac{1}{\mu E_2} \right)^{j-i} \binom{n-i}{n-j} \delta_{n-i,n-j}\]
\[= \left( \frac{1}{\mu E_2} \right)^{j-i} \delta_{ij} = \delta_{ij}.\]
This completes the proof of the theorem.

2.3 A criterion for determining quasimodular forms

In this short (but important) section we show a way to determine the exact form (in terms of $E_2$, $E_4$ and $E_6$) of a quasimodular form of given weight on $\Gamma = SL_2(\mathbb{Z})$.

Recall that the space of quasimodular form on $\Gamma$ is a graded ring $\mathbb{C}[E_2, E_4, E_6] = \bigoplus_{k \geq 0} \tilde{M}_k$ where

$$\tilde{M}_k = \sum_{a, b, c \geq 0} \mathbb{C} \cdot E_2^a E_4^b E_6^c.$$

Let $f$ be a quasimodular form of weight $k$ with a $q$-expansion $f = a_0 + a_1 q + \cdots + c_n q^n + \cdots$. Suppose that

$$E_2^a E_4^b E_6^c = c_0^{(a, b, c)} + c_1^{(a, b, c)} q + \cdots + c_n^{(a, b, c)} q^n + \cdots.$$

Then we have a system of linear equations

$$\sum_{(a, b, c)} c_i^{(a, b, c)} x^{(a, b, c)} = a_i \quad (0 \leq i \leq d_k - 1), \quad (17)$$

where $d_k = \dim_{\mathbb{C}} \tilde{M}_k$. Suppose that (17) has a solution $(x^{(a, b, c)})$ and that the coefficient of $q^{d_k}$ in

$$f - \sum_{a, b, c \geq 0} x^{(a, b, c)} E_2^a E_4^b E_6^c \quad (18)$$

vanishes. Then we can conclude that

$$f = \sum_{a, b, c \geq 0} x^{(a, b, c)} E_2^a E_4^b E_6^c.$$

3 Applications to the theory of vertex operator algebras

In this section we give several applications of our theorems (Theorem 4 and Theorem 5) to the theory of vertex operator algebras (or equivalently 2DCFT).

Let $V$ be a vertex operator algebra and $\mathcal{C}(V)$ the space of one-point functions on a family of elliptic curves (one-point functions for short). Then it is known that the space $\mathcal{C}_V(1)$ of one-point functions evaluated at the vacuum $1$ is invariant with respect to the slash $|_0$-action of $\Gamma = SL_2(\mathbb{Z})$ and then each basis of $\mathcal{C}_V(1)$ is a (meromorphic) vector-valued modular form of weight 0. This aspect is one of motivations of our research on vector-valued modular forms. One of difficulties of study on one-point functions is that it is rather hard to find a basis of the space $\mathcal{C}(V)$. In fact, there are only few examples of which basis are known in (especially) non-rational theory (cf. [2]). Each element of the space $\mathcal{C}_V(1)$ is called pseudo-characters of $V$. Of course, determining a pseudo-character is not easy in general, however,
formal characters of a vertex operator algebra and its simple modules, which are a typical examples of pseudo-characters is mostly computable (at least several terms of $q$-expansion and therefore one can know if they are quasimodular forms on $\Gamma$). By a short calculation we have

**Lemma 6.** Let $f$ be a quasimodular form of weight $k$ and depth $d$ (at most $n$) on $SL_2(\mathbb{Z})$. Then

$$f(-1/\tau) = \tau^k g(\tau) E_2(\tau)^d + \left( \tau^k g_1(\tau) + \mu \tau^{k-1} g_0(\tau) \right) E_2^{d-1} + \cdots,$$

where $g_i$ ($i = 1, 2$) is a modular form of weight $k - 2d + 2i$ on $SL_2(\mathbb{Z})$.

**Definition.** Let $V$ be a vertex operator algebra and $C^2(V)$ the vector subspace of $V$ that is linearly generated by $a(-2)b$ for all $a, b \in V$. If the quotient space $V/C(V)$ is finite-dimensional, then $V$ is called a $C_2$-cofinite vertex operator algebra.

**Proposition 7.** Let $V$ be a $C_2$-cofinite vertex operator algebra and suppose that there is a quasimodular pseudo-character. Then $V$ is not rational.

**Proof.** Suppose that there is a quasimodular pseudo-character. Then by Lemma 6 there is a pseudo-character with logarithmic terms of $q$. Therefore $V$ is not rational since every element of $\mathbb{C}(1)$ has no logarithmic form if $V$ is rational and $C_2$-cofinite. \qed

**Corollary.** Let $V$ be a $C_2$-cofinite vertex operator algebra and suppose that the formal character of $V$ is a quasimodular form. Then $V$ is not rational.

It is well known that if $V$ is $C_2$-cofinite then the space of pseudo-characters is finite-dimensional, and central charge and conformal weights are rational numbers (see [6] and [7]). Then the following theorem is useful under most circumstances.

**Theorem 8.** Let $V$ be a $C_2$-cofinite vertex operator algebra and $\dim \mathbb{C}_V(1) = n + 1$ where $n$ is a nonnegative integer. Suppose that there is a pseudo-character $f$ which is a quasimodular form of weight $n$ on $\Gamma = SL_2(\mathbb{Z})$. Then

$$f_i(\tau) = \sum_{\ell=0}^i \binom{n}{\ell}^{-1} \binom{i}{\ell} \mu^\ell \tau^{i-\ell} \partial^{(\ell)} f, \quad \mu = \frac{6}{\pi i}$$

is an element of $V$ for each $0 \leq i \leq n$ and the set $\{f_0, f_1, \ldots, f_n\}$ is a basis of $V$. Moreover, the $|0\rangle$-action of $\Gamma$ with respect to this basis is given by

$$\mathbb{F}|0\rangle \gamma = \rho^{(n)}(\gamma) \mathbb{F} \text{ for every } \gamma \in \Gamma,$$

where $\mathbb{F} = \tau(f_0, f_1, \ldots, f_n)$.

**Proof.** Since the space $\mathbb{C}_V(1)$ is invariant with respect to the $|0\rangle$-action of $\Gamma$, it follows from Theorem 4 that there are holomorphic functions $f_i$ ($0 \leq i \leq n$) such that $\mathbb{F}|0\rangle \gamma = \rho^{(n)}(\gamma) \mathbb{F}$ for every $\gamma \in \Gamma$ where $\mathbb{F} = \tau(f_0, f_1, \ldots, f_n)$ and that $\{f_0, f_1, \ldots, f_n\}$ is a basis of $\mathbb{C}_V(1)$. This completes the proof of the theorem. \qed
Theorem 9. Let $V$ be a $C_2$-cofinite vertex operator algebra and $\dim \mathcal{C}_V(1) = n + 1$. Suppose that each element of $\mathcal{C}_V(1)$ belongs to $\mathcal{F}$ and is bounded as $\text{Im}(\tau) \to \infty$, and that there is a basis $(f_0, f_1, \ldots, f_n)$ of $\mathcal{C}_V(1)$, of which $(n+1) \times (n+1)$ matrix representing the $|0\rangle$-action of $SL_2(\mathbb{Z})$ is the symmetric representation $\rho^{(n)}$. Then $f_0$ is a quasimodular pseudo-character of weight $n$. In particular $V$ is not rational.

Proof. It follows from Theorem 5 that there is a quasimodular pseudo-character of weight $n$ (it is $g_0$ in Theorem 5). Then Lemma 6 completes the proof of theorem. \qed

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